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Rigidity of graph products of groups

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Abstract We show that if a group can be represented as a graph product of finite directly indecomposable groups, then this representation is unique.

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1 Introduction

Given a simple graph with nontrivial groups as vertices, a group is formed by taking the free product of the vertex groups, with added relations implying that elements of adjacent groups commute. This group is said to be the *graph product* of the vertex groups. If the graph is discrete then the graph product is the free product of the vertex groups; while if the graph is complete then the graph product is the restricted direct product[1] of the vertex groups. Graph products were first defined in Elisabeth Green's Ph.D. thesis [8], and have been studied by other authors [9, 10, 11].

Important special cases of graph products arise when we specify the vertex groups. If all vertex groups are infinite cyclic, then the graph product is called a *graph group* or a *right-angled Artin group*. Graph groups have been studied by many authors [7, 16, 17]. If all vertex groups have order two, then the graph product is called a *right-angled Coxeter group*. These groups were first studied by Ian Chiswell [2], and they have been studied by many other authors [4, 5, 6].

In this article we investigate the question of uniqueness for graph product decompositions. Carl Droms [7] proved that two graph products of infinite cyclic groups are isomorphic if and only if their graphs are isomorphic. Elisabeth Green [8] proved that if a group can be represented as a graph product of cyclic groups of prime order, then this representation is unique. This result was extended to primary cyclic groups by the present author [14]. Our main result is the following: If a group can be represented as a graph product of directly indecomposable finite groups, then this representation is unique.

[1] By "restricted" we mean that all but finitely many entries are the identity.

2 Graphs and modular partitions

A graph is an ordered pair of sets $(V; E)$ where E is a set of two-element subsets of V. Elements of V are called *vertices*, and elements of E are called *edges*. For the remainder of this paper we shall assume that V is finite. A clique is a maximal complete subgraph, or (by abuse of terminology) the set of vertices of a maximal complete subgraph.

A *module* of a graph $(V; E)$ is a subset X of V such that for every $v \in V - X$, either v is adjacent to every element of X or v is adjacent to no element of X. A *modular partition* is a partition of V into non-empty modules. A modular partition induces a quotient graph $(\bar{V}; \bar{E})$ where $\bar{V}$ is the set of partition classes and $\{\alpha; \beta\} \in \bar{E}$ if and only $\{u; v\} \in E$ for some (and hence for all) $u \in \alpha$ and $v \in \beta$. We may regard $(\bar{V}; \bar{E})$ as a compressed version of the original graph. Given the quotient graph and the subgraphs induced by the partition classes, it is possible to reconstruct the original graph. For this reason, modular partitions have been studied extensively by computer scientists [12].

We say that a graph $(V; E)$ is T_0 if no edge is a module. This means that for all $\{u; v\} \in E$, there exists $w \in V - \{u; v\}$ so that w is adjacent to u or v but not both. A graph is T_0 if and only if vertices are distinguished by the cliques to which they belong. That is, a graph is T_0 if and only if the following condition holds: for every pair of distinct vertices, there exists a clique which contains exactly one of them.

Let us say that two vertices are *equivalent* if they cannot be distinguished by the cliques. Then the set of equivalence classes is a modular partition. The quotient graph resulting from this partition satisfies the T_0 condition, and it will be called the *T_0 quotient*.

Similarly, a graph $(V; E)$ is T_1 if for all $\{u; v\} \in E$ there exists $w \in V - \{u; v\}$ so that $\{u; w\} \in E$ and $\{v; w\} \notin E$. Equivalently, a graph is T_1 if and only if every vertex is the intersection of the set of cliques to which it belongs. Note that this condition is stronger than the T_0 condition.

3 Graph products of groups

Let $\Gamma = (V; E)$ be a graph, and let $\{G_v\}_{v \in V}$ be a collection of groups which is indexed by the vertex set of Γ. We say that $(\Gamma; G_v)$ is a *graph of groups.*[2] Two

[2] This differs from the usual definition, which has vertex groups and edge groups, together with monomorphisms from the edge groups to the vertex groups [1].

graphs of groups, $(\Gamma; G_v)$ and $(\Gamma'; G'_v)$, are *isomorphic* if there exists a graph isomorphism $\phi\colon \Gamma \to \Gamma'$ so that G_v and $G'_{\phi(v)}$ are isomorphic for all $v \in V$.

The *graph product* G of a graph of groups is the quotient of the free product of the vertex groups by the normal subgroup generated by all commutators of elements taken from pairs of adjacent groups. That is, $G = F/N$ where $F = \ast_{v\in V} G_v$ and N is the normal closure in F of

$$\{g^{-1}h^{-1}gh : g \in G_u, h \in G_v, \{u, v\} \in E\}.$$

The canonical monomorphism from G_v to F induces a monomorphism from G_v to G. We may thus identify each vertex group G_v with its image in G, in which case we say that G is an *internal* graph product.

The graph product can also be described in terms of generators and relations. Choose a presentation $(\gamma_v; \mathcal{R}_v)$ for each vertex group, so that the generating sets γ_v are pairwise disjoint. Then G has a presentation $(\bigcup \gamma_v; \bigcup \mathcal{R}_v \cup \mathcal{R})$ where $\mathcal{R} = \{a^{-1}b^{-1}ab : a \in \gamma_u, b \in \gamma_v, \{u, v\} \in E\}$.

If A is a subset of V, then we denote by Γ_A the subgraph of Γ that is induced by A, and we denote by $G(A)$ the subgroup of G that is generated by $\bigcup_{a\in A} G_a$.

Theorem 3.1 [8, 14] *If A is a subset of V then $G(A)$ is the internal graph product of $(\Gamma_A; G_a)$.* □

Corollary 3.2 *If A is complete then $G(A)$ is the (restricted) direct product of the G_a, and if A is discrete then $G(A)$ is the free product of the G_a.* □

Theorem 3.3 *If $A \subseteq V$ then there is a homomorphism $\rho_A\colon G \to G(A)$ so that $\rho_A(x) = x$ for all $x \in G(A)$ and $\rho_A(x) = 1$ for all $x \in G(V - A)$. We call ρ_A a* retraction homomorphism.

Proof For each $a \in A$ let $h_a\colon G_a \to G(A)$ be the inclusion homomorphism, and for $b \in V - A$ let $h_b\colon G_b \to G(A)$ be the trivial homomorphism. Then there exists a homomorphism ρ_A which extends h_v for all $v \in V$. It is clear that $\rho_A(x) = x$ for all $x \in G(A)$, since $\rho_A(x) = x$ for all $x \in \bigcup_{a\in A} G_a$, and likewise that $\rho_A(x) = 1$ for all $x \in G(V - A)$. □

Theorem 3.4 *If A and B are subsets of V then $G(A \cup B) = \langle G(A) \cup G(B)\rangle$.*

Proof $G(A) = \left\langle \bigcup_{a\in A} G_a \right\rangle$ and $G(B) = \left\langle \bigcup_{b\in B} G_b \right\rangle$, thus

$$\langle G(A) \cup G(B)\rangle = \left\langle \bigcup_{c\in A\cup B} G_c \right\rangle = G(A \cup B).$$ □

Theorem 3.5 *If A and B are subsets of V then $G(A) \cap G(B) = G(A \cap B)$.*

Proof It is clear that $G(A \cap B) \subseteq G(A) \cap G(B)$. For the reverse inclusion, let $\rho_A \colon G \to G(A)$ be the retraction homomorphism of Theorem 3.3, and let $x \in G(A) \cap G(B)$. It remains to prove that $x \in G(A \cap B)$. If $b \in A \cap B$ then $\rho_A(y) = y$ for all $y \in G_b$. If $b \in B - A$ then $\rho_A(y) = 1$ for all $x \in G_b$. In either case $\rho_A(y) \in G(A \cap B)$ for all $y \in G_b$ and all $b \in B$. Therefore $\rho_A(x) \in G(A \cap B)$. But $\rho_A(x) = x$ since $x \in G(A)$. Therefore $x \in G(A \cap B)$ as claimed. □

Theorem 3.6 *Let $A, B \subseteq V$ be complete. If $x \in G(A)$ and x is conjugate to an element $y \in G(B)$, then $x \in G(A \cap B)$.*

Proof Let $\rho \colon G \to G(V - B)$ be the retraction homomorphism of Theorem 3.3. Then $\rho(y) = 1$, so $\rho(x) = 1$ as well, since the kernel is a normal subgroup. By Corollary 3.2, we may express x uniquely as $x = \prod_{a \in A} x_a$, where $x_a \in G_a$ for all $a \in A$. Then $\rho(x) = \prod_{a \in A} \rho(x_a) = \prod_{a \in A - B} x_a$. But $\rho(x) = 1$, so $x_a = 1$ for all $a \in A - B$. Therefore $x \in G(A \cap B)$. □

Corollary 3.7 *If $A, B \subseteq V$ are complete and $G(A)$ is conjugate to $G(B)$ then $A = B$.* □

The proof of the following theorem is left to the reader.

Theorem 3.8 *Let $\Gamma = (V, E)$ be the quotient graph resulting from a modular partition of $\Gamma = (V, E)$. Then G is the graph product of $(\Gamma, G(A))$, where A varies over the modules of Γ.* □

Since the partition of a graph into its components (or co-components) is modular, we obtain the following corollary. (Recall that a co-component of a graph is a component of the complement.)

Corollary 3.9 *If the components of Γ are $A_1, \ldots, A_n$ then $G = \coprod_i G(A_i)$. If the co-components of Γ are $B_1, \ldots, B_m$ then $G = \bigoplus_i G(B_i)$.* □

We also require the following result, which is proved in [8].

Theorem 3.10 *For every finite subgroup F of G there exists a complete subgraph C so that F is conjugate to a subgroup of $G(C)$.* □

Corollary 3.11 *If all vertex groups are finite, then F is a maximal finite subgroup of G if and only if there exists a clique C so that F is conjugate to $G(C)$.*

Proof Let F be a maximal finite subgroup of G. By the previous theorem, there exists a clique C so that F is conjugate to a subgroup of $G(C)$. However, $G(C)$ itself is a finite subgroup of G. Therefore F is conjugate to $G(C)$.

Conversely, let C be a clique, and let F be a conjugate of $G(C)$. Let F' be a finite subgroup of G so that $F' \geq F$. By the previous theorem there exists a clique D so that F' is conjugate to a subgroup of $G(D)$. Therefore $G(C)$ is conjugate to a subgroup of $G(D)$. It follows from Theorem 3.6 that C is a subset of D. But C is a clique, hence $C = D$ and $F' = F$. Consequently F is a maximal finite subgroup. □

Remark 3.12 An alternate proof of this corollary can be obtained by considering the action of G on the CAT(0) cube complex defined by John Meier and other authors [13, 3, 11]. Any finite group acting on cellularly on a CAT(0) complex fixes some cell. Since stabilizers of cubes in this complex are conjugates of the groups G(C), the corollary follows.

4 Conjugacy classes of finite subgroups

Let G be the internal graph product of $(\Gamma; G_v)$. We assume for the remainder of this article that $\Gamma = (V; E)$ is a finite graph, and that each vertex group G_v is finite.

Let $\mathcal{F}$ denote the set of conjugacy classes of finite subgroups of G. We write $[F]$ to denote the set of subgroups of G which are conjugate to a given finite subgroup F. We define a partial ordering on $\mathcal{F}$ as follows: If A and B are finite subgroups of G, then $[A] \leq [B]$ if and only if there exists $g \in G$ so that $A \leq gBg^{-1}$.

Theorem 4.1 *The relation $\leq$ is a well-defined partial ordering on $\mathcal{F}$.*

Proof *$\leq$ is well defined:* Let A and B be finite subgroups of G, and suppose that there exists $g \in G$ so that $A \leq gBg^{-1}$. Let A' and B' be subgroups of G which are conjugate to A and B respectively. There exist $h, k \in G$ so that $A' = hAh^{-1}$ and $B' = kBk^{-1}$. Now $A' \leq h(gBg^{-1})h^{-1} = hgk^{-1}B'kg^{-1}h^{-1}$, so $A' \leq mB'm^{-1}$ where $m = hgk^{-1}$.

$\prec$ *is transitive:* Let A, B, C be finite subgroups of G so that $[A] \prec [B]$ and $[B] \prec [C]$. There exist $g, h \in G$ so that $A \subseteq gBg^{-1}$ and $B \subseteq hCh^{-1}$. Then $A \subseteq ghC(gh)^{-1}$, hence $[A] \prec [C]$.

$\prec$ *is irreflexive:* Let A and B be finite subgroups of G so that $[A] \prec [B]$ and $[B] \prec [A]$. Then there exist g, h so that $A \subseteq gBg^{-1}$ and $B \subseteq hAh^{-1}$. Since A and B are finite, it follows that $|A| = |B|$ and $A = gBg^{-1}$. Therefore $[A] = [B]$. □

Recall that if A is a subset of a partially ordered set $(X, \preceq)$ then the *least upper bound* of A, denoted $\bigvee A$, is an element $x \in X$ so that $a \preceq x$ for all $a \in A$, and if $a \preceq y$ for all $a \in A$ then $x \preceq y$. The least upper bound is unique when it exists. Similarly, the *greatest lower bound* of A, denoted $\bigwedge A$, is an element $x \in X$ so that $x \preceq a$ for all $a \in A$, and if $y \preceq a$ for all $a \in A$ then $y \preceq x$.

Theorem 4.2 *If $A, B \in \mathcal{C}$ then $[G(A \cap B)] = [G(A)] \wedge [G(B)]$.*

Proof It is obvious that $[G(A \cap B)] \preceq [G(A)]$ and $[G(A \cap B)] \preceq [G(B)]$. Suppose that F is a finite subgroup of G so that $[F] \preceq [G(A)]$ and $[F] \preceq [G(B)]$. We need to show that $[F] \preceq [G(A \cap B)]$.

We may assume without loss of generality that $F \subseteq G(A)$. If $x \in F$ then x is conjugate to an element of $G(B)$, hence $x \in G(B)$ by Theorem 3.6. Therefore $F \subseteq G(A) \cap G(B) = G(A \cap B)$ by Theorem 3.5, so we are done. □

Theorem 4.3 *Let $A, B \in \mathcal{C}$. If $A \cup B \in \mathcal{C}$ then $[G(A \cup B)] = [G(A)] \vee [G(B)]$. If $A \cup B \notin \mathcal{C}$, then $[G(A)]$ and $[G(B)]$ do not have a common upper bound.*

Proof Suppose that $A \cup B \in \mathcal{C}$. Then $[G(A \cup B)]$ is an upper bound for $[G(A)]$ and $[G(B)]$. We wish to show that it is the least upper bound.

Let $[F]$ be another upper bound of $[G(A)]$ and $[G(B)]$. By Theorem 3.10, there exists $C \in \mathcal{C}$ and $h \in G$ so that $hFh^{-1} \subseteq G(C)$. Then $C \supseteq A \cup B$ by Theorem 3.6.

Now hFh^{-1} contains conjugates of $G(A)$ and $G(B)$, so hFh^{-1} contains both $G(A)$ and $G(B)$ by Theorem 3.6. Therefore $hFh^{-1} \supseteq G(A \cup B)$, and hence $[G(A \cup B)] \preceq [F]$.

Now suppose that $A \cup B \notin \mathcal{C}$. If $[F]$ is an upper bound for $[G(A)]$ and $[G(B)]$, then (since F is finite) there exists a complete subgraph D so that $F \subseteq G(D)$. By Theorem 3.6, D contains $A \cup B$. But $A \cup B$ is not complete, and this is a contradiction. □

5 Uniqueness of graph product decompositions

Let $\Gamma = (V, E)$ and $\Gamma' = (V', E')$ be finite graphs so that $V \cap V' = \emptyset$. Let G be a group, and suppose that G_v is a nontrivial finite subgroup of G for each $v \in V \cup V'$. Finally, suppose that G is the internal graph product of both (Γ, G_v) and $(\Gamma', G_{v'})$.

Theorem 5.1 *For each clique C of Γ there is a unique clique C' of Γ' so that $[G(C)] = [G(C')]$.*

Proof If C is a clique of Γ then $G(C)$ is a maximal finite subgroup of G by Corollary 3.11. Again by Corollary 3.11, $G(C)$ is conjugate to $G(C')$ for some clique C' of Γ'. Therefore $[G(C)] = [G(C')]$.

Uniqueness of C' follows from Corollary 3.7. □

Theorem 5.2 *If Γ and Γ' are T_1 then there is a graph isomorphism $\phi\colon \Gamma \to \Gamma'$ so that $[G_v] = [G_{\phi(v)}]$ for all $v \in V$. In particular, (Γ, G_v) and $(\Gamma', G_{v'})$ are isomorphic graphs of groups.*

Proof Let $v \in V$, and let $\{C_1, \ldots, C_n\}$ be the set of all cliques of Γ which contain v. Then $\{v\} = \bigcap_{i=1}^n C_i$ since Γ is T_1.

For each i there exists a clique C_i' of Γ' so that $[G(C_i)] = [G(C_i')]$ by Theorem 5.1. Now $[G_v] = \bigwedge_i [G(C_i)] = \bigwedge_i [G(C_i')] = [G(C')]$, where $C' = \bigcap_i C_i'$. In particular, C' is not empty since $[G(C')]$ is non-trivial.

I claim that C' has only one element. Suppose that C' contains two distinct elements r', s'. Since Γ' is T_1, there is a clique D' of Γ' which contains r' but not s', and by Theorem 5.1 there is a corresponding clique D of Γ so that $[G(D)] = [G(D')]$.

Now $D \cap \bigcap_{i=1}^n C_i = \emptyset$, since $v \notin D$. But $r' \in D' \cap \bigcap_{i=1}^n C_i'$, which is a contradiction. Therefore C' has a unique element $v' = \phi(v)$ as claimed, and so $[G_v] = [G_{v'}]$.

Similarly, there is a function $\phi'\colon V' \to V$ so that $[G_v] = [G_{\phi'(v)}]$ for all $v \in V'$. Then $\phi'\phi = id_V$ and $\phi\phi' = id_{V'}$, so ϕ is a bijection and $\phi' = \phi^{-1}$.

If $\{u, v\} \in E$ then there is a clique C of Γ so that $\{u, v\} \subseteq C$. Then $\{\phi(u), \phi(v)\} \subseteq C'$, therefore $\{\phi(u), \phi(v)\} \in E'$. Conversely, if $\{\phi(u), \phi(v)\} \in E'$ then there exists a clique C' of Γ' so that $\{\phi(u), \phi(v)\} \subseteq C'$. Hence $\{u, v\} \subseteq C$ and so $\{u, v\} \in E$. Therefore ϕ is an isomorphism of graphs, as claimed. □

Theorem 5.3 *If Γ and Γ' are T_0 graphs, then $(\Gamma; G_v)$ and $(\Gamma'; G_{v'})$ are isomorphic graphs of groups.*

Proof Let $v \in V$. Let $\{C_1, \ldots, C_n\}$ be the set of cliques of Γ which contain v, and let $\{D_1, \ldots, D_m\}$ be the set of cliques of Γ which do not contain v. It follows from the T_0 hypothesis that $\{v\} = \bigcap_i C_i - \bigcup_i D_i$.

For each C_i and each D_i there are cliques C'_i and D'_i of Γ' so that $[G(C_i)] = [G(C'_i)]$ and $[G(D_i)] = [G(D'_i)]$. Observe that $C - \{v\} = \bigcup_i (C \cap D_i)$. Let $C' = \bigcap_i C'_i$ and $D' = \bigcup (C' \cap D'_i)$.

Now

$$[G(C)] = [G(\textstyle\bigcap_i C_i)] = \bigwedge_i [G(C_i)] = \bigwedge_i [G(C'_i)] = [G(\textstyle\bigcap_i C'_i)] = [G(C')]$$

and

$$\begin{aligned}[G(C - \{v\})] &= [G(\textstyle\bigcup_i (C \cap D_i))] = \bigvee_i [G(C \cap D_i)] = \bigvee_i ([G(C)] \wedge [G(D_i)]) \\ &= \bigvee_i ([G(C')] \wedge [G(D'_i)]) = \bigvee_i [G(C' \cap D'_i)] = [G(\textstyle\bigcup_i (C' \cap D'_i))] = [G(D')].\end{aligned}$$

Choose $h \in G$ so that $G(C) = hG(C')h^{-1}$. Then $hG(C - \{v\})h^{-1}$ is a subgroup of $G(C')$ that is conjugate to $G(D')$. Theorem 3.6 implies that $hG(C - \{v\})h^{-1} = G(D')$. Therefore,

$$G_v = G(C)/G(C - \{v\}) \cong G(C')/G(D') \cong G(C' - D').$$

In particular, $C' - D'$ is nonempty. But the T_0 hypothesis prevents $C' - D'$ from having more than one element, since two elements of $C' - D'$ would belong to the same cliques of Γ'. So $C' - D'$ has a unique element $v' = \phi(v)$.

In a similar manner, we can associate to each $v' \in V'$ a unique element $v = \phi'(v')$ of V. Now $\phi'\phi(v)$ belongs to the same cliques that v does, so $\phi'\phi = id_V$, and likewise $\phi\phi' = id_{V'}$.

Therefore ϕ is a bijection from V to V' such that $G_v \cong G_{\phi(v)}$ for all $v \in V$. It remains only to prove that ϕ is a graph isomorphism. Now, if $\{u, v\} \in E$ then there exists a clique C so that $\{u, v\} \subseteq C$. Then $\{\phi(u), \phi(v)\} \subseteq C'$, where C' is a clique of Γ' and $[G(C)] = [G(C')]$. Therefore $\{\phi(u), \phi(v)\} \in E'$. A similar argument shows that if $\{\phi(u), \phi(v)\} \in E'$ then $\{u, v\} \in E$. So ϕ is a graph isomorphism, and we are done. □

Theorem 5.4 *If G_v is directly indecomposable for all $v \in V \cup V'$ then $(\Gamma; G_v)$ and $(\Gamma'; G_{v'})$ are isomorphic graphs of groups.*

Proof Let $\Gamma = (\mathcal{V}, \mathcal{E})$ and $\Gamma' = (\mathcal{V}', \mathcal{E}')$ be the T_0 quotients of $\Gamma = (V, E)$ and $\Gamma' = (V', E')$ respectively. Then G is the internal graph product of both $(\Gamma, G(\mathcal{A}))$ and $(\Gamma', G(\mathcal{A}'))$.

If $A \in \mathcal{V}$ then $G(A) = \prod_{a \in A} G_a$, so $G(A)$ is a finite group. Likewise each $G(A')$ is a finite group. By the previous theorem, there exists a graph isomorphism $\psi : \mathcal{V} \to \mathcal{V}'$ so that $G(A) \cong G(\psi(A))$ for all $A \in \mathcal{V}$.

It is well-known that every finite group has a unique factorization as a direct product of directly indecomposable groups, up to isomorphism and order of factors [15]. Thus, for each $A \in \mathcal{V}$ there is a bijection $\phi_A : A \to \psi(A)$ so that $G_v \cong G_{\phi_A(v)}$ for all $v \in A$.

Let $\phi : V \to V'$ be the union of the ϕ_A's. Then ϕ is clearly a graph isomorphism between Γ and Γ', and $G_v \cong G_{\phi(v)}$ for all $v \in V$. Therefore ϕ is an isomorphism between the two graphs of groups. □

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