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Resolutions of p-stratifolds with isolated singularities

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Abstract Recently M. Kreck introduced a class of stratified spaces called p-stratifolds [Kr3]. He defined and investigated resolutions of p-stratifolds analogously to resolutions of algebraic varieties. In this note we study a very special case of resolutions, so called optimal resolutions, for p-stratifolds with isolated singularities. We give necessary and sufficient conditions for existence and analyze their classification.

AMS Classification 58A32; 58K60

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1 Introduction

Roughly speaking, p-stratifolds are topological spaces which are constructed by attaching manifolds with boundary by a map to the already inductively constructed space. The attaching map has to fulfill some subtle properties. There is a more general notion of stratifolds introduced by M. Kreck [Kr3]. However, the only results concerning the resolution of stratifolds exist after going over to the subclass of p-stratifolds.

The situation simplifies very much, if we consider only p-stratifolds with isolated singularities, where the construction is done in two steps only. The first step is the choice of a countable number of points $\{x_i\}_{i \in I \subset \mathbb{N}}$ which will become the isolated singularities. The second step is the choice of a smooth manifold N of dimension m, together with a proper map $g : \partial N \longrightarrow \{x_i\}_{i \in I}$, where $\{x_i\}_{i \in I}$ is considered as 0-dimensional manifold and the collection of boundary components $f^{-1}(x_i)$ is equipped with a germ of collars. The p-stratifold is obtained by forming

$$\mathscr{S} = N \cup_g \{x_i\}_{i \in I}.$$

We reformulate this in a slightly different way.

Definition An m-dimensional *p-stratifold with isolated singularities* is a topological space $\mathscr{S}$ together with a proper map $f : N \longrightarrow \mathscr{S}$, where

N is an m-dimensional manifold with boundary,

$f|_N$ is a homeomorphism onto its image,

$\mathscr{S} - f(N)$ is a discrete countable set, denoted by Σ, the *singular set*,

$f^{-1}(x)$ is equipped with a germ of collars for all $x \in \Sigma$,

$U \subset \mathscr{S}$ is open if and only if $U \cap \Sigma$ is open and $f^{-1}(U)$ is open in N.

The manifold $f(N)$ is called the *top stratum*, Σ is called the 0-stratum of $\mathscr{S}$. Choose an identification $\Sigma = \{x_i\}_{i \in I \subset \mathbb{N}}$ and denote the collection of boundary components mapped to a singular point $L_i := f^{-1}(x_i)$ the *link of $\mathscr{S}$ at $x_i \in \Sigma$*.

A *collar* around L_i is a diffeomorphism $c_i : L_i \times [0, \delta_i) \longrightarrow U_i$, where U_i is an open neighbourhood of L_i in N and $\delta_i > 0$, such that $c_i|_{L_i \times \{0\}}$ is the identity map on L_i. The *germ* is an equivalence class of collars, where two collars $c_i : L_i \times [0, \delta_i) \longrightarrow U_i$ and $\tilde{c}_i : L_i \times [0, \tilde{\delta}_i) \longrightarrow \tilde{U}_i$ are called equivalent if there is a positive $\delta \leq \min\{\delta_i, \tilde{\delta}_i\}$, such that $c_i|_{L_i \times [0,\delta)} \equiv \tilde{c}_i|_{L_i \times [0,\delta)}$. The role of the collars becomes clear if we define smooth maps from a smooth manifold to a p-stratifold.

Definition Let $\mathscr{S}$ be a p-stratifold with isolated singularities and $\rho : \mathscr{S} \longrightarrow \mathbb{R}$ a continuous map. The map ρ is called *smooth* if $\rho f|_N : N \longrightarrow \mathbb{R}$ is smooth and there are representatives of the germ of collars $c_i : L_i \times [0, \varepsilon_i) \longrightarrow N$ satisfying for all i:

$$\rho f(c_i(x,t)) = \rho f(x) \text{ for all } x \in L_i.$$

Let M be a smooth manifold. A continuous map $g : M \longrightarrow \mathscr{S}$ is called *smooth*, if for all smooth maps $\rho : \mathscr{S} \longrightarrow \mathbb{R}$ the composition ρg is again smooth.

It is not hard to verify that the map g is smooth if and only if the restriction $g|_{g^{-1}(\mathscr{S} - \Sigma)}$ is smooth.

The most important examples of p-stratifolds as defined above are algebraic varieties with isolated singularities.

Example (Algebraic varieties with isolated singularities)

Consider an algebraic variety $V \subset \mathbb{R}^n$ with isolated singularities, i.e. the singular set Σ is zero-dimensional. Let $s_i \in \Sigma$ be a singular point. There is nothing to do if s_i is open in V. Otherwise consider the distance function ρ_i on $\mathbb{R}^n$ given by $\rho_i(x) := ||x - s_i||^2$. It is well known that there is an $\varepsilon_i > 0$ such that on $V_{\varepsilon_i}(s_i) := V \cap D_{\varepsilon_i}(s_i)$ the restriction $\rho_i|_{V_{\varepsilon_i} - \{s_i\}}$ has no critical values. Here $D_{\varepsilon_i}(s_i)$ denotes the closed ball in $\mathbb{R}^n$ of radius ε_i centered at s_i. Set $\partial V_{\varepsilon_i}(s_i) := V_{\varepsilon_i}(s_i) \cap \partial D_{\varepsilon_i}(s_i)$.

By following the integral curves of the gradient vector field of $\rho_i|_{V_{\varepsilon_i}-\{s_i\}}$, we obtain a diffeomorphism

$$\begin{array}{ccc} h : \partial V_{\varepsilon_i}(s_i) \times [0,\delta_i) & \longrightarrow & V_{\varepsilon_i} - \{s_i\} \\ & {\scriptstyle \mathrm{pr}_2} \searrow \quad \swarrow {\scriptstyle \delta_i - \rho_i} & \\ & [0,\delta_i) & \end{array}$$

being the identity on $\partial V_{\varepsilon_i}(s_i) \times \{0\}$, see [H, §6.2]. We extend this map to a continuous map

$$h : \partial V_{\varepsilon_i}(s_i) \times [0,\delta_i] \longrightarrow V_{\varepsilon_i}.$$

Finally, we define the manifold N (with obvious collar) by setting

$$N := V - (\sqcup_i D_{\varepsilon_i}(s_i)) \cup_{\mathrm{id}} \partial V_{\varepsilon_i}(s_i) \times [0,\delta_i].$$

The map $f = \mathrm{id} \cup h : N \longrightarrow V$ gives V the structure of a p-stratifold with isolated singularities.

Since every complex algebraic variety is in particular a real one, we obtain the same result for a complex algebraic variety with isolated singularities.

From now on all p-stratifolds are p-stratifolds with isolated singularities. To simplify the notation combine the representatives of the collars $\mathrm{c}_i : L_i \times [0,\varepsilon_i)$ to a single map $\mathrm{c} : \sqcup_i L_i \times [0,\varepsilon_i) \longrightarrow N$. Using this map the singular set Σ is equipped with the germ of neighbourhoods $[U_\Sigma]$ by taking $U_\Sigma := f(\mathrm{im}\ \mathrm{c}) \sqcup (\Sigma - f(\partial N))$. The collars also give us a retraction $r : U_\Sigma \longrightarrow \Sigma$.

We also introduce the germ of closed neighbourhoods $[\overline{U}_\Sigma]$ by setting $\overline{U}_\Sigma := f(\mathrm{c}(\sqcup_i L_i \times [0,\varepsilon_i/2])) \sqcup (\Sigma - f(\partial N))$. If we want to make the dependency on the representative of the germ of collars clear, we sometimes write $U_{\Sigma_{\mathbf{c}}}$ and $\overline{U}_{\Sigma_{\mathbf{c}}}$.

Definition Let $\mathscr{S}$ be an n-dimensional p-stratifold with top stratum $f(N)$. A *resolution* of $\mathscr{S}$ is a proper map $p : \hat{\mathscr{S}} \longrightarrow \mathscr{S}$ such that

- $\hat{\mathscr{S}}$ is a smooth manifold;
- p is a proper smooth map;
- the restriction of p on $p^{-1}(f(N))$ is a diffeomorphism on $f(N)$;
- $p^{-1}(f(N))$ is dense in $\hat{\mathscr{S}}$;
- the inclusion $\hat{\Sigma} := p^{-1}(\Sigma) \hookrightarrow \hat{U}_\Sigma := p^{-1}(U_\Sigma)$ is a homotopy equivalence for a representative of the neighbourhood U_Σ of Σ.

A resolution $p : \hat{\mathscr{S}} \longrightarrow \mathscr{S}$ is called *optimal*, if $p|_{\hat{\Sigma}} : \hat{\Sigma} \longrightarrow \Sigma$ is an $[n/2]$-equivalence. In particular, it follows that $p : \hat{\mathscr{S}} \longrightarrow \mathscr{S}$ is an $[n/2]$-equivalence as well.

If the manifold N is equipped with more structure, e.g. orientation or spin-structure, we introduce corresponding resolutions, which have more structure.

Definition Let $\mathscr{S} = f(N) \cup \{x_i\}_i$ be a p-stratifold with oriented N. A resolution $p : \hat{\mathscr{S}} \longrightarrow \mathscr{S}$ is called an *oriented resolution*, if $\hat{\mathscr{S}}$ is oriented and $p|_{p^{-1}(f(N))}$ is orientation preserving. Analogously, if N is spin, then $p : \hat{\mathscr{S}} \longrightarrow \mathscr{S}$ is called a *spin resolution* if $\hat{\mathscr{S}}$ is spin and $p|_{p^{-1}(f(N))}$ preserves the spin structure.

If V is an algebraic variety, Hironaka has shown [Hi] that there is a resolution of singularities in the sense of algebraic geometry. The above topological definition is modelled on the one from algebraic geometry. All conditions are analogous except the last one, which is always fulfilled in the context of algebraic geometry. As explained in [BR], a neighbourhood U of the singular set Σ of an algebraic variety V such that the inclusion $\Sigma \hookrightarrow U$ is a homotopy equivalence can be obtained from a proper algebraic map $\rho : V \longrightarrow \mathbb{R}$ with $\Sigma = \rho^{-1}(0)$ by taking $U = \rho^{-1}[0, r)$, provided $r > 0$ is small enough. Thus for a resolution $p : \hat{V} \longrightarrow V$ the preimage $\hat{U} := p^{-1}(U)$ is a neighbourhood of $\hat{\Sigma} := p^{-1}(\Sigma)$ in $\hat{V}$ obtained from $\hat{\rho} := \rho p$, hence the inclusion $\hat{\Sigma} \hookrightarrow \hat{U}$ is a homotopy equivalence.

Note that $\overline{U}^{\wedge}$ is a smooth manifold with boundary diffeomorphic to ∂N. Consider the preimage of the neighbourhood of each singularity and set $\overline{U}^{\wedge}_i := p^{-1}(f\,c_i(L_i \times [0, \varepsilon_i/2]))$, where c_i is the restriction of the collar to $L_i \times [0, \varepsilon_i/2]$.

It is not hard to verify that a resolution $p : \hat{\mathscr{S}} \longrightarrow \mathscr{S}$ is optimal if and only if the manifolds $\overline{U}^{\wedge}_i$ are $([n/2] - 1)$-connected.

In contrast to algebraic varieties, resolutions of stratifolds in general do not exist, not even for isolated singularities. But in this case there is a simple necessary and sufficient condition, see [Kr3] and §6 for a proof.

Theorem 1 *An n-dimensional p-stratifold with isolated singularities admits a resolution if and only if each link of the singularity L_i, vanishes in the bordism group Ω_{n-1}.*

Example The p-stratifold $\mathscr{S} = \mathbb{C}P^2 \times I \cup_f \{x_0, x_1\}$ with the obvious stratification, such that $f(\mathbb{C}P^2 \times \{0\}) = x_0$ and $f(\mathbb{C}P^2 \times \{1\}) = x_1$, does not admit a resolution.

To give a feeling of the result concerning optimal resolution, we formulate the following special case which will be derived as Corollary 7 of Theorem 5 (cf. §2).

Corollary *Let $\mathscr{S}$ be a p-stratifold with parallelizable links of singularities L_i. Assume L_i is bounded by a parallelizable manifold, then $\mathscr{S}$ admits an optimal resolution.*

We have shown above that every algebraic variety with isolated singularities admits a structure of a p-stratifold. One may ask the converse question. When does a p-stratifold with isolated singularities admit an algebraic structure? The following Theorem of Akbulut and King [AK, Thm. 4.1] clarifies the situation in the case of a real algebraic structure.

Theorem 2 *A topological space X is homeomorphic to a real algebraic set with isolated singularities if and only if X is obtained by taking a smooth compact manifold M with boundary $\partial M = \cup_{i=1}^{r} L_i$, where each L_i bounds, then crushing some L_i's to points and deleting the remaining L_i's.*

Combining this result with Theorem 1 we immediately obtain:

Corollary 3 *A compact p-stratifold $\mathscr{S}$ with isolated singularities is homeomorphic to a real algebraic set with isolated singularities if and only if $\mathscr{S}$ admits a resolution.*

Example (Resolutions of hypersurfaces with isolated singularities)

Let $p : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$ be a polynomial with isolated singularities $\{s_i\}$, i.e. $s_i \in V := p^{-1}(0)$ and s_i is an isolated critical point of p. Assume further that the points s_i are not open. According to a previous example, the hypersurface V admits a canonical structure of a p-stratifold. We have to investigate the link of the singularity, which is given by $\partial V_{\varepsilon_i}(s_i)$.

Choose a $\delta > 0$ such that all c with $|c| \leq \delta$ are regular values of p and take c such that $p^{-1}(c) \neq \emptyset$. Then $p^{-1}(c)$ is a smooth manifold with trivial normal bundle. With the help of the gradient vector field we see that $p^{-1}(c) \cap S^n_{\varepsilon_i}(s_i)$ is diffeomorphic to $p^{-1}(0) \cap S^n_{\varepsilon_i}(s_i) = \partial V_{\varepsilon_i}(s_i)$. Thus, $\partial V_{\varepsilon_i}(s_i) = p^{-1}(c) \cap S^n_{\varepsilon_i}(s_i) = \partial(p^{-1}(c) \cap D^{n+1}_{\varepsilon_i}(s_i))$. We see that a resolution always exists, and since the bounding manifolds are automatically parallelizable, we even obtain an optimal resolution after choosing an appropriate bordism (compare with Figure 1).

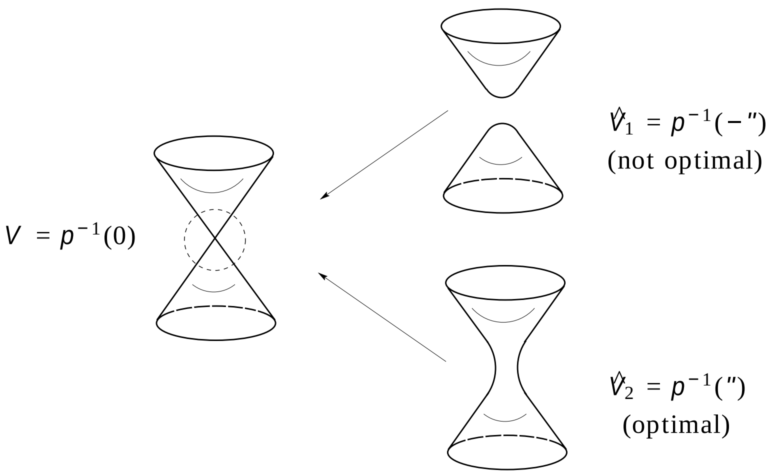


Figure 1: $p(x, y, z) = x^2 + y^2 - z^2$

In the case of a complex polynomial $p : \mathbb{C}^{n+1} \longrightarrow \mathbb{C}$ $(n > 0)$, every deformation $p^{-1}(c)$ gives us an optimal resolution, provided $\|c\|$ is small enough. This follows from a result of Milnor [Mi4, Thm. 6.5] which states that $M := p^{-1}(c) \cap D^{2n+2}_{\varepsilon}(s_i)$ is homotopy equivalent to a wedge of 1 copies of S^n and thus $(n-1)$-connected.

Consider another interesting class of p-stratifolds with isolated singularities, namely those arising from a smooth group action.

Definition A smooth S^1-action on a smooth manifold M is called *semi-free* if the action is free outside of the fixed point set, i.e. if $gx = x$ for a $g \in S^1, g \neq 1$ and $x \in M$, then $hx = x$ for all $h \in S^1$.

Lemma 4 *Let M be a closed oriented manifold with semi-free S^1-action with only isolated fixed points. Then M/S^1 admits an oriented resolution if and only if* $\dim M \equiv 0$ (*mod* 4).

Proof Let $\dim M = n$. There is nothing to show if the action is free. Thus let $x \in M$ be a fixed point. The differential of the action gives a representation of S^1 on $T_x M$ and there is an equivariant local diffeomorphism from $T_x M$ onto a neighbourhood of x in M. According to [BT, Prop. (II.8.1)], every irreducible representation of S^1 on $\mathbb{R}^n$ is equivalent to:

$$\begin{pmatrix} z^{n_1} & 0 & \dots & \dots & 0 \\ 0 & & \ddots & & \vdots \\ \vdots & \ddots & & \ddots & \vdots \\ \vdots & & \ddots & z^{n_{[n/2]}} & 0 \\ 0 & \dots & \dots & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} z^{n_1} & 0 & \dots & \dots & 0 \\ 0 & & \ddots & & \vdots \\ \vdots & \ddots & & \ddots & \vdots \\ \vdots & & \ddots & & 0 \\ 0 & \dots & \dots & 0 & z^{n_{n/2}} \end{pmatrix}$$

considered as a representation on $\mathbb{C} \oplus \dots \oplus \mathbb{C} \oplus \mathbb{R}$, if n is odd; considered as a representation on $\mathbb{C} \oplus \dots \oplus \mathbb{C}$, if n is even.

Since the action is semi-free and x an isolated fixed point we conclude that $\dim M$ is even. We can further assume $n_i = 1$ for all $i \in \{1, \dots, n/2\}$. Let $\dim M = 2m$ and let $\{x_1, \dots, x_k\}$ be the set of fixed points. Choose equivariant disks D_{x_i} around x_i. In this situation we have

$$M/S^1 = (M - \sqcup_i D_{x_i})/S^1 \cup \{x_1, \dots, x_k\}.$$

The domain of the top stratum is then given by $N := (M - \sqcup_i D_{x_i})/S^1$ and the singular set is $\Sigma := \{x_1, \dots, x_k\}$. The links of singularities are given by $L_i = S^{2m-1}/S^1 = \mathbb{C}P^{m-1}$. Using Theorem 1 we conclude that the resolution exists if and only if $[\mathbb{C}P^{m-1}]$ vanishes in Ω^{SO}_{2m-2}. For $m = 2l+1$ the signature of $\mathbb{C}P^{m-1}$ is equal to 1, hence $\mathbb{C}P^{m-1}$ does not bound. In the case of an even $m = 2l+2$ we have $\mathbb{C}P^{2l+1} = S^{4l+3}/S^1 = S(\mathbb{H}^{l+1})/S^1$, where $S(\mathbb{H}^{l+1})/S^1$ is the sphere bundle

$$\begin{array}{ccc} S^2 = S^3/S^1 & \hookrightarrow & S(\mathbb{H}^{l+1})/S^1 \\ & & \downarrow \\ & & S(\mathbb{H}^{l+1})/S^3 = \mathbb{H}P^l \end{array}$$

and the associated disk bundle bounds. □

As mentioned before, we are particularly interested in the classification of resolutions. Thus we have to decide when we are going to consider two resolutions as equivalent. We can restrict our attention to the resolving manifolds and introduce a relation on them, e.g. diffeomorphism, but in this case we completely ignore an important part of the resolution data, namely the resolving map. Hence, one can ask for diffeomorphisms between the resolving manifolds commuting with the resolving maps. This relation is very strong and, therefore, very hard to control. In the following definition, we combine these two ideas.

Definition Let $\mathscr{S}$ be a p-stratifold and $p : \hat{\mathscr{S}} \longrightarrow \mathscr{S}$ and $p' : \hat{\mathscr{S}}' \longrightarrow \mathscr{S}$ two resolutions of $\mathscr{S}$. We call the resolutions *equivalent*, if, for every representative of the neighbourhood germ $\overline{U}(\Sigma)_{\mathbf{c}}$, there is a diffeomorphism $\varphi_{\mathbf{c}} : \hat{\mathscr{S}} \longrightarrow \hat{\mathscr{S}}'$ such that the following holds:

> $p'\varphi_{\mathbf{c}} = p$ on $\hat{\mathscr{S}} - \hat{\overline{U}}_{\mathbf{c}}$ and
>
> $rp'\varphi_{\mathbf{c}} = rp$ on $\hat{\overline{U}}_{\mathbf{c}}$, where $r : \overline{U}(\Sigma)_{\mathbf{c}} \longrightarrow \Sigma$ is the neighbourhood's retraction.

This means outside of an arbitrary small neighbourhood of the singularity, the diffeomorphism commutes with the resolving maps and near Σ it only commutes after the composition with the retraction.

Observe that $\varphi_{\mathbf{c}}$ gives a diffeomorphism $\partial\hat{\overline{U}}_{\mathbf{c}} \longrightarrow \partial\hat{\overline{U}}'_{\mathbf{c}}$.

The classification of optimal resolutions is quite a difficult problem. For if $\hat{\mathscr{S}}$ is an optimal resolution of $\mathscr{S}$, then $\hat{\mathscr{S}} \# S$ is again optimal for an arbitrary homotopy sphere S. In particular, consider the sphere S^n stratified as $D^n \cup pt$, then every homotopy sphere S^n gives us a resolution of S^n. Thus, we weaken the problem and ask for the equivalence up to a homotopy sphere.

Definition Two resolutions $\hat{\mathscr{S}} \longrightarrow \mathscr{S}$ and $\hat{\mathscr{S}}' \longrightarrow \mathscr{S}$ are called *almost equivalent* if $\hat{\mathscr{S}} \# S$ is equivalent to $\hat{\mathscr{S}}'$ for a homotopy sphere S.

A special case of the classification result is the following.

Corollary *Let $\hat{\mathscr{S}} \longrightarrow \mathscr{S}$ and $\hat{\mathscr{S}}' \longrightarrow \mathscr{S}$ be two resolutions of a $2n$-dimensional p-stratifold $\mathscr{S}$ having $(n-2)$-connected links of isolated singularities. Assume that $n \equiv 6 \pmod 8$ and that $\hat{\overline{U}}_i$ and $\hat{\overline{U}}'_i$ are parallelizable with compatible parallelizations on the boundary. Let further $e(\hat{\overline{U}}_i) = e(\hat{\overline{U}}'_i)$ and $\mathrm{sign}(\hat{\overline{U}}_i \cup_\partial \hat{\overline{U}}'_i) = 0$. Then there is a $k \in \{0, 1\}$ such that $\hat{\mathscr{S}} \# k(S^n \times S^n)$ is almost equivalent to $\hat{\mathscr{S}}' \# k(S^n \times S^n)$.*

2 Existence of optimal resolutions

Before proceeding with the existence of an optimal resolution we need to introduce some notation. For a topological space X let $X\langle k\rangle$ be the k-connected cover of X, which always comes with a fibration $p : X\langle k\rangle \longrightarrow X$. For further information see for example [Ba]. We take X to be the classifying space BO and denote by $\Omega_n^{BO\langle k\rangle}$ the bordism group of closed n-dimensional manifolds together with a lift of the normal Gauss map, compare [St, Chap. I].

Theorem 5 *An n-dimensional p-stratifold with isolated singularities admits an optimal resolution if and only if the normal Gauss map $\nu_j : L_j \longrightarrow BO$ admits a lift over $BO\langle[n/2]-1\rangle$, such that $[L_j, \bar{\nu}_j] = 0$ in $\Omega_{n-1}^{BO\langle[n/2]-1\rangle}$.*

Proof Let $\mathcal{S}$ be a p-stratifold with isolated singularities $\{x_i\}_{i\in I}$ and $\hat{\mathcal{S}}$ an optimal resolution of $\mathcal{S}$. Set $W_i := \overline{\hat{U}_i}$, then W_i is a smooth manifold with boundary L_i for $i \in I \subset \mathbb{N}$. Consider the normal Gauss map, together with the $([n/2]-1)$-connected cover over BO.

$$
\begin{array}{ccc}
 & & BO\langle[n/2]-1\rangle \hookleftarrow F \\
 & \overset{\bar{\nu}_j}{\nearrow} & \downarrow \\
\nu_j : W_j & \longrightarrow & BO
\end{array}
$$

The obstructions for the existence of a lift lie in $\tilde{H}^r(W_j; \pi_{r-1}(F))$. Note that we can use global coefficients since the fibration $BO\langle[n/2]-1\rangle \longrightarrow BO$ is simple.

Since the resolution is optimal the manifold W_i is $([n/2]-1)$-connected, hence $\tilde{H}^r(W_i) = 0$ for $r < [n/2]$.

Using the properties of the connected cover it follows from the long exact homotopy sequence that $\pi_r(F) = 0$ for $r \geq [n/2]-1$.

Hence, there are no obstructions for the lifting of the normal Gauss map, thus $[L_j, \bar{\nu}_j]$ vanishes in $\Omega_{n-1}^{BO\langle[n/2]-1\rangle}$.

The fact that the condition is also sufficient is an immediate consequence of the following result from [Kr1, Prop. 4]. □

Theorem 6 *Let $\xi : B \longrightarrow BO$ be a fibration and assume that B is connected and has a finite $[m/2]$-skeleton. Let $\bar{\nu} : M \longrightarrow B$ be a lift of the normal Gauss map of an m-dimensional compact manifold M. Then if $m \geq 4$, by a finite sequence of surgeries $(M, \bar{\nu})$ can be replaced by $(M', \bar{\nu}')$ so that $\bar{\nu}' : M' \longrightarrow B$ is an $[m/2]$-equivalence.*

For example, we obtain the following:

Corollary 7 *Let $\mathcal{S}$ be a p-stratifold with parallelizable links of singularities L_i. Assume L_i is bounded by a parallelizable manifold, then $\mathcal{S}$ admits an optimal resolution.*

3 Classification of almost equivalent resolutions

Now we turn to the main result of this note. In this section we consider p-stratifolds with isolated singularities of dimension $2n > 4$ having $(n-2)$-connected links of singularities. The classification is based on the following result from [Kr1, Thm. 2].

Theorem 8 *For $n > 2$ let W_1 and W_2 be two compact connected $2n$-manifolds with normal $(n-1)$-smoothings in a fibration B. Let $g : \partial W_1 \longrightarrow \partial W_2$ be a diffeomorphism compatible with the normal $(n-1)$-smoothings $\bar\nu_1$ and $\bar\nu_2$. Let further*

$$e(W_1) = e(W_2),$$

$$[W_1 \cup_g (-W_2); \bar\nu_1 \cup \bar\nu_2] = 0 \in \Omega_{2n}(B).$$

Then g can be extended to a diffeomorphism $G : W_1 \# k(S^n \times S^n) \longrightarrow W_2 \# k(S^n \times S^n)$ for $k \in \mathbb{N}$. Moreover, if B is 1-connected then $k \in \{0, 1\}$.

If W_1 is simply connected and n is odd, then k can be chosen as 0, i.e. we obtain diffeomorphism instead of stable diffeomorphism.

We have to explain some terms appearing in the last theorem. Let B be a fibration over BO, a *normal B-structure* on a manifold M is a lift $\bar\nu$ of the normal Gauss map $\nu : M \longrightarrow BO$ to B.

Definition Let B be a fibration over BO.

(1) A normal B-structure $\bar\nu : M \longrightarrow B$ of a manifold M in B is a *normal k-smoothing*, if it is a $(k+1)$-equivalence.

(2) We say that B is *k-universal* if the fiber of the map $B \longrightarrow BO$ is connected and its homotopy groups vanish in dimension $\geq k+1$.

Obstruction theory implies that if B and B' are both k-universal and admit a normal k-smoothing of the same manifold M, then the two fibrations are fiber homotopy equivalent. Furthermore, the theory of Moore-Postnikov decompositions implies that for each manifold M there is a k-universal fibration B^k over BO admitting a normal k-smoothing, compare [Ba, §5.2]. Thus, the fiber homotopy type of the fibration B^k over BO is an invariant of the manifold M and we call it the *normal k-type of M*.

There is an obvious bordism relation on closed n-dimensional manifolds with normal B-structures and the corresponding bordism group is denoted $\Omega_n(B)$.

Applying the theorem to our situation, we first have to determine the normal $(n-1)$-type of an $(n-1)$-connected $2n$ manifold.

Consider a subgroup H of $G := \pi_n(BO)$. Since the last group is always cyclic, the group H is determined by an integer k, such that $H = \langle kx \rangle$ where x is the generator of $\pi_n(BO)$. We call this integer the *index* of H.

Every subgroup H of $G = \pi_n(BO)$ gives us a fibration

$$\begin{array}{ccc} B & \longrightarrow & P(K(G/H;n)) \\ \downarrow{\scriptstyle p} & & \downarrow \\ BO\langle n-1\rangle & \xrightarrow{\ \varphi\ } & K(G/H;n) \end{array}$$

where the map φ corresponds to the canonical epimorphism $G \longrightarrow G/H$. We denote the space B belonging to the index-k group B_k. The composition $p_k : B_k \xrightarrow{p} BO\langle n-1\rangle \longrightarrow BO$ gives us a fibration over BO with fiber F_k.

Definition A $2n$-dimensional manifold M is said to have the *index* k, if $\nu(\pi_n(M))$ is a subgroup of index k in $\pi_n(BO)$.

Theorem 9 *The fibration $B_k \longrightarrow BO$ is the normal $(n-1)$-type of an $(n-1)$-connected $2n$-dimensional manifold M, if and only if M is of index k.*

The proof can be found in §6, which also contains proofs of the following two theorems.

Now we look for conditions implying an $(n-1)$-connected $2n$-manifold to be bordant to a homotopy sphere. Note first that as an easy consequence from the universal coefficient theorem the first non-trivial homology group is free. The homological information of M is stored in the triple $(H_n(M);\lambda;\alpha)$ where λ denotes the intersection product $\lambda : H_n(M) \longrightarrow \mathbb{Z}$, we often simply write $x \cdot y$ for $\lambda(x;y)$. The last data α is the normal bundle information, described in the following way. According to a theorem of Haefliger [Hae] every element of $H_n(M)$ is represented by an embedding $S^n \hookrightarrow M$, and two embeddings corresponding to the same homotopy class are regular homotopic. Thus, assigning to an embedded sphere its normal bundle gives us a well defined map $\alpha : H_n(M) \longrightarrow \pi_{n-1}(SO(n))$.

Definition An $(n-1)$-connected $2n$-dimensional manifold M is called *elementary* if $H_n(M)$ admits a Lagrangian L w.r.t. λ, such that $\alpha|_L \equiv 0$.

Theorem 10 *Let M be an $(n-1)$-connected manifold of dimension $2n$. Then M is bordant to a homotopy sphere if and only if M is elementary.*

Our main result, based on the last two theorems is:

Theorem 11 *For $n > 2$ let $\hat{\mathscr{S}} \longrightarrow \mathscr{S}$ and $\hat{\mathscr{S}}' \longrightarrow \mathscr{S}$ be two optimal resolutions of a $2n$-dimensional p-stratifold $\mathscr{S}$ with isolated singularities $\{x_i\}_{i\in I}$, such that each link L_i is $(n-2)$-connected. Assume further that for a suitable representative $\overline{U}$ the following conditions hold for all $i \in I$:*

- *$e(\widehat{\overline{U}}_i) = e(\widehat{\overline{U}}'_i)$;*
- *$\widehat{\overline{U}}_i$ and $\widehat{\overline{U}}'_i$ have the same index k_i;*
- *there exits normal $(n-1)$-smoothings ν_i and ν'_i of $\widehat{\overline{U}}_i$ and $\widehat{\overline{U}}'_i$ in the fibration $B_{k_i} \longrightarrow BO$, such that $\nu_i|_{\partial\widehat{\overline{U}}_i} = \nu'_i|_{\partial\widehat{\overline{U}}'_i}$;*
- *$\widehat{\overline{U}}_i \cup_\partial \widehat{\overline{U}}'_i$ is elementary.*

If n is odd, then $\hat{\mathscr{S}}$ is almost equivalent to $\hat{\mathscr{S}}'$.

If n is even, then $\hat{\mathscr{S}}\sharp k(S^n \times S^n)$ is almost equivalent to $\hat{\mathscr{S}}'\sharp k(S^n \times S^n)$ for a $k \in \{0, 1\}$.

4 Algebraic invariants

In this section we will find algebraic invariants, which allow us to decide whether an $(n-1)$-connected closed $2n$-dimensional manifold is elementary or not ($n > 2$). Some proofs can be found in §6.7.

Recall the algebraic data corresponding to such a manifold M. We have a triple (H, λ, α), where $H = H_n(M)$ is a free $\mathbb{Z}$-module, $\lambda : H \times H \longrightarrow \mathbb{Z}$ is the intersection product and $\alpha : H \longrightarrow \pi_{n-1}(SO_n)$ is a normal bundle map, described in the previous section. The map α is not a homomorphism, but satisfies the following equation:

$$\alpha(x + y) = \alpha(x) + \alpha(y) + \partial\lambda(x, y); \qquad (*)$$

where $\partial : \mathbb{Z} = \pi_n(S^n) \longrightarrow \pi_{n-1}(SO_n)$ is the boundary map from the long exact homotopy sequence of the fibration $SO(n) \hookrightarrow SO(n+1) \longrightarrow S^n$, see [W1].

Thus, we obtain an algebraic object, the set T_n of triples (H, λ, α), where H is a free $\mathbb{Z}$-module, $\lambda : H \times H \longrightarrow \mathbb{Z}$ is an $(-1)^n$-symmetric unimodular quadratic form and $\alpha : H \longrightarrow \pi_{n-1}(SO_n)$ is a map satisfying $(*)$. We want to investigate the assumptions under which an element $(H,\lambda,\mu) \in T_n$ is elementary, i. e. when H possesses a Lagrangian L with respect to λ such that $\mu|_L \equiv 0$.

We begin with an observation that for a $4k$-dimensional manifold, the normal bundle information can be replaced by the stable normal bundle map.

Lemma 12 *Let n be even and let $S^n \hookrightarrow M^{2n}$ be an embedding. The normal bundle $\nu(S^n)$ of S^n in M is trivial if and only if $\nu \oplus \mathbb{R}$ is trivial and the Euler class of $\nu(S^n)$ vanishes.*

Thus, instead of considering $(H,\lambda,\mu) \in T_n$ we can go over to (H,λ,s_μ), where $s_\mu : H \longrightarrow \pi_{n-1}(SO)$ corresponds to the stable normal bundle map. Since the Euler class of an embedded sphere representing $x \in H$ can be identified with the self intersection class we conclude:

Lemma 13 *Let n be even. Then $(H,\lambda,\mu) \in T_n$ is elementary if and only if (H,λ,s_μ) is elementary.*

Let T_n^s denote the set of triples (H,λ,s_μ), with H and λ as above and $s_\mu : H \longrightarrow \pi_{n-1}(SO)$ a homomorphism. According to the different possibilities for $\pi_{n-1}(SO)$ we distinguish 3 cases.

(1) $\pi_{n-1}(SO) = 0$.

Claim $(H,\lambda,s_\mu) \in T_n^s$ is elementary if and only if $\text{sign}(\lambda) = 0$, where sign denotes the signature of a quadratic form.

(2) $\pi_{n-1}(SO) = \mathbb{Z}$. Since λ is unimodular it induces an isomorphism $H \xrightarrow{\cong} H^*$, which we also denote by λ. The map s_μ gives an element of H^* and we consider $\alpha_s := \lambda^{-1}(s_\mu) \in H$.

Claim $(H,\lambda,s_\mu) \in T_n^s$ is elementary if and only if $\text{sign}(\lambda) = 0$ and $\alpha_s^2 = 0$.

(3) $\pi_{n-1}(SO) = \mathbb{Z}_2$. Let (H,λ,s_μ) be an element of T_n^s with vanishing signature and suppose λ is of type II, i.e. $\lambda(x,x) = 0$ (mod 2) for all $x \in H$. Note that since $n \neq 8$ in this case, an elementary element corresponding to a manifold always has a type II quadratic form. Thus, the dimension of H is even and according to [Mi1, Lem. 9] we can choose a basis $\{\alpha_1,\dots,\alpha_k,\beta_1,\dots,\beta_k\}$ satisfying

$$\lambda(\alpha_i,\alpha_j) = 0, \quad \lambda(\beta_i,\beta_j) = 0 \text{ and } \lambda(\alpha_i,\beta_j) = \delta_{ij}.$$

Consider the set of all elements $x \in H$ with $\lambda(x,x) = 0$ and denote its image under canonical projection on $H \otimes \mathbb{Z}_2$ by H^0. The class $\Phi(H,\lambda,s_\mu) :=$

$\sum_{i=1}^{k} s(\ )s(\ ) \ 2 \ \mathbb{Z}_2$ is well-de ned and is equal to the value s takes most frequently on the nite set H^0, the class is called Arf invariant.

Claim An element $(H; \ ; s\ ) \ 2 \ T_n^s$ with type II form is elementary if and only if sign( ) = 0 and $(H; \ ; s\ ) = 0$.

Consider now the case of an odd n. The quadratic form is now skew symmetric. Depending on the values of there are again three di erent cases (compare [Ke]), which were completely investigated in [W1].

(4) $_{n-1}(SO_n) = 0$. In this case every element of T_n is elementary.

(5) $_{n-1}(SO_n) = \mathbb{Z}_2$. As in (3), we can de ne the Arf invariant $(H; \ ; \ ) = \sum_{i=1}^{k} (\ _i)(\ _i) \ 2 \ \mathbb{Z}_2$, using a symplectic basis $f\ _1; \ldots; \ _k; \ _1; \ldots; \ _k g$ of H.

Claim An element $(H; \ ; \ ) \ 2 \ T_n$ is elementary if and only if $(H; \ ; \ ) = 0$.

(6) $_{n-1}(SO_n) = \mathbb{Z}_2 \ \ \mathbb{Z}_2$. We consider again the stable normal bundle map $s\ : H \ -! \ \mathbb{Z}_2$, the projection on the rst component. As in (2) using , we obtain an element (determined mod $2H$) with $s\ (x) = (\ ; x)$ (mod 2) for all $x \ 2 \ H$.

Claim An element $(H; \ ; \ ) \ 2 \ T_n$ is elementary if and only if $(H; \ ; \ ) = 0$ and $\mathrm{pr}_2 (\ ) = 0$, where pr_2 denotes the projection on the second component.

Knowing the algebraic description of elementary manifolds, we formulate a special case of Theorem 11.

Corollary 14 *Let $\hat{\mathscr{S}} \ -! \ \mathscr{S}$ and $\hat{\mathscr{S}}^0 \ -! \ \mathscr{S}$ be two resolutions of a $2n$-dimensional p-stratifold $\mathscr{S}$ having $(n-2)$-connected links of isolated singularities. Assume that $n \ \ 6$ (mod 8) and that $\overline{U}^{\wedge}{}_i$ and $\overline{U}^{\wedge 0}_i$ are parallelizable with compatible parallelizations on the boundary. Let further $e(\overline{U}^{\wedge}{}_i) = e(\overline{U}^{\wedge 0}_i)$ and sign$(\overline{U}^{\wedge}{}_i \ [_@ \ \overline{U}^{\wedge 0}_i) = 0$. Then there is a $k \ 2 \ f0; 1g$ such that $\hat{\mathscr{S}}]k(S^n \ \ S^n)$ is almost equivalent to $\hat{\mathscr{S}}^0]k(S^n \ \ S^n)$.*

5 4-dimensional results

In this section we consider the exceptional case of a 4-dimensional p-stratifold and give a similar classi cation result in that situation. The proof of the main theorem can be found in x6.8.

For a 4-dimensional stratifold $\mathscr{S}$, every link of the singularity L_i is a 3-dimensional manifold. According to the computation of Ω by Thom [Th] we immediately obtain from Theorem 1:

Corollary 15 *A four-dimensional p-stratifold with isolated singularities always admits a resolution.*

If we further assume the links to be oriented we can use the following well-known result, which can be proved easily.

Proposition 16 *Every orientable 3-manifold is parallelizable, hence in particular spin.*

The normal 1-type of a simply connected 4-dimensional spin-manifold is given by $B\mathrm{Spin}$. Since $\Omega_3^{\mathrm{spin}} = 0$ (see [Mi3, Lem. 9]) we obtain the following corollary from Theorem 5.

Corollary 17 *A four-dimensional p-stratifold with isolated singularities admits an optimal resolution if and only if all links of singularities are orientable.*

We have to develop some notation in the topological category. Use $B\mathrm{TOP}$ to denote the classifying space of topological vector bundles and let $B\mathrm{TOPSpin}$ be the 2-connected cover over $B\mathrm{TOP}$. Let M be a simply connected 4-manifold. Using the Wu-Formula we can explain the Stiefel-Whitney-classes of M. We call the topological manifold M *spin* if $w_2(M)$ vanishes. One can show that the topological Gauss map of M lifts to $B\mathrm{TOPSpin}$ if and only if M is spin. Note further that if such a lift exists, it is unique.

Using [Kr1, Thm. 2] and the h-cobordism-Theorem in dimension 4 [F, Thm. 10.3] we formulate:

Theorem 18 *Let M_1 and M_2 be compact 4-dimensional topological spin manifolds with $e(M_1) = e(M_2)$ and let $g : \partial M_1 \longrightarrow \partial M_2$ be a homeomorphism compatible with the induced spin-structures on the boundaries. If $M_1 \cup_g M_2$ vanishes in $\Omega_4^{BTOPSpin}$, then g can be extended to a homeomorphism $G : M_1 \sharp k(S^2 \times S^2) \longrightarrow M_2 \sharp k(S^2 \times S^2)$ for $k \in \{0, 1\}$.*

We call two resolutions *topologically equivalent* if the diffeomorphism φ_c in the definition of equivalent resolutions in §1 is replaced by a homeomorphism. Using this notation, we obtain the following classification result in dimension four:

Theorem 19 *Let $\hat{\mathscr{S}} \longrightarrow \mathscr{S}$ and $\hat{\mathscr{S}}' \longrightarrow \mathscr{S}$ be two optimal resolutions of a 4-dimensional p-stratifold $\mathscr{S}$ with isolated singularities $\{x_i\}_{i \in I}$, such that each link L_i is connected. Assume that both $\hat{\mathscr{S}}$ and $\hat{\mathscr{S}}'$ are spin and that for a suitable representative $\overline{U}$ of the neighbourhood germ, the following conditions hold for all $i \in I$:*

$e(\overline{U}^\wedge_i) = e(\overline{U}^{\wedge\prime}_i)$,

the spin-structures of $\overline{U}^\wedge_i$ *and* $\overline{U}^{\wedge\prime}_i$ *coincide on the boundary,*

$sign(\overline{U}^\wedge_i \cup_\partial \overline{U}^{\wedge\prime}_i) = 0$.

Then $\hat{\mathscr{S}} \sharp k(S^2 \times S^2)$ *is topologically equivalent to* $\hat{\mathscr{S}}' \sharp k(S^2 \times S^2)$ *for a* $k \in \{0,1\}$.

6 Outline of the proofs

6.1 Proof of Theorem 1

Although the proof can be found in [Kr3], it is useful to understand its nature for the succeeding results.

One of the basic tools for constructing a resolving map is the following lemma, which can be proved with the help of Morse theory, cf. [Kr3].

Lemma 20 *Let* W *be a smooth compact manifold with boundary. Then there is a codense compact subspace* X *of* W *and a continuous map* $f : \partial W \longrightarrow X$ *such that* W *is homeomorphic to* $\partial W \times [0,1] \cup_f X$, *where on* $\partial W \times [0,1)$ *the homeomorphism can be chosen to be a di eomorphism.*

In other words, every smooth manifold with boundary arises from its collar by attaching a codense set. The notation *codense* stands for the complement of a dense subset. With this information we are ready to prove Theorem 1.

Proof Let $p : \hat{\mathscr{S}} \longrightarrow \mathscr{S}$ be a resolution. Set $W_i := \overline{U}^\wedge_i$. Since W_i is a compact manifold with boundary L_i, we obtain $[L_i] = 0$ in Ω_{n-1}.

Let on the other hand W_i be a compact manifold bounding L_i and let $f(N)$ be the top stratum of $\mathscr{S}$. Set $\hat{\mathscr{S}} := N \cup (\sqcup_i W_i)$ and construct with the help of the last lemma the following resolving map:

$$\hat{\mathscr{S}} = N \cup_i (\partial W_i \times [0,\varepsilon_i] \cup_{f_i} X_i) \xrightarrow{p} N \cup_i (L_i \times [0,\varepsilon_i] \cup \{x_i\}) = \mathscr{S} :$$

□

6.2 Proof of Theorem 9

We consider a $2n$-dimensional manifold M, which is $(n-1)$-connected, and want to determine its $(n-1)$ type. We begin with the classification up to fiber homotopy equivalence of fibrations $p : B \longrightarrow BO$, with a CW-complex B, fulfilling

(1) B is $(n-1)$-connected and

(2) $\pi_i(F) = 0$ for $i \geq n$, where F is the fiber.

Compare such a fibration with the $(n-1)$-connected cover of BO:

$$\begin{array}{ccc} & & BO\langle n-1\rangle \hookleftarrow F_{\langle n-1\rangle} \\ & \nearrow & \downarrow p_{\langle n-1\rangle} \\ B & \xrightarrow{p} & BO \end{array}$$

Since all obstructions vanish, we obtain a lift $\bar{p} : B \longrightarrow BO\langle n-1\rangle$, which without loss of generality may be assumed to be a fibration. From the long exact homotopy sequence we see that the homotopy groups of the fiber vanish, except in dimension $(n-1)$, where the group is $\Gamma := \operatorname{coker}(p_* : \pi_n(B) \longrightarrow \pi_n(BO))$. Thus, $\bar{p} : B \longrightarrow BO\langle n-1\rangle$ is a fibration with fiber $K(\Gamma; n-1)$. Such fibrations are classified in [Ba, §5.2] as follows:

$$\begin{array}{ccc} [BO\langle n-1\rangle; K(\Gamma; n)]/[Aut(\Gamma)] & \xrightarrow{\cong} & \mathcal{F}(K(\Gamma; n-1); BO\langle n-1\rangle) \\ [f] & \longmapsto & f^*(P(K(\Gamma; n))) \end{array}$$

Here $\mathcal{F}(K(\Gamma; n-1); BO\langle n-1\rangle)$ denotes the set of all fibrations over $BO\langle n-1\rangle$ with fiber $K(\Gamma; n-1)$ up to fiber homotopy equivalence. Thus $\bar{p} : B \longrightarrow BO\langle n-1\rangle$ is a pull back

$$\begin{array}{ccc} B & \longrightarrow & P(K(\Gamma; n)) \\ \downarrow \bar{p} & & \downarrow \\ BO\langle n-1\rangle & \xrightarrow{\varphi} & K(\Gamma; n) \end{array}$$

with an appropriate map φ. The definition of Γ force the induced map $\varphi_* : \pi_n(BO\langle n-1\rangle) \longrightarrow \pi_n(K(\Gamma; n)) = \Gamma$ to be surjective. Therefore we can assume φ_* to be the canonical projection to the factor group Γ. On the other hand, each factor group of $\pi_n(BO)$ leads to a fibration with the claimed properties. We summarize this discussion in

Lemma 21 *Fibrations with properties 1. and 2. are given by $p_k : B_k \longrightarrow BO$ up to fiber homotopy equivalence ($k \in \mathbb{N}$, $0 \leq k < |\pi_n(BO)|$).*

Consider now the fibration $p : B_k \longrightarrow BO$ and ask for a lift:

$$\begin{array}{ccc} & & B_k \\ & \nearrow & \downarrow p \\ \nu : M & \longrightarrow & BO \end{array}$$

With the help of obstruction theory we see that such a lift exists if and only if

$$\mathrm{im}\,(\nu_* : \pi_n(M) \longrightarrow \pi_n(BO)) \subset \langle kx \rangle,$$

where $\pi_n(BO) = \langle x \rangle$. Combining this discussion with Lemma 21, the statement of Theorem 9 follows immediately.

6.3 Surgery in the middle dimension

First we give a brief introduction in surgery, for more details compare [W2].

Surgery is a tool to eliminate homotopy classes in the category of manifolds. Let M be a compact m-dimensional manifold. One starts with an embedding $f : S^r \times D^{m-r} \hookrightarrow M$ and define $T := D^{r+1} \times D^{m-r} \cup_f (M \times I)$, where f is considered as a map to $M \times 1$. The corners of the manifold T can always be straighten, according to [CF]. This construction is called *attaching an* $(r+1)$-*handle* and T the *trace of a surgery via* f.

The boundary of T is $M \cup (\partial M \times I) \cup M'$ and we call M' the *result of a surgery of index* $r+1$ *via* f. It is not difficult to see that T can also be viewed as the trace of a surgery on M' via the obvious embedding of $D^{r+1} \times S^{m-r-1}$ into M', compare [Mi2].

Since we are working in the category of manifolds with B-structures we have to ask, whether the result of surgery via an embedding f is equipped with a B-structure. For general results see [Kr1]. In our situation we are only looking at fibrations $p_k : B_k \longrightarrow B$ defined in §3.

Lemma 22 *Let M be a manifold of dimension $m \in \{2n, 2n+1\}$ with B_k-structure $\bar\nu$ and $f : S^n \times D^{m-n} \hookrightarrow M$ an embedding. Then $\bar\nu : M \longrightarrow B$ extends to a normal B_k-structure on T, the trace of the surgery via f.*

Proof The embedding $f : S^n \times D^{m-n} \hookrightarrow M$ induces a normal B_k-structure on $S^n \times D^{m-n}$ denoted by $f^*\bar\nu$. There is a unique (up to homotopy) B_k-structure on $D^{n+1} \times D^{m-n}$ and we have to show that its restriction to $S^n \times D^{m-n}$ is

f . Let F_k be the fiber of $p_k : B_k \longrightarrow BO$. From the long exact homotopy sequence we see that the different B_k structures on $S^n \times D^{m-n}$ are classified by $\pi_n(F_k)$. But the construction of B_k implies that $\pi_n(F_k) = 0$, thus the restriction on $S^n \times D^{m-n}$ coincides with the given structure and we obtain a B_k-structure on T. □

Lemma 23 *Let M be an $(n-1)$-connected $2n$-dimensional manifold. Let $H_n(M)$ have a free basis $\{\alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_r\}$ with*

$\alpha_i \cdot \alpha_j = 0$ *for all* i, j *and*

$\alpha_i \cdot \beta_j = \delta_{ij}$ *for all* i, j.

If further each embedded sphere representing a generator α_i has a trivial normal bundle, then by a finite sequence of surgeries the homology group $H_n(M)$ can be eliminated.

Remark The last two lemmas imply that the condition in Theorem 10 is sufficient.

Proof According to a result of Haefliger ([Hae]), every element of $H_n(M)$ can be represented by an embedding $S^n \hookrightarrow M$. Let $\alpha := \alpha_r$ and $\varphi_0 : S^n \hookrightarrow M$ an embedding representing α. Since α is assumed to have a trivial normal bundle, the embedding can be extended to $\varphi : S^n \times D^n \hookrightarrow M$. Set $M_0 := M - \varphi((S^n \times D^n)^\circ)$ and let M' denote the result of surgery via φ, i.e. $M' = M_0 \cup_\varphi (D^{n+1} \times S^{n-1})$. Combine now the long exact homology sequences of pairs $(M; M_0)$ and $(M'; M_0)$ and obtain the following commutative diagram:

$$\begin{array}{ccccccccccc}
 & & \mathbb{Z} & \xrightarrow{1 \mapsto \beta} & & & & & & & \\
 & & \downarrow{\scriptstyle \partial'} & & & & & & & & \\
0 & \longrightarrow & H_n(M_0) & \xrightarrow[i]{} & H_n(M) & \longrightarrow & \mathbb{Z} & \xrightarrow{\partial} & H_{n-1}(M_0) & \longrightarrow & 0 \\
 & & \downarrow & & & & & & & & \\
 & & H_n(M') & & & & & & & & \\
 & & \downarrow & & & & & & & & \\
 & & 0 & & & & & & & &
\end{array}$$

The excision, together with the surjectivity of $H_n(M) \longrightarrow \mathbb{Z}$, implies M_0 is $(n-1)$-connected, further we see $H_n(M_0) = \langle \alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_{r-1} \rangle$.

Thus M^0 is $(n-1)$-connected as well and

$$H_n(M^0) = h\ _1; : : : ;\ _{r-1};\ _1; : : : ;\ _{r-1} i;$$

where the generators are given by $\ _i = \ _i + \ \mathbb{Z}$ and $\ _i = \ _i + \ \mathbb{Z}$. We can always deform the embedding of the generator $\ _i$ to M_0, such that it represents the class $\ _i + \ _r \ 2\ H_n(M_0)$. Thus we conclude $\ _i \ \ _j = 0$ und $\ _i \ \ _j = \ _{ij}$. Since the intersection product $\ _i \ \ _r$ vanishes we obtain $(\ _i + \ _r) = (\ _i) + (\ _r)$ (cf. [W1]). Now we proceed with the manifold M^0 and inductively obtain the desired statement. □

6.4 Surgery on odd-dimensional manifolds

Lemma 24 *Let T be a bordism in $\Omega_{2n}(B_k)$ between a manifold M of index k and a homotopy sphere S. Then T is bordant in $\Omega_{2n}(B_k)$ rel. boundary to T^0, such that its homology groups $H_n(T^0)$ and $H_{n+1}(T^0)$ are free and $H_i(T^0) = 0$ for $i\ 2\ f0; n; n+1; 2n+1g$.*

Proof According to Theorem 6, we can assume that T is $(n-1)$-connected, further the Universal Coe cient Theorem implies that $H_{n+1}(T)$ is free and $\mathrm{Tor}(H_n(T)) = \mathrm{Tor}(H_n(T; M))$. We will show that the torsion of $H_n(T; M)$ can be eliminated by a nite sequence of surgeries.

From the long exact homology sequence of the pair $(T; M)$ we see, that every torsion element $\ ^0\ 2\ H_n(T; M)$ comes from an element $\ 2\ H_n(T)$. After possible correction of by an element of $H_n(M) = \ _n(B_k)$ we achieve $(\ ) = 0$.

Let $' : S^n \ D^{n+1}\ ,!\ T$ be an embedding representing . As in the previous proof, we set $T_0 = T - '((S^n \ D^{n+1})\ )$ and $T^0 = T_0 \ [\ (D^{n+1} \ S^n)$. We combine now the exact triple sequences for $(T; T_0; M)$ and $(T^0; T_0; M)$ to obtain:

$$
\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
 & & & & H_{n+1}(T_0;M) & & & & \\
 & & & & \downarrow & & & & \\
 & & & & H_{n+1}(T';M) & & & & \\
 & & & & \downarrow & & & & \\
 & & & & \mathbb{Z} & \xrightarrow{1\mapsto\ '} & & & \\
 & & & & \downarrow d' & & & & \\
H_{n+1}(T;M) & \longrightarrow & \mathbb{Z} & \xrightarrow{d} & H_n(T_0;M) & \longrightarrow & H_n(T;M) & \longrightarrow & 0 \\
 & & & \searrow^{1\mapsto\ '} & \downarrow j' & & & & \\
 & & & & H_n(T';M) & & & & \\
 & & & & \downarrow & & & & \\
 & & & & 0 & & & &
\end{array}
$$

The element $\in H_n(T')$ is given by the embedding $: D^{n+1} \quad S^n \hookrightarrow T'$ and corresponds to the homotopy class $[\ j_0 \ {}_{S^n}]$.

We consider the two cases, where is free or a torsion element modulo $i\ (H_n(\partial T))$, separately.

Case 1 is primitive (mod $i\ (H_n(\partial T))$).

In this case the Poincare duality implies that the map $H_{n+1}(T;M) \longrightarrow \mathbb{Z}$ is surjective, therefore

$$H_n(T';M) = H_n(T;M)/\langle\ '\ \rangle.$$

Hence the torsion group of $H_n(T';M)$ has been reduced.

Case 2 is torsion (mod $i\ (H_n(\partial T))$).

The map $H_{n+1}(T;M) \longrightarrow \mathbb{Z}$ is trivial now. Denote with $o(x)$ the order of a torsion element x. From the sequence above we see that $o(\ ')d(1)$ im d, thus there exists a $b' \in \mathbb{Z}$ such that

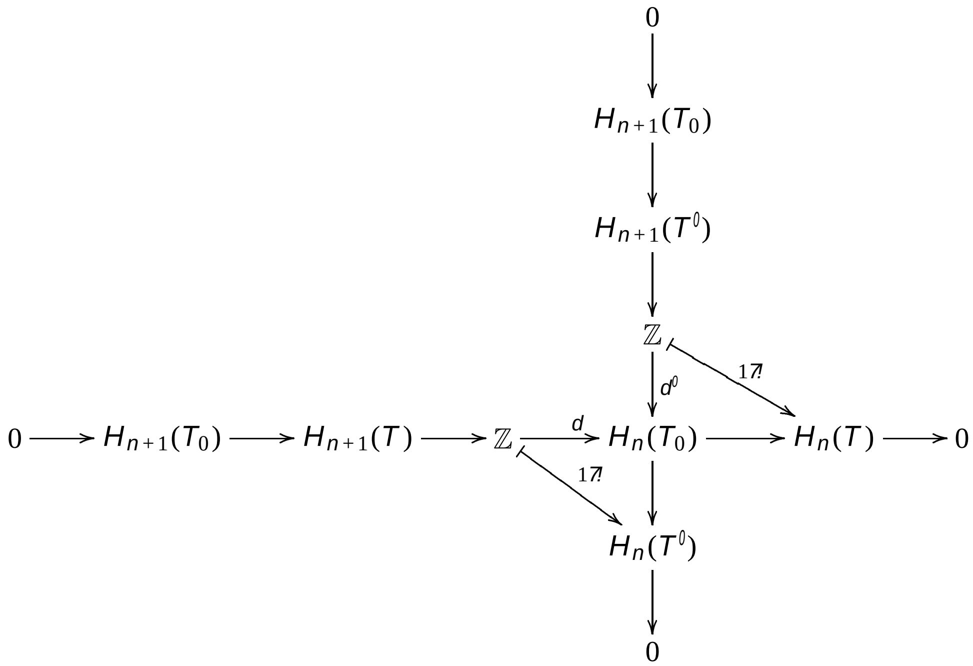
$$o(\ ')d'(1) = b'd(1). \qquad (\ )$$

If $b' = 0$, then the element $'$, corresponding to $'$, has infinite order, and the torsion rang again decreases.

If $b' \neq 0$, then $(\ker j') \subset \mathrm{Tor}(H_n(T_0; M))$, thus j' is injective on the free part of $H_n(T_0; M)$, therefore the element $d'(1)$ has infinite order and $o(\alpha') \leq |b'|$. We need a finer case differentiation.

Claim If n is even and α a torsion element of order a in $H_n(T)$, then $\alpha' \in H_n(T')$ is an element of infinite order.

Consider the pair sequences $(T; T_0)$ and $(T'; T_0)$ and obtain the following diagram



As in the previous case, there exists a $b \in \mathbb{Z}$ such that

$$a \cdot d'(1) = b \cdot d(1) \text{ in } H_n(T_0)$$

or equivalently after the identification of the generators

$$(\varphi'_0)_*(a \cdot ([S^n] \times 1) - b \cdot (1 \times [S^n])) = 0.$$

By computing the self intersection number of $a \cdot ([S^n] \times 1) - b \cdot (1 \times [S^n])$ we obtain $2ab([S^n] \times [S^n])^* \in \mathrm{im}\,((\varphi'_0)^* : H^{2n}(W_0) \longrightarrow H^{2n}(S^n \times S^n))$. Thus $2ab = 0$, and since $a \neq 0$ it follows $b = 0$. We conclude again that α' is of infinite order, as described previously.

Claim If n is even and α has infinite order, then the element α' will be eliminated.

In this situation we have $= \tilde{} + i\,(y)$, where $\tilde{}$ is a torsion element and $y \in H_n(M)$. The normal bundle map is given by

$$: H_n(T) \longrightarrow {}_{n-1}(SO(n+1)) = {}_{n-1}(SO).$$

If ${}_{n-1}(SO) = 0$ or $\mathbb{Z}$ the fact that the map $: H_n(T) \longrightarrow {}_{n-1}(SO)$ is a homomorphism implies, that $\tilde{}$ already has a trivial normal bundle, this leads us in the situation of the last claim. It remains to study the situation ${}_{n-1}(SO) = \mathbb{Z}/2$ with $(\tilde{}) = 1$. Observe that without loss of generality we can assume $i\,(y)$ to be primitive. Thus the map $H_{n+1}(T) \longrightarrow \mathbb{Z}$ from the sequence of the last claim is surjective and we obtain

$$H_n(T^0) = H_n(T)/\langle \ \rangle \Rightarrow H_n(T^0;M) = H_n(T;M)/\langle {}^0 \rangle .$$

Claim If n is odd then the torsion group $\mathrm{Tor}(H_n(T;M))$ can be reduced.

The group ${}_n(SO_{n+1})$ acts on the trivializations via

$$(\omega; \varphi) \mapsto \ : S^n \ D^{n+1} \hookrightarrow T$$
$$(x;y) \mapsto \varphi(x; \omega(x)y)$$

Note, that the change of trivialization does not affect the B-structure on S^n D^{n+1} since the induced B-structures φ and differ by an element from ${}_n(F_k) = 0$.

Let y_0 be the basis point of S^n, then we see

$${}_0(x;y_0) = \varphi_0(x;\omega(x)(y_0)) = \varphi_0(x;pw(x)) = \varphi_0(\mathrm{id} \ \ p\omega) \ (x);$$

where the map $p : SO(n+1) \longrightarrow S^n$ is the canonical fiber bundle and $: S^n \longrightarrow S^n \ S^n$ the diagonal. Denote by $i_l : S^n \longrightarrow S^n \ S^n$ the inclusion in the l-th component and let be the generator of ${}_n(S^n)$, set ${}_l := (i_l)\ (\ )$. We pass to homotopy and use the fact that the map $: {}_{n+1}(S^{n+1}) \longrightarrow {}_n(S^n)$ from the long exact homotopy sequence for p is given by multiplication with 2 [Ste]. Thus we compute

$$\begin{aligned} (\ {}_0)\ (\ {}_1) &= (\varphi_0)\ (\mathrm{id} \ \ p\omega)\ \ (\ ) \\ &= (\varphi_0)\ (\mathrm{id} \ \ p\omega)\ (\ {}_1 + \ {}_2) \\ &= (\varphi_0)\ (\ {}_1 + 2k \ \ {}_2) \\ &= (\varphi_0)\ (\ {}_1) + 2k\ \ (\varphi_0)\ (\ {}_2) \end{aligned}$$

Since M is $(n-1)$-connected we obtain the corresponding statement in homology:

$$(\ {}_0)\ ([S^n] \ \ 1) = (\varphi_0)\ ([S^n] \ \ 1) + 2k\ \ (\varphi_0)\ (1 \ \ [S^n])$$

The equality ( ) now becomes

$$b\ (\ _0)\ (1\quad [S^n]) = a\ ((\ _0)\ ([S^n]\quad 1) - 2k(\ _0)\ (1\quad [S^n]))$$
$$a\ (\ _0)\ ([S^n]\quad 1) = (b + 2ka)(\ _0)\ (1\quad [S^n])$$

Since the case $b + 2ka = 0$ has already been treated, we assume $b + 2ka \neq 0$. By choosing k appropriately we can achieve

$$o(\ ^0)\quad o(\ ^0).$$

Let p be a prime number such that $(\ ^0)_p \neq 0$ in $H_n(T; M; \mathbb{F}_p)$. From the analogous sequences with $\mathbb{F}_p$-coe cients we conclude

$$H_n(T^0; M; \mathbb{F}_p) = H_n(T_0; M; \mathbb{F}_p)/\mathrm{im}\, d^0 = H_n(T; M; \mathbb{F}_p)/<\ ^0>$$

and with universal coe cient theorem

$$|\mathrm{Tor}(H_n(T^0; M))| < |\mathrm{Tor}(H_n(T; M))|.$$

Combining all cases together we see, that a torsion element of $H_n(T; M)$ can either be eliminated by a nite sequence of surgeries or can be replaced by an element of in nite order. Inductively we obtain the desired statement. □

6.5 Proof of Theorem 10

Proof We only have to prove that every manifold normally B_k bordant to a homology sphere is elementary.

Let T be a bordism between M and a homology sphere S. According to Theorem 6, we can without loss of generality assume that T is $(n-1)$-connected and that the homology groups $H_n(T)$ and $H_{n+1}(T)$ are free. The long exact sequence of the pair $(T; M)$ together with Poincare duality leads to the following commutative diagram:

$$\begin{array}{ccccccccc}
H_{n+1}(T) & \longrightarrow & H_{n+1}(T; M) & \longrightarrow & H_n(M) & \xrightarrow{i} & H_n(T) & \xrightarrow{j} & H_n(T; M) \longrightarrow 0 \\
\downarrow {\scriptstyle P.-D.} & & \downarrow {\scriptstyle P.-D.} & & \downarrow {\scriptstyle P.-D.} & & \downarrow {\scriptstyle P.-D.} & & \\
H^n(T; M) & \xrightarrow{j} & H^n(T) & \xrightarrow{i} & H^n(M) & \xrightarrow{d} & H^{n+1}(T; M) & &
\end{array}$$

From this we see

$$\dim H_n(M) = 2(\dim \ker i\ ).$$

Thus $\ker i$ is a direct summand of $H_n(M)$. It is not hard to verify, that $\ker i$ is isotropic. Let now $\ : S^n \hookrightarrow M$ be a representative of an element of $\ker i$, then the map can be extended to $\ : D^n \longrightarrow T$ and with the help of the Whitney-trick [Hae] this map can be assumed to be an embedding. Thus the restriction of the normal bundle of D^n to the boundary gives us a trivialization of the normal bundle of . □

6.6 Proof of Theorem 11

Proof Using the statement of Theorem 10 we see, that the conditions of Theorem 6 are fulfilled for the manifolds $\overline{U}^\wedge{}_i \sharp S$ and $\overline{U}^{\wedge\prime}_i$ for a homotopy sphere S. Thus, the diffeomorphism on the boundary can be extended to the the diffeomorphism $\overline{U}^\wedge{}_i \sharp S \sharp k(S^n \times S^n) \longrightarrow \overline{U}^{\wedge\prime}_i \sharp k(S^n \times S^n)$ for $k \in \{0, 1\}$. From the definition of resolutions and from construction the diffeomorphism on the boundaries we see that the obtained diffeomorphisms extend in the obvious way to a diffeomorphism $\mathscr{S} \sharp S \sharp k(S^n \times S^n) \longrightarrow \mathscr{S} \sharp k(S^n \times S^n)$ having the desired properties.

It remains to show, that the conditions of the theorem are true for every pair of representatives of the neighbourhood germs $[\overline{U}^\wedge{}_i]$ and $[\overline{U}^{\wedge\prime}_i]$, once we have checked them on a single representative pair. If $\mathrm{c}_i : L_i \times [0, \varepsilon_i/2] \longrightarrow N$ and $\mathrm{d}_i : L_i \times [0, \varepsilon'_i/2] \longrightarrow N$ with $\varepsilon_i < \varepsilon'_i$ are two representatives of the germ of collars around L_i, then there exists a $\delta_i > 0$ such that c_i coincides with d'_i on $L_i \times [0, \delta_i]$. We choose a diffeomorphism $\varphi_i : [0, \varepsilon_i/2] \longrightarrow [0, \varepsilon'_i/2]$ with $\varphi_i \equiv \mathrm{id}$ on $[0, \delta_i]$. The map induces an isomorphism

$$\begin{array}{ccc} \overline{U}_{i\mathbf{c}} & \longrightarrow & \overline{U}_{i\mathbf{d}} \\ f(\mathrm{c}_i(x, t)) & \longmapsto & f(\mathrm{d}_i(x, \varphi(t))) \end{array}$$

being the identity on a small neighbourhood of $x_i \in \Sigma$. This gives us a diffeomorphism between $\overline{U}^\wedge{}_{i\mathbf{c}}$ and $\overline{U}^\wedge{}_{i\mathbf{d}}$ making the following diagram commutative:

$$\begin{array}{ccccccc} \partial\overline{U}^\wedge{}_{i\mathbf{c}} & \hookrightarrow & \overline{U}^\wedge{}_{i\mathbf{c}} & \xrightarrow{\cong} & \overline{U}^\wedge{}_{i\mathbf{d}} & \hookleftarrow & \partial\overline{U}^\wedge{}_{i\mathbf{d}} \\ \downarrow\cong & & & & & & \downarrow\cong \\ \partial\overline{U}^{\wedge\prime}{}_{i\mathbf{c}} & \hookrightarrow & \overline{U}^{\wedge\prime}{}_{i\mathbf{c}} & \xrightarrow{\cong} & \overline{U}^{\wedge\prime}{}_{i\mathbf{d}} & \hookleftarrow & \partial\overline{U}^{\wedge\prime}{}_{i\mathbf{d}} \end{array}$$

This completes the proof. □

6.7 Algebraic invariants

ad (2) Let (H, λ, s) be elementary and $L = \langle \alpha_1, \ldots, \alpha_k \rangle$ a Lagrangian with $s|_L \equiv 0$. Thus $\mathrm{sign}(\lambda) = 0$ and $0 = s(\alpha_i) = \langle \alpha_s, \alpha_i \rangle$ for all $1 \leq i \leq k$. Since L is maximal it follows that $\alpha_s \in L$ and therefore $s(\alpha_s) = 0$.

On the other hand let $\mathrm{sign}(\lambda) = 0$ and $s(\alpha_s) = 0$. Choose a basis $\{\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_k\}$ of H such that $\lambda(\alpha_i, \alpha_j) = 0$ and $\lambda(\alpha_i, \beta_j) = \delta_{ij}$.

There is nothing to show if $\alpha_s = 0$. Otherwise we can without loss of generality assume that $s(\alpha_i) = 0$ for all $i > 1$, since s is a homomorphism.

Recall the equality $s(v) = (\ ,v)\ \forall v \in H$. Since $\ _s \in H$ there are $a_i, b_i \in \mathbb{Z}$ such that $\ _s = \sum_{i=1}^{k}(a_i\ _i + b_i\ _i)$. Consider a sub-Lagrangian $L' := \langle\ _2, \ldots, \ _k\rangle$. If $\ _s \notin L'$, build $L := \langle\ _2, \ldots, \ _k, \tilde{}_s\rangle$, where $\tilde{}_s$ is a primitive element of H with $\ _s \in \langle\tilde{}_s\rangle$. This is a Lagrangian, satisfying $s|_L \equiv 0$. In the case of $\ _s \in L'$, the coefficients $a_1, b_1, \ldots, b_k$ have to be zero, thus $L = \langle\ _1, \ldots, \ _k\rangle$ is a Lagrangian with the desired property.

ad (3) The conditions are obviously necessary. To see that they are also sufficient choose a symplectic basis $\{\ _1, \ldots, \ _k, \ _1, \ldots, \ _k\}$ of H. Sort the generators in the following way

$$s(\ _i) = s(\ _i) = 1 \qquad \text{for } i \le s;$$
$$s(\ _i) = 0 \qquad \text{for } i > s;$$

where s is an integer between 0 and k. The assumption

$$(H) = \sum_{i=1}^{k} s(\ _i)s(\ _i) = 0$$

implies that $s \equiv 0 \pmod 2$. Construct a new basis $\{\ '_1, \ldots, \ '_k\}$ for H by the substitution

$$\ '_{2i-1} = \ _{2i-1} + \ _{2i}, \qquad \ '_{2i} = \ _{2i-1} - \ _{2i};$$
$$\ '_{2i-1} = \ _{2i} \qquad \ '_{2i} = \ _{2i}$$

for $2i \le s$, and

$$\ '_i = \ _i \qquad \ '_i = \ _i$$

for $i > s$. This new basis is again symplectic and satisfies the condition

$$s(\ '_1) = \cdots = s(\ '_k) = 0.$$

6.8 Proof of Theorem 19

Proof As in the proof of Theorem 11 we conclude that it is enough to show that the diffeomorphism on the boundary $\partial\overline{U}^{\wedge}_i \longrightarrow \partial\overline{U}^{\wedge\prime}_i$ can be extended to a homeomorphism on $\overline{U}^{\wedge}_i \# k(S^2 \times S^2) \longrightarrow \overline{U}^{\wedge\prime}_i \# k(S^2 \times S^2)$. Since the resolutions are optimal, the manifolds $\overline{U}^{\wedge}_i$ and $\overline{U}^{\wedge\prime}_i$ are 1-connected, hence $M := \overline{U}^{\wedge}_i \cup_{\partial} \overline{U}^{\wedge\prime}_i$ is again 1-connected. In order to apply Theorem 18 we have to show that the closed 4-dimensional spin manifold with vanishing signature is bordant to a homotopy sphere. Then we apply the topological 4-dimensional Poincare conjecture proved by Freedman [F, Thm. 1.6] and obtain the desired statement.

We want to use surgery to prove that M is bordant to a homotopy sphere. Since M is spin and $\operatorname{sign}(M) = 0$, there is a basis $\{\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_k\}$ of $H_2(M)$ satisfying

$$(\alpha_i, \alpha_j) = 0 \quad (\beta_i, \beta_j) = 0 \quad (\alpha_i, \beta_j) = \delta_{ij}.$$

We can not use the Haefliger's embedding theorem in dimension 4, but according to [F, Thm. 3.1, 1.1] every generator α_i is represented by a topological embedding $S^2 \times D^2 \hookrightarrow M$. Knowing this we can proceed in exactly the same way as in the proof of Lemma 23, working in the category TOP.

Lemma 24 is still valid in dimension 4 as well as the arguments in §6.6.

To complete the proof observe that according to arguments from §6.6 it is enough to check the conditions for a single representative of the neighbourhood germ. □

References

[AK] S. Akbulut, H. King, *Topology of Real Algebraic Sets.* L'Enseignement Mathematique, 29 (1983), 221-261.

[Ba] H. H. Baues, *Obstruction Theory, on Homotopy Classification of Maps.* Springer Lectures Notes 628, 1977.

[BR] R. Benedetti and Jean-Jacques Risler, *Real Algebraic and Semi-Algebraic Sets.* Hermann, Editeurs des science et des arts, 1990.

[BT] T. Bröcker and T. tom Dieck, *Representations of Compact Lie Groups.* Springer-Verlag, 1985.

[CF] P. Conner and E. E. Floyd, *Differentiable Periodic Maps.* Springer-Verlag, 1972.

[F] M. H. Freedman, *The Topology of Four-Dimensional Manifolds.* J. Differ. Geom. 17 (1982), 357-453.

[H] M. W. Hirsch, *Differential Topology.* Springer-Verlag, 1976.

[Hae] A. Haefliger, *Differentiable Imbeddings.* Bull. Amer. Soc. 67 (1961), 109-112.

[Hi] H. Hironaka, *Resolutions of Singularities of an Algebraic Variety over a Field of Characteristic Zero: I,II.* Ann. of Math. 79 (1964), 109 - 326.

[Ke] M. A. Kervaire, *Some Non-Stable Homotopy Groups of Lie Groups.* Illinois J. Math. Vol 4 (1960), 161-169.

[Kr1] M. Kreck, *Surgery and Duality.* Annals of Mathematics Vol. 149 (1999), 704-754.

[Kr2] M. Kreck, *Differential Algebraic Topology.* Preprint.

[Kr3] M. Kreck, *Stratifolds.* Preprint.

[Mi1] J. Milnor, *A Procedure for Killing the Homotopy Groups of Di erentiable Manifolds.* Symposia in Pure Math. A.M.S., Vol. III (1961), 39-55.

[Mi2] J. Milnor, *Morse Theory.* Annals of Math. Studies 51, Princeton University Press, 1965.

[Mi3] J. Milnor, *Remarks Concerning Spin Manifolds.* Di er. and Combinat. Topology, Sympos. Marston Morse, Princeton, 1965, 55-62.

[Mi4] J. Milnor, *Singular Points of Complex Hypersurfaces.* Princeton University Press, 1968.

[St] R. E. Stong, *Notes on Cobordism Theory.* Mathematical Notes, Princeton University Press, 1968.

[Ste] N.Steenrod, *The Topology of Fibre Bundles.* Princeton University Press, 1951.

[Th] R. Thom, *Quelques Properietes Globales des Varietetes Di erentiables.* Comm. Mathe. Helv. Vol 28 (1954), 17-86.

[W1] C. T. C. Wall, *Classi cation of* $(n-1)$*-Connected* $2n$*-Manifolds.* Annals of Mathematics Vol. 75 (1962), 163-189.

[W2] C. T. C. Wall, *Surgery on Compact Manifolds.* London Math. Soc. Monographs 1, Academic Press, New York, 1970.

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