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On the slice genus of links

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Abstract We de ne Casson-Gordon -invariants for links and give a lower bound of the slice genus of a link in terms of these invariants. We study as an example a family of two component links of genus h and show that their slice genus is h, whereas the Murasugi-Tristram inequality does not obstruct this link from bounding an annulus in the 4-ball.

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1 Introduction

A knot in S^3 is slice if it bounds a smooth 2-disk in the 4-ball B^4. Levine showed [Le] that a slice knot is algebraically slice, i.e. any Seifert form of a slice knot is metabolic. In this case, the Tristram-Levine signatures at the prime power order roots of unity of a slice knot must be zero. Levine showed also that the converse holds in high odd dimensions, i.e. any algebraically slice knot is slice. This is false in dimension 3: Casson and Gordon [CG1, CG2, G] showed that certain two-bridge knots in S^3, which are algebraically slice, are not slice knots. For this purpose, they de ned several knot and 3-manifold invariants, closely related to the Tristram-Levine signatures of associated links. Further methods to calculate these invariants were developed by Gilmer [Gi3, Gi4], Litherland [Li], Gilmer-Livingston [GL], and Naik [N]. Lines [L] also computed some of these invariants for some bered knots, which are algebraically slice but not slice. The slice genus of a link is the minimal genus for a smooth oriented connected surface properly embedded in B^4 with boundary the given link.

The Murasugi-Tristram inequality (see Theorem 2.1 below) gives a lower bound on the slice genus of a link in terms of the link's Tristram-Levine signatures and related nullity invariants. The second author [Gi1] used Casson-Gordon invariants to give another lower bound on the slice genus of a knot. In particular he gave examples of algebraically slice knots whose slice genus is arbitrarily large. We apply these methods to restrict the slice genus of a link.

We study as an example a family of two component links, which have genus h Seifert surfaces. Using Theorem 4.1, we show that these links cannot bound a smoothly embedded surface in B^4 with genus lower than h, while the Murasugi-Tristram inequality does not show this. In fact there are some links with the same Seifert form that bound annuli in B^4. We work in the smooth category.

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2 Preliminaries

2.1 The Tristram-Levine signatures

Let L be an oriented link in S^3, with μ components, and θ_S be the Seifert pairing corresponding to a connected Seifert surface S of the link. For any complex number ω with $|\omega| = 1$, one considers the hermitian form $H_S := (1-\omega)\theta_S + (1-\bar{\omega})(\theta_S)^T$. The Tristram signature $\sigma_L(\omega)$ and nullity $n_L(\omega)$ of L are defined as the signature and nullity of H_S. Levine defined these same signatures for knots [Le]. The Alexander polynomial of L is $\Delta_L(t) := \mathrm{Det}(\theta_S - t(\theta_S)^T)$. As is well-known, σ_L is a locally constant map on the complement in S^1 of the roots of Δ_L and n_L is zero on this complement. If $\Delta_L = 0$, it is still true that the signature and nullity are locally constant functions on the complement of some finite collection of points.

The Murasugi-Tristram inequality allows one to estimate the slice genus of L, in terms of the values of $\sigma_L(\omega)$ and $n_L(\omega)$.

Theorem 2.1 [M, T] *Suppose that L is the boundary of a properly embedded connected oriented surface F of genus g in B^4. Then, if ω is a prime power order root of unity, we have*

$$|\sigma_L(\omega)| + n_L(\omega) \leq 2g + \mu - 1.$$

2.2 The Casson-Gordon σ-invariant

In this section, for the reader convenience, we review the definition and some of the properties of the simplest kind of Casson-Gordon invariant. It is a reformulation of the Atiyah-Singer α-invariant.

Let M be an oriented compact three manifold and $\chi: H_1(M) \to \mathbb{C}$ be a character of finite order. For some $q \in \mathbb{N}$, the image of χ is contained a cyclic subgroup of order q generated by $\omega = e^{2\pi i/q}$. As $\mathrm{Hom}(H_1(M); C_q) = [M; B(C_q)]$, it follows that χ induces q-fold covering of M, denoted $\widetilde{M}$, with a canonical deck transformation. We will denote this transformation also by τ. If χ maps onto C_q, the canonical deck transformation sends x to the other endpoint of the arc

that begins at x and covers a loop representing an element of $(\chi)^{-1}(\omega)$.

As the bordism group $\Omega_3(B(C_q)) = C_q$, we may conclude that n disjoint copies of M, for some integer n, bounds bound a compact 4-manifold W over $B(C_q)$. Note n can be taken to be q. Let $\widetilde{W}$ be the induced covering with the deck transformation, denoted also by τ, that restricts to τ on the boundary. This induces a $\mathbb{Z}[C_q]$- module structure on $C_*(\widetilde{W})$, where the multiplication by $\tau \in \mathbb{Z}[C_q]$ corresponds to the action of τ on $\widetilde{W}$.

The cyclotomic field $\mathbb{Q}(C_q)$ is a natural $\mathbb{Z}[C_q]$-module and the twisted homology $H^t_*(W; \mathbb{Q}(C_q))$ is defined as the homology of

$$C_*(\widetilde{W}) \otimes_{\mathbb{Z}[C_q]} \mathbb{Q}(C_q).$$

Since $\mathbb{Q}(C_q)$ is flat over $\mathbb{Z}[C_q]$, we get an isomorphism

$$H^t_*(W; \mathbb{Q}(C_q)) \simeq H_*(\widetilde{W}) \otimes_{\mathbb{Z}[C_q]} \mathbb{Q}(C_q).$$

Similarly, the twisted homology $H^t_*(M; \mathbb{Q}(C_q))$ is defined as the homology of

$$C_*(\widetilde{M}) \otimes_{\mathbb{Z}[C_q]} \mathbb{Q}(C_q).$$

Let $\widetilde{\lambda}$ be the intersection form on $H_2(\widetilde{W}; \mathbb{Q})$ and define

$$\lambda(W) : H^t_2(W; \mathbb{Q}(C_q)) \otimes H^t_2(W; \mathbb{Q}(C_q)) \to \mathbb{Q}(C_q)$$

so that, for all a, b in $\mathbb{Q}(C_q)$ and x, y in $H_2(\widetilde{W})$,

$$\lambda(W)(x \otimes a, y \otimes b) = \bar{a} b \sum_{i=1}^{q} \widetilde{\lambda}(x, \tau^i y) \omega^{-i},$$

where $a \to \bar{a}$ denotes the involution on $\mathbb{Q}(C_q)$ induced by complex conjugation.

Definition 2.2 The Casson-Gordon σ-invariant of (M, χ) and the related nullity are

$$\sigma(M, \chi) := \frac{1}{n}\left(\mathrm{Sign}(\lambda(W)) - \mathrm{Sign}(W)\right)$$

$$\eta(M, \chi) := \dim H^t_1(M; \mathbb{Q}(C_q)).$$

If U is a closed 4-manifold and $\chi : H_1(U) \to C_q$ we may define $\sigma(U)$ as above. One has that modulo torsion the bordism group $\Omega_4(B(C_q))$ is generated by the constant map from $CP(2)$ to $B(C_q)$. If χ is trivial, one has that $\mathrm{Sign}(\chi(U)) = \mathrm{Sign}(U)$. Since both signatures are invariant under cobordism, one has in general that $\mathrm{Sign}(\chi(U)) = \mathrm{Sign}(U)$. The independence of $\sigma(M,\chi)$ from the choice of W and n follows from this and Novikov additivity. One may see directly that these invariants do not depend on the choice of q. In this way Casson and Gordon argued that $\sigma(M,\chi)$ is an invariant. Alternatively one may use the Atiyah-Singer G-Signature theorem and Novikov additivity [AS].

We now describe a way to compute $\sigma(M,\chi)$ for a given surgery presentation of (M,χ).

Definition 2.3 Let K be an oriented knot in S^3. Let A be an embedded annulus such that $\partial A = K \cup K'$ with $lk(K,K') = f$. A *p-cable on K with twist f* is defined to be the union of oriented parallel copies of K lying in A such that the number of copies with the same orientation minus the number with opposite orientation is equal to p.

Let us suppose that M is obtained by surgery on a framed link $L = L_1 \cup \cdots \cup L_\mu$ with framings $f_1,\dots,f_\mu$. One shows that the linking matrix Λ of L with framings in the diagonal is a presentation matrix of $H_1(M)$ and a character on $H_1(M)$ is determined by $\omega^{p_i} = \chi(m_{L_i}) \in C_q$ where m_{L_i} denotes the class of the meridian of L_i. Let $\vec{p} = (p_1,\dots,p_\mu)$. We use the following generalization of a formula in [CG2, Lemma (3.1)], where all p_i are assumed to be 1, that is given in [Gi2, Theorem(3.6)].

Proposition 2.4 *Suppose χ maps onto C_q. Let L' with μ' components be the link obtained from L by replacing each component by a non-empty algebraic p_i-cable with twist f_i along this component. Then, if $\omega = e^{2\pi i r/q}$, for $(r,q) = 1$, one has*

$$\sigma(M,\chi^r) = \sigma_{L'}(\omega) - \mathrm{Sign}(\Lambda) + 2\frac{r(q-r)}{q^2}\vec{p}^{\,>}\Lambda\vec{p},$$

$$\eta(M,\chi^r) = \eta_{L'}(\omega) - \mu' + \mu.$$

The following proposition collects some easy additivity properties of the σ-invariant and the nullity under the connected sum.

Proposition 2.5 *Suppose that M_1, M_2 are connected. Then, for all $\chi_i \in H^1(M_i,C_q)$, $i = 1,2$, we have*

$$\sigma(M_1 \# M_2, \chi_1 \oplus \chi_2) = \sigma(M_1,\chi_1) + \sigma(M_2,\chi_2).$$

If both $_i$ are non-trivial, then

$$(M_1 \# M_2; \ _1 \quad _2) = (M_1; \ _1) + (M_2; \ _2) + 1.$$

If one $_i$ is trivial, then

$$(M_1 \# M_2; \ _1 \quad _2) = (M_1; \ _1) + (M_2; \ _2).$$

Proposition 2.6 *For all* $\in H_1(S^1 \quad S^2; C_q)$, *we have*

$$(S^1 \quad S^2; \ ) = 0$$

If $\neq 0$, *then* $(S^1 \quad S^2; \ ) = 0$. *If* $= 0$, *then* $(S^1 \quad S^2; \ ) = 1$.

Proposition 2.6 for non-trivial can be proved for example by the use of Proposition 2.4, since $S^1 \quad S^2$ is obtained by surgery on the unknot framed 0. However it is simplest to derive this result directly from the definitions.

2.3 The Casson-Gordon -invariant

In this section, we recall the definition and some of the properties of the Casson-Gordon -invariant. Let C_∞ denote a multiplicative infinite cyclic group generated by t. For $^+ : H_1(M) \to C_q \quad C_\infty$, we denote $: H_1(M) \to C_q$ the character obtained by composing $^+$ with projection on the first factor. The character $^+$ induces a $C_q \quad C_\infty$-covering $\tilde{M}_\infty$ of M.

Since the bordism group $\Omega_3(B(C_q \quad C_\infty)) = C_q$; bounds a compact 4-manifold W over $B(C_q \quad C_\infty)$ Again n can be taken from to be q.

If we identify $\mathbb{Z}[C_q \quad C_\infty]$ with the Laurent polynomial ring $\mathbb{Z}[C_q][t; t^{-1}]$, the field $\mathbb{Q}(C_q)(t)$ of rational functions over the cyclotomic field $\mathbb{Q}(C_q)$ is a flat $\mathbb{Z}[C_q \quad C_\infty]$-module. We consider the chain complex $C(\tilde{W}_\infty)$ as a $\mathbb{Z}[C_q \quad C_\infty]$-module given by the deck transformation of the covering. Since W is compact, the vector space $H_2^t(W; \mathbb{Q}(C_q)(t)) \cong H_2(\tilde{W}_\infty) \quad _{\mathbb{Z}[C_q][t;t^{-1}]} \mathbb{Q}(C_q)(t)$ is finite dimensional.

We let J denote the involution on $\mathbb{Q}(C_q)(t)$ that is linear over $\mathbb{Q}$ sends t^i to t^{-i} and ^{i} to $^{-i}$. As in [G], one defines a hermitian form, with respect to J,

$$_+ : H_2^t(W; \mathbb{Q}(C_q)(t)) \quad H_2^t(W; \mathbb{Q}(C_q)(t)) \to \mathbb{Q}(C_q)(t);$$

such that

$$_+(x \quad a; y \quad b) = J(a) \ b \sum_{i \in \mathbb{Z}} \sum_{j=1} \tilde{}_+(x; t^i \ ^j y)^{-j} t^{-i}.$$

Here $\tilde{f}_+$ denotes the ordinary intersection form on $\tilde{W}_\infty$. Let $W(\mathbb{Q}(C_q)(t))$ be the Witt group of non-singular hermitian forms on finite dimensional $\mathbb{Q}(C_q)(t)$ vector spaces. Let us consider $H_2^t(W;\mathbb{Q}(C_q)(t))/(\text{Radical}(f_+))$. The induced form on it represents an element in $W(\mathbb{Q}(C_q)(t))$, which we denote $w(W)$. Furthermore, the ordinary intersection form on $H_2(W;\mathbb{Q})$ represents an element of $W(\mathbb{Q})$. Let $w_0(W)$ be the image of this element in $W(\mathbb{Q}(C_q)(t))$.

Definition 2.7 The Casson-Gordon τ-invariant of (M,χ^+) is

$$\tau(M,\chi^+) := \frac{1}{n}\left(w(W) - w_0(W)\right) \in W(\mathbb{Q}(C_q)(t)) \otimes \mathbb{Q}.$$

Suppose that nM bounds another compact 4-manifold W' over $B(C_q \times C_\infty)$. Form the closed compact manifold over $B(C_q \times C_\infty)$, $U := W \cup -W'$ by gluing along the boundary. By Novikov additivity, we get $w(U) - w_0(U) = \left(w(W) - w_0(W)\right) - \left(w(W') - w_0(W')\right)$. Using [CF], the bordism group $\Omega_4(B(C_q \times C_\infty))$, modulo torsion, is generated by $CP(2)$, with the constant map to $B(C_q \times C_\infty)$. We have that $w(CP(2)) = w_0(CP(2))$. Since $w(U)$, and $w_0(U)$ only depend on the bordism class of U over $B(C_q \times C_\infty)$, it follows that $w(U) = w_0(U)$ and $\tau(M,\chi^+)$ is independent of the choice of W. Using the above techniques, one may check $\tau(M,\chi^+)$ is independent of n.

If $A \in W(\mathbb{Q}(C_q)(t))$, let $A(t)$ be a matrix representative for A. The entries of $A(t)$ are Laurent polynomials with coefficients in $\mathbb{Q}(C_q)$. If ξ is in $S^1 \subset \mathbb{C}$, then $A(\xi)$ is hermitian and has a well defined signature $\sigma_\xi(A)$. One can view $\sigma_\xi(A)$ as a locally constant map on the complement of the set of the zeros of $\det A(\xi)$. As in [CG1], we re-define $\sigma_\xi(A)$ at each point of discontinuity as the average of the one-sided limits at the point.

We have the following estimate [Gi3, Equation (3.1)].

Proposition 2.8 *Let $\chi^+ : H_1(M) \to C_q \times C_\infty$ and $\chi : H_1(M) \to C_q$ be χ^+ followed by the projection to C_q. We have*

$$\left|\sigma_1\left(\tau(M,\chi^+)\right) - \sigma(M,\chi)\right| \le \eta(M,\chi).$$

2.4 Linking forms

Let M be a rational homology 3-sphere with linking form

$$l : H_1(M) \times H_1(M) \to \mathbb{Q}/\mathbb{Z}.$$

We have that l is non-singular, that is the adjoint of l is an isomorphism $\phi : H_1(M) \to \mathrm{Hom}(H_1(M),\mathbb{Q}/\mathbb{Z})$. Let $H_1(M)^\wedge$ denote $\mathrm{Hom}(H_1(M),\mathbb{C}^*)$. Let

χ denote the map $\mathbb{Q}/\mathbb{Z} \to \mathbb{C}$ that sends $\frac{a}{b}$ to $e^{\frac{2\pi i a}{b}}$. So we have an isomorphism $\imath\colon H_1(M) \to \widehat{H_1(M)}$ given by $x \mapsto \chi \circ \lambda(x, \cdot)$. Let $\hat\lambda\colon \widehat{H_1(M)} \times \widehat{H_1(M)} \to \mathbb{Q}/\mathbb{Z}$ be the dual form defined by $\hat\lambda(\imath x, \imath y) = -l(x, y)$.

Definition 2.9 The form $\hat\lambda$ is metabolic with metabolizer H if there exists a subgroup H of $\widehat{H_1(M)}$ such that $H^{\perp} = H$.

Lemma 2.10 [Gi1] *If M bounds a spin 4-manifold W then $\hat\lambda = \lambda_1 \oplus \lambda_2$ where λ_2 is metabolic and λ_1 has an even presentation with rank dim $H_2(W;\mathbb{Q})$ and signature Sign(W). Moreover, the set of characters that extend to $H_1(W)$ forms a metabolizer for λ_2.*

2.5 Link invariants

Let $L = L_1 \cup \dots \cup L_\mu$ be an oriented link in S^3. Let N_2 be the two-fold covering of S^3 branched along L and λ_L be the linking form on $\widehat{H_1(N_2)}$, see previous section.

We suppose that the Alexander polynomial of L satisfies

$$\Delta_L(-1) \neq 0.$$

Hence, N_2 is a rational homology sphere. Note that if $\Delta_L(-1) \neq 1$, then $H_1(N_2;\mathbb{Z})$ is non-trivial.

Definition 2.11 For all characters χ in $\widehat{H_1(N_2)}$, the Casson-Gordon σ-invariant of L and the related nullity are (see Definition 2.2):

$$\sigma(L, \chi) := \sigma(N_2, \chi);$$

$$\eta(L, \chi) := \eta(N_2, \chi).$$

Remark 2.12 If L is a knot, then Definition 2.11 coincides with $\sigma(L, \chi)$ defined in [CG1, p.183].

3 Framed link descriptions

In this section, we study the Casson-Gordon -invariants of the two-fold cover M_2 of the manifold M_0 described below.

Let $S^3 - T(L)$ be the complement in S^3 of an open tubular neighborhood of L in S^3 and P be a planar surface with boundary components.

Let S be a Seifert surface for L and γ_i for $i = 1, \dots,$ be the curves where S intersects the boundary of $S^3 - T(L)$. We define M_0 as the result of gluing $P \; S^1$ to $S^3 - T(L)$, where $P \; 1$ is glued along the curves γ_i. Let be a point in the boundary of P.

A recipe for drawing a framed link description for M_0 is given in the proof of Proposition 3.1.

Proposition 3.1

$$H_1(M_0) \simeq \mathbb{Z} \quad \mathbb{Z}^{-1} \simeq \langle m \rangle \quad \mathbb{Z}^{-1};$$

where m denotes the class of S^1 in $P \; S^1$.

Proof Form a 4-manifold X by gluing $P \; D^2$ to D^4 along S^3 in such a way that the total framing on L agrees with the Seifert surface S. The boundary of this 4-manifold is M_0. We can get a surgery description of M_0 in the following way: pick -1 paths of S joining up the components of L in a chain. Deleting open neighborhoods of these paths in S gives a Seifert surface for a knot L' obtained by doing a fusion of L along bands that are neighborhoods of the original paths. Put a circle with a dot around each of these bands (representing a 4-dimensional 1-handle in Kirby's [K] notation), and the framing zero on L'. This describes a handlebody decomposition of X.

One can then get a standard framed link description of M_0 by replacing the circle with dots with unknots $T_1, \dots, T_{-1}$ framed zero. This changes the 4-manifold but not the boundary. Note also that $lk(T_i, T_j) = 0$ and $lk(T_i, L') = 0$ for all $i = 1, \dots, -1$. Hence $H_1(M_0) \simeq \mathbb{Z}$ and m represents one of the generators. □

We now consider an infinite cyclic covering M_∞ of M_0, defined by a character $H_1(M_0) \to C_\infty = \langle t \rangle$ that sends m to t and the other generators to zero. Let us denote by M_2 the intermediate two-fold covering obtained by composing this character with the quotient map $C_\infty \to C_2$ sending t to -1. Let m_2 denote the loop in M_2 given by the inverse image of m. A recipe for drawing a framed link description for M_2 is given in the proof of Remark 3.3.

Proposition 3.2 *There is an isomorphism between $H_1(N_2)$ and the torsion subgroup of $H_1(M_2)$, which only depends on L. Moreover*

$$H_1(M_2) \cong H_1(N_2) \oplus \mathbb{Z} \cong H_1(N_2) \oplus \langle m_2 \rangle \oplus \mathbb{Z}^{\mu-1}.$$

Proof Let R be the result of gluing $P \times D^2$ to $S^3 \times I$ along $L \times 1 \subset S^3 \times 1$ using the framing given by the Seifert surface. Thus R is the result of adding $\mu - 1$ 1-handles to $S^3 \times I$ and then one 2-handle along L', as in the proof above. Then X in the proof above can be obtained by gluing D^4 to R along $S^3 \times 0.$ Since D^2 is the double branched cover of itself along the origin, $P \times D^2$ is the double branched cover of itself along $P \times 0$. Let R_2 denote the double branched cover of R that is obtained by gluing $P \times D^2$ to $N_2 \times I$ along a neighborhood of the lift of $L \times 1 \subset S^3 \times 1.$ We have that $\partial R_2 = -N_2 \sqcup M_2$, where R_2 is the result of adding $\mu - 1$ 1-handles to $N_2 \times I$ and then one 2-handle along the lift $L'.$ Moreover this lift of L' is null-homologous in $N_2.$ It follows that $H_1(R_2)$ is isomorphic to $H_1(N_2) \oplus \mathbb{Z}^{\mu-1};$ with the inclusion of N_2 into R_2 inducing an isomorphism i_N of $H_1(N_2)$ to the torsion subgroup of $H_1(R_2).$ Turning this handle decomposition upside down we have that R_2 is the result of adding to $M_2 \times I$ one 2-handle along a neighborhood of m_2 and then $\mu - 1$ 3-handles. It follows that $H_1(R_2) \oplus \mathbb{Z} = H_1(R_2) \oplus \langle m_2 \rangle$ is isomorphic to $H_1(M_2)$ with the inclusion of M_2 in R_2 inducing an isomorphism i_M of the torsion subgroup $H_1(M_2)$ to the torsion subgroup of $H_1(R_2).$ Thus $(i_M)^{-1} \circ i_N$ is an isomorphism from $H_1(N_2)$ to the torsion subgroup of $H_1(M_2)$ and this isomorphism is constructed without any arbitrary choices. □

Remark 3.3 We could have proved Proposition 3.1 in a similar way to the proof of Proposition 3.2. We could have also proved Proposition 3.2 (except for the isomorphism only depending on L) in a similar way to the proof of Proposition 3.1 as follows. We can find a surgery description of M_2 from a surgery description of N_2. The procedure of how to visualize a lift of L and the surface S in N_2 is given in [AK]. One considers the lifts of the paths chosen in the proof of Proposition 3.1, on the lift of $S.$ One then fuses the components of the lift of L along these paths, obtaining a lift of $L'.$ The surgery description of M_2 is obtained by adding to the surgery description of N_2 the lift of L' with zero framing together with $\mu - 1$ more unknotted zero-framed components encircling each fusion. The linking matrix of this link is a direct sum of that of N_2 and a $\mu \times \mu$ zero matrix.

Let i_T denote the inclusion of the torsion subgroup of $H_1(M_2)$ into $H_1(M_2)$; and let : $H_1(N_2)$! $H_1(M_2)$ denote the monomorphism given by i_T $(i_M)^{-1}$ i_N:

Theorem 3.4 *Let* $^+$: $H_1(M_2)$! C_q C_1: *Let* : $H_1(N_2)$! C_q *be* $^+$ *composed with the projection to* C_q: *We have that:*

$$j \;\; _1(\;\; (M_2; \;\; ^+)) - \;\; (L; \;\;)j \qquad (L; \;\;) + \;\; :$$

Remark 3.5 If L is a knot, then $(M_2; \;\; ^+)$ coincides with $(L; \;\;)$ de ned in [CG1, p.189].

Proof of Theorem 3.4 We use the surgery description of M_2 given in Remark 3.3. Let P be given by the surgery description of M_2 but with the component corresponding to L^0 deleted. Hence,

$$P = N_2]_{(\;\; -1)} S^1 \quad S^2:$$

$+$ induces some character ^{0} on $H_1(P)$.

According to Section 2.3, we let $^-$ 2 $H^1(M_2;C_q)$ and $^{-0}$ 2 $H^1(P;C_q)$ denote the characters $^+$ and ^{0} followed by the projection C_q C_1 ! C_q. Using Propositions 2.5 and 2.6, one has that

$$(P; {}^{-0}) = \;\; (L; \;\;) \text{ and } \;\; (P; {}^{-0}) = \;\; (L; \;\;) + \;\; - 1:$$

Moreover, since M_2 is obtained by surgery on L^0 in P, it follows from [Gi3, Proposition (3.3)] that

$$j \;\; (P; {}^{-0}) - \;\; (M_2; {}^-)j + j \;\; (M_2; {}^-) - \;\; (P; {}^{-0})j \quad 1 \text{ or}$$

$$j \;\; (L; \;\;) - \;\; (M_2; {}^-)j + j \;\; (M_2; {}^-) - \;\; (L; \;\;) - \;\; + 1j \quad 1:$$

Thus

$$j \;\; (L; \;\;) - \;\; (M_2; {}^-)j \qquad (L; \;\;) + \;\; - \;\; (M_2; {}^-):$$

Finally, one gets, by Theorem 2.8,

$$j \;\; _1(\;\; (M_2; \;\; ^+)) - \;\; (L; \;\;)j \quad j \;\; _1(\;\; (M_2; \;\; ^+)) - \;\; (M_2; {}^-)j + j \;\; (M_2; {}^-) - \;\; (L; \;\;)j$$

$$(M_2; {}^-) + \;\; (L; \;\;) + \;\; - \;\; (M_2; {}^-) = \;\; (L; \;\;) + \;\; : \qquad \square$$

4 The slice genus of links

See Section 2.5 for notations.

Theorem 4.1 *Suppose L is the boundary of a connected oriented properly embedded surface F of genus g in B^4; and that $_L(-1) \neq 0$. Then, $_L$ can be written as a direct sum $_1$ $_2$ such that the following two conditions hold:*

1) $_1$ *has an even presentation of rank $2g + \ - 1$ and signature $_L(-1)$, and* $_2$ *is metabolic.*

2) *There is a metabolizer for $_2$ such that for all characters of prime power order in this metabolizer,*

$$j \ (L; \ ) + \ _L(-1)j \quad (L; \ ) + 4g + 3 \ - 2:$$

Proof We let $b_i(X)$ denote the ith Betti number of a space X. We have $b_1(F) = 2g + \ - 1:$

Let W_0^0, with boundary M_0^0, be the complement of an open tubular neighborhood of F in B^4. By the Thom isomorphism, excision, and the long exact sequence of the pair $(B^4; W_0^0)$; W_0^0 has the homology of S^1 wedge $b_1(F)$ 2-spheres. Let W_2^0 with boundary M_2^0 be the two-fold covering of W_0^0. Note that if F is planar, $M_0^0 = M_0$; and $M_2^0 = M_2$ (see Section 3).

Let V_2 be the two-fold covering of B^4 with branched set F. Note that V_2 is spin as $w_2(V_2)$ is the pull-up of a class in $H^2(B^4; \mathbb{Z}_2)$, by [Gi5, Theorem 7], for instance. The boundary of V_2 is N_2. As in [Gi1], one calculates that $b_2(V_2) = 2g + \ - 1$. One has $\mathrm{Sign}(V_2) = \ _L(-1)$ by [V].

By Lemma 2.10, $_L$ can be written as a direct sum $_1$ $_2$ as in condition 1) above, such that the characters on $H_1(N_2)$ that extend to $H_1(V_2)$ form a metabolizer H for $_2$. We now suppose $2 H$ and show that Condition 2) holds for $:$

We also let denote an extension of to $H_1(V_2)$ with image some cyclic group C_q where q is a power of a prime integer (possibly larger than those corresponding to the character on $H_1(N_2)$). Of course $2 H^1(V_2; C_q)$ restricted to W_2^0 extends restricted to M_2^0. We simply denote all these restrictions by .

Let W_1^0 denote the in nite cyclic cover of W_0^0. Note that W_2^0 is a quotient of this covering space. induces a C_q-covering of V_2 and thus of W_2^0. If we pull the C_q-covering of W_2^0 up to W_1^0, we obtain $\hat{W}_1^0$, a $C_q \ C_1$-covering of W_2^0. If we identify properly $F \ S^1$ in M_2^0; this covering restricted to $F \ S^1$ is given by a character $H_1(F \times S^1) \cong H_1(F) \oplus H_1(S^1) \to C_q \oplus C_\infty$ that maps $H_1(F)$ to zero in C_∞, $H_1(S^1)$ to zero in C_q and isomorphically onto C_∞. For this note: since $\mathrm{Hom}(H_1(F),\mathbb{Z}) = H^1(F) = [F, S^1]$, we may define diffeomorphisms of $F \times S^1$ that induce the identity on the second factor of $H_1(F \times S^1) \cong H_1(F) \oplus \mathbb{Z}$, and send $(x, 0) \in H_1(F) \oplus \mathbb{Z}$, to $(x, f(x)) \in H_1(F) \oplus \mathbb{Z}$, for any $f \in \mathrm{Hom}(H_1(F),\mathbb{Z})$.

As in [Gi1], choose inductively a collection of g disjoint curves in the kernel of χ that form a metabolizer for the intersection form on $H_1(F)/H_1(\partial F)$. By taking a tubular neighborhood of these curves in F, we obtain a collection of $S^1 \times I$ embedded in F. Using these embeddings we can attach round 2-handles $(B^2 \times I) \times S^1$ along $(S^1 \times I) \times S^1$ to the trivial cobordism $M_2' \times I$ and obtain a cobordism Ω between M_2 and M_2'.

Let $U = W_2' \cup_{M_2'} \Omega$ with boundary M_2. The $C_q \oplus C_\infty$-covering of W_2' extends uniquely to U. Note that Ω may also be viewed as the result of attaching round 1-handles to $M_2 \times I$.

As in [Gi1], $\mathrm{Sign}(W_2') = \mathrm{Sign}(V_2)$. Since the intersection form on Ω is zero, we get $\mathrm{Sign}(U) = \mathrm{Sign}(W_2') = \mathrm{Sign}(V_2) = \sigma_L(-1)$. The $C_q \oplus C_\infty$-covering of Ω, restricted to each round 2-handle is q copies of $B^2 \times I \times \mathbb{R}$ attached to the trivial cobordism $\tilde{M}'_\infty \times I$ along q copies of $S^1 \times I \times \mathbb{R}$. Using a Mayer-Vietoris sequence, one sees that the inclusion induces an isomorphism (which preserves the Hermitian form)

$$H_2^t(U;\mathbb{Q}(C_q)(t)) \cong H_2^t(W_2';\mathbb{Q}(C_q)(t)).$$

Thus, if $w(W_2')$ denotes the image of the intersection form on $H_2^t(W_2';\mathbb{Q}(C_q)(t))$ in $W(\mathbb{Q}(C_q)(t))$, we get $\sigma_1(\tau(M_2,\chi^+)) = \sigma_1(w(W_2')) - \sigma_L(-1)$.

If q is a prime power, we may apply Lemma 2 of [Gi1] and conclude that $H_i(\tilde{W}'_\infty;\mathbb{Q})$ is finite dimensional for all $i \neq 2$. Thus, $H_i^t(W_2';\mathbb{Q}(C_q)(t))$ is zero for all $i \neq 2$. Since the Euler characteristic of W_2' with coefficients in $\mathbb{Q}(C_q)(t)$ coincides with those with coefficients in $\mathbb{Q}$, we get $\dim H_2^t(W_2';\mathbb{Q}(C_q)(t)) = \chi(W_2') = 2\chi(W_0') = 2(1-\chi(F)) = 2b_1(F)$. Thus $|\sigma_1(\tau(M_2,\chi^+)) + \sigma_L(-1)| \le 2b_1(F)$. Hence,

$$|\sigma(L,\chi) + \sigma_L(-1)| \le |\sigma(L,\chi) - \sigma_1(\tau(M_2,\chi)^+)| + |\sigma_1(\tau(M_2,\chi)^+) + \sigma_L(-1)|$$
$$\le \eta(L,\chi) + \mu + 2(2g + \mu - 1) = \eta(L,\chi) + 4g + 3\mu - 2 \text{ by Theorem 3.4.} \qquad \square$$

5 Examples

Let $L = L_1 \cup L_2$ be the link with two components of Figure 1 and S be the Seifert surface of L given by the picture. The squares with K denote two parallel copies with linking number 0 of an arc tied in the knot K. Note that L is actually a family of examples. Speci c links are determined by the choice of two parameters: a knot K and a positive integer h: Since S has genus h, the slice genus of L is at most h.

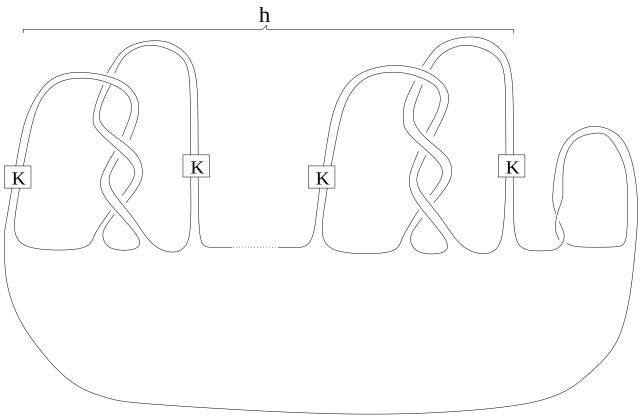

Figure 1: The link L

One calculates that $_L(\ ) = 1$, and $n_L(\ ) = 0$ for all . Thus, the Murasugi-Tristram inequality says nothing about the slice genus of L. In fact, if K is a slice knot, then one can surger this surface to obtain a smooth cylinder in the 4-ball with boundary L. Thus there can be no arguments based solely on a Seifert pairing for L that would imply that the slice genus is non-zero.

Theorem 5.1 *If* $_K(e^{2i\ =3})\ \ 2h$ *or* $_K(e^{2i\ =3})\ \ -2h-2$*; then* L *has slice genus* h.

Proof Using [AK], a surgery presentation of N_2 as surgery on a framed link of $2h+1$ components can be obtained from the surface S (see Figure 2).

Let Q be the 3-manifold obtained from the link pictured in Figure 2. Here K^0 denotes K with the string orientation reversed. Since $RP(3)$ is obtained by surgery on the unknot framed 2, we get:

$$N_2 = RP(3)\#_h Q:$$

The linking matrix of the framed link of the surgery presentation of N_2 is

$$= [2] \bigoplus_h \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix}.$$

is a presentation matrix of $(H_1(N_2)\ ;\ _L)$; we obtain

$$H_1(N_2)\ \ '\ \mathbb{Z}_2 \bigoplus {}^{2h}\mathbb{Z}_3$$

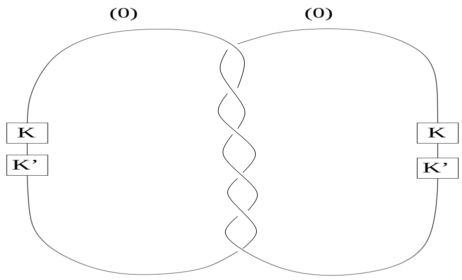

Figure 2: Surgery presentation of Q

and $_L$ is given by the following matrix, with entries in $\mathbb{Q}/\mathbb{Z}$:

$$[1/2] \bigoplus_h \begin{pmatrix} 0 & 1/3 \\ 1/3 & 0 \end{pmatrix}.$$

By Theorem 4.1, if L bounds a surface of genus $h-1$ in B^4, then $_L$ must be decomposed as $_1$ $_2$ where:

1) $_1$ has an even presentation matrix of rank $2h-1$, and signature 1 (all we really need here is that it has a rank $2h-1$ presentation.)

2) $_2$ is metabolic and for all characters of prime power order in some metabolizer of $_2$, the following inequality holds:

( ) $j\ (L;\ ) + 1j -\ (L;\ )\quad 4h.$

As $\mathbb{Z}_2 \bigoplus^{2h} \mathbb{Z}_3$ does not have a rank $2h-1$ presentation, $_2$ is non-trivial. As metabolic forms are de ned on groups whose cardinality is a square, $_2$ is de ned on a group with no 2-torsion. Thus the metabolizer contains a non-trivial character of order three satisfying $_L(\ ;\ ) = 0.$

The rst homology of Q is $\mathbb{Z}_3$ $\mathbb{Z}_3$, generated by, say, m_1 and m_2, positive meridians of these components. Each of these components is oriented counter-clockwise. We rst work out $(Q;\ )$ and $(Q;\ )$ for characters of order three. Let $_{(a_1,a_2)}$ denote the character on $H_1(Q)$ sending m_j to $e^{\frac{2i\ a_j}{3}}$, where the a_j take the values zero and $1.$

We use Proposition 2.4 to compute $(Q;\ _{(1,0)})$ and $(Q;\ _{(1,0)})$ assuming that K is trivial. For this, one may adapt the trick illustrated on a link with 2 twists between the components [Gi2, Fig (3.3), Remark (3.65b)]. In the case K is the unknot, we obtain

$$(Q;\ _{(1,0)}) = 1 \quad \text{and} \quad (Q;\ _{(1,0)}) = 0.$$

It is not di cult to see that inserting the knots of the type K changes the result as follows (note that K and K^0 have the same Tristram-Levine signatures):

$$(Q; \ _{(1;0)}) = 1 + 2 \ _K(e^{2 \ i=3}) \quad \text{and} \quad (Q; \ _{(1;0)}) = 0:$$

These same values hold for the characters $_{(-1;0)}$ and $_{(0; \ 1)}$ by symmetry. Using Proposition 2.4

$$(Q; \ _{(1;1)}) = -1 - 24=9 + 4 \ _K(e^{2 \ i=3}); \qquad (Q; \ _{(1;1)}) = 0$$

$$(Q; \ _{(1;-1)}) = 4 + 24=9 + 4 \ _K(e^{2 \ i=3}) \quad \text{and} \quad (Q; \ _{(1;-1)}) = 1:$$

One also has

$$(Q; \ _{(0;0)}) = 0 \quad \text{and} \quad (Q; \ _{(0;0)}) = 0:$$

Any order three character on N_2 that is self annihilating under the linking form is given as the sum of the trivial character on $RP(3)$ and characters of type $_{(0;0)}$, $_{(\ 1;0)}$ and $_{(0; \ 1)}$ on Q and characters of type $_{(1;1)} + \ _{(1;-1)}$ on $Q\#Q$. Using Proposition 2.5, one can calculate $(L; \ )$ and $(L; \ )$ for all these characters . It is now a trivial matter to check that for every non-trivial character with $(\ ; \ ) = 0$, the inequality (*) is not satis ed. □

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