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A flat plane that is not the limit of periodic flat planes

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Abstract We construct a compact nonpositively curved squared 2-complex whose universal cover contains a flat plane that is not the limit of periodic flat planes.

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1 Introduction

Gromov raised the question of which "semi-hyperbolic spaces" have the property that their flats can be approximated by periodic flats [4, §6.B$_3$]. In this note we construct an example of a compact nonpositively curved squared 2-complex Z whose universal cover $\widetilde{Z}$ contains an isometrically embedded flat plane that is not the limit of a sequence of periodic flat planes.

A flat plane $\mathbb{E} \hookrightarrow \widetilde{Z}$ is *periodic* if the map $\mathbb{E} \hookrightarrow Z$ factors as $\mathbb{E} \to T \to Z$ where $\mathbb{E} \to T$ is a covering map of a torus T. Equivalently, $\pi_1 Z$ contains a subgroup isomorphic to $\mathbb{Z} \times \mathbb{Z}$ which stabilizes $\mathbb{E}$ and acts cocompactly on it. A flat plane $f\colon \mathbb{E} \hookrightarrow \widetilde{Z}$ is the *limit of periodic flat planes* if there is a sequence of periodic flat planes $f_i\colon \mathbb{E} \hookrightarrow \widetilde{Z}$ which converge pointwise to $f\colon \mathbb{E} \to \widetilde{Z}$. In our setting, $\widetilde{Z}$ is a 2-dimensional complex, and so $\mathbb{E} \hookrightarrow \widetilde{Z}$ is the limit of periodic flat planes if and only if every compact subcomplex of $\mathbb{E}$ is contained in a periodic flat plane.

In Section 2 we describe a compact nonpositively curved 2-complex X whose universal cover contains a certain aperiodic plane called an "anti-torus". In Section 3 we construct Z from X by strategically gluing tori and cylinders to X so that $\widetilde{Z}$ contains a flat plane which is a mixture of the anti-torus and periodic planes. This flat plane is not approximable by periodic flats because it contains a square that does not lie in any periodic flat. Our example Z is a $K(\pi,1)$ for a negatively curved square of groups, and in Section 4 we describe an interesting related triangle of groups.

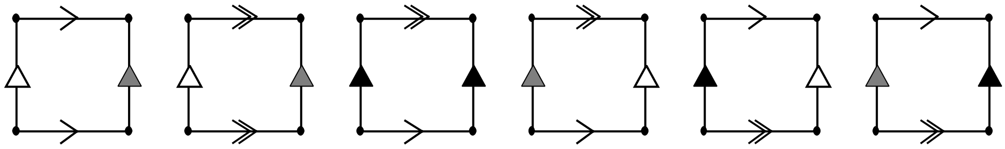

Figure 1: The figure above indicates the gluing pattern for the six squares of X. The three vertical edges colored white, grey, and black are denoted a, b, and c respectively. The two horizontal edges, single and double arrow, are denoted x and y respectively.

2 The anti-torus in X

2.1 The 2-complex X

Let X denote the complex consisting of the six squares indicated in Figure 1. The squares are glued together as indicated by the oriented labels on the edges. Note that X has a unique 0-cell, and that the notion of vertical and horizontal are preserved by the edge identifications. Let H denote the subcomplex consisting of the 2 horizontal edges, and let V denote the subcomplex consisting of the 3 vertical edges.

The complex X, which was first studied in [8], has a number of interesting properties that we record here: The link of the unique 0-cell in X is a complete bipartite graph. It follows that the universal cover $\tilde{X}$ is the product of two trees $\tilde{H} \times \tilde{V}$ where $\tilde{H}$ and $\tilde{V}$ are the universal covers of H and V. In particular, the link contains no cycle of length < 4 and so X satisfies the combinatorial nonpositive curvature condition for squared 2-complexes [3, 1] which is a special case of the $C(4)$-$T(4)$ small-cancellation condition [6].

The 2-complex X was used in [8] to produce the first examples of non-residually finite groups which are fundamental groups of spaces with the above properties. The connection to finite index subgroups arises because while $\tilde{X}$ is isomorphic to the cartesian product of two trees, X does not have a finite cover which is the product of two graphs.

2.2 The anti-torus

The exotic behavior of X can be attributed to the existence of a strangely aperiodic plane in $\tilde{X}$ that we shall now describe. Let $\tilde{x} \in \tilde{X}^0$ be the basepoint of $\tilde{X}$. Let c^{∞} denote the infinite periodic vertical line in $\tilde{X}$ which is the based component of the preimage of the loop labeled by c in X. Define y^{∞}

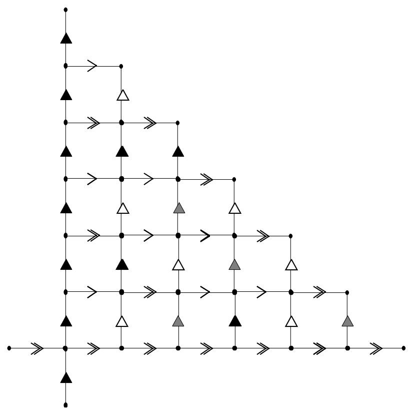

Figure 2: The Anti-Torus : The plane above is the convex hull of two periodically labeled lines in $\widetilde{X}$. A small region of the northeast quadrant has been tiled by the squares of X.

analogously. Let denote the convex hull in $\widetilde{X}$ of the infinite geodesics labeled by c^∞ and y^∞, so $= y^\infty \; c^\infty$. The plane is tiled by the six orbits of squares in $\widetilde{X}$ as in Figure 2. The reader can extend $c^\infty \cup y^\infty$ to a flat plane by successively adding squares wherever there is a pair of vertical and horizontal edges meeting at a vertex. From a combinatorial point of view, the existence and uniqueness of this extension is guaranteed by the fact that the link of X is a complete bipartite graph.

The "axes" c^∞ and y^∞ of are obviously periodic, and using that X is compact, it is easy to verify that for any $n \in \mathbb{N}$, the infinite strips $[-n;n] \times \mathbb{R}$ and $\mathbb{R} \times [-n;n]$ are periodic. However, the period of these infinite strips increases exponentially with n. Thus, the entire plane is aperiodic. Note that to say that $[-n;n] \times \mathbb{R}$ is *periodic* means that the immersion $[-n;n] \times \mathbb{R} \looparrowright X$ factors as $[-n;n] \times \mathbb{R} \to C \looparrowright X$ where $[-n;n] \times \mathbb{R} \to C$ is the universal covering map of a cylinder. The map $\looparrowright X$ is *aperiodic* in the sense that it does not factor through an immersed torus.

We conclude this section by giving a brief explanation of the aperiodicity of . A complete proof that is aperiodic is given in [8]. Let $W_n(m)$ denote the word corresponding to the length n horizontal positive path in beginning at the endpoint of the vertical path c^m. Thus, $W_n(m)$ is the label of the side opposite y^n in the rectangle which is the combinatorial convex hull of y^n and c^m. Equivalently, $W_n(m)$ occupies the interval $\{m\} \times [0;n]$. For each n, the words $\{W_n(m) \mid 0 \leq m \leq 2^n - 1\}$ are all distinct! Consequently every positive length n word in x and y is $W_n(m)$ for some m. This implies that the infinite

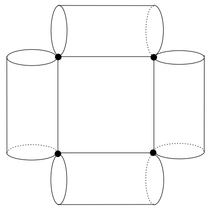

Figure 3: The complex Y is formed by gluing four cylinders to a square.

strip $[0,n] \times \mathbb{R}$ has period 2^n, and in particular cannot be periodic.

We refer to as an *anti-torus* because the aperiodicity of implies that c and y do not have nonzero powers which commute. Indeed, if c^p and y^q commuted for $p,q \neq 0$ then the flat torus theorem (see [1]) would imply that c^{1} and y^{1} meet in a periodic flat plane, which would contradict that is aperiodic.

3 The 2-complex Z with a nonapproximable flat

We first construct a new complex Y as follows: Start with a square s, and then attach four cylinders each of which is isomorphic to $S^1 \times I$. One such cylinder is attached along each side of s. The resulting complex Y containing exactly five squares is illustrated in Figure 3.

Let T^2 denote the torus $S^1 \times S^1$ with the usual product cell structure consisting of one 0-cell, two 1-cells, and a single square 2-cell. We let $\widetilde{T}^2$ denote the universal cover and we shall identify $\widetilde{T}^2$ with $\mathbb{R}^2$.

At each corner of $s \subset Y$, there is a pair of intersecting circles in Y^1, which are boundary circles of distinct cylinders. Note that they meet at an angle of $\frac{3\pi}{2}$ in Y. At each of three (NW, SW, & SE) corners of $s \subset Y$ we attach a copy of T^2 by identifying the pair of circles in the 1-skeleton of T^2 with the pair of intersecting circles noted above at the respective corner of s. At the fourth (NE) corner of s, we attach a copy of the complex X. Here we identify the pair of circles meeting at the corner of s with the pair of perpendicular circles c and y of X. We denote the resulting complex by Z. Thus, $Z = T^2 \cup T^2 \cup T^2 \cup Y \cup X$. See Figure 4 for a depiction of the 8 squares of $Z - X$ and their gluing patterns.

Definition 3.1 *Infinite cross* An *infinite cross* is a squared 2-complex isomorphic to the subcomplex of $\widetilde{T}^2$ consisting of $([0,1] \times \mathbb{R}) \cup (\mathbb{R} \times [0,1])$. The *base square* of the infinite cross is the square $[0,1] \times [0,1]$.

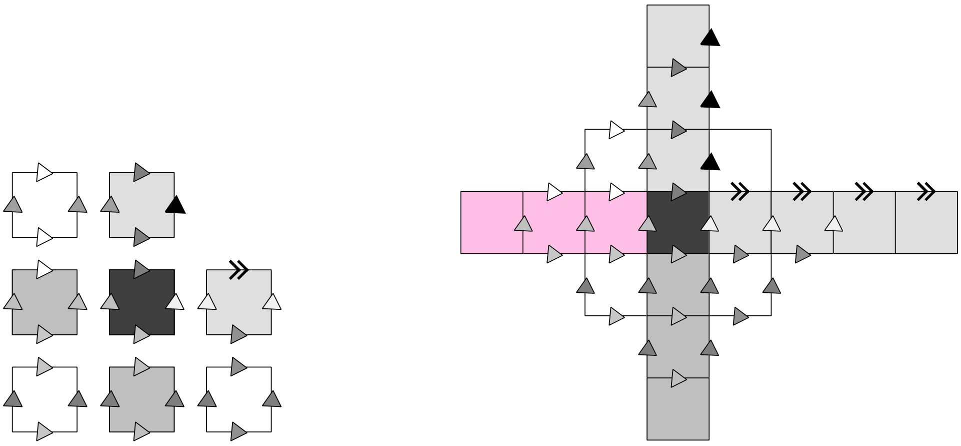

Figure 4: $Z - X$ and Z: The eight squares of the gure on the left are glued together following the gluing pattern to form $Z - X$. To form Z, we add a copy of X at the NE corner, identifying the loops in X labeled by c and y, with the black single and double arrows of the diagram. The gure on the right represents an in nite cross whose convex hull in Z is not approximable by any periodic plane. Note that while the NW, SW, and SE quarters of this plane are periodic, the NE quarter is an aperiodic quarter of .

The planes containing s: Observe that Y contains various immersions of an in nite cross whose base square maps to s. In particular, there are exactly 16 distinct immersed in nite crosses $C \looparrowright Y$ that pass through s exactly once. Each of these in nite crosses extends uniquely to an immersed flat plane in Z. Each such flat plane fails to be periodic because its four quarters map to distinct parts of Z. Our main result is that these immersed flat planes are not approximable by periodic flat planes because of the following:

Theorem 3.2 (No periodic approximation) *There is no immersion of a torus T^2 ! Z which contains s. Equivalently, there is no periodic plane in $\tilde{Z}$ containing $\tilde{s}$.*

Proof We argue by contradiction. Suppose that there is an immersed periodic plane Ω containing s. We shall now produce a rectangle as in Figure 5 that will yield a contradiction. We may assume that a copy of s in Ω is oriented as in Figure 4. We begin at this copy of s and travel north inside the northern cylinder until we reach another copy s_n of s. The existence of s_n is guaranteed by our assumption that Ω is periodic. Similarly, we travel east from s to reach a square s_e. Travelling north from s_e and east from s_n, we trace out the boundary of a rectangle whose NE corner is a square s_{ne} (see Figure 5).

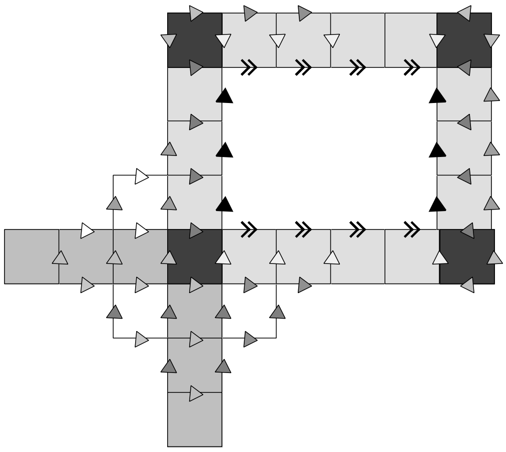

Figure 5: The figure above illustrates one of the four possible contradictions which explain why no periodic plane contains the square s.

This yields a contradiction because the inside of this rectangle is tiled by squares in X, yet the boundary of this rectangle is a commutator $[c^n, y^m]$. As explained in Section 2, such a word cannot be trivial in $\pi_1 X$ because of the anti-torus. □

Remark 3.3 Using an argument similar to the above proof, one can show that these sixteen planes are the *only* flat planes in $\tilde{Z}$ containing $\tilde{s}$. One considers the pair of "axes" intersecting at $\tilde{s}$ in a plane containing $\tilde{s}$. If this plane is different from each of the 16 mentioned above, then some translate of $\tilde{s}$ must appear along one of these "axes". The infinite strip in the plane whose corners are these two s squares yields a contradiction similar to the one obtained above.

Remark 3.4 While X is a rather pathological complex, we note that every flat plane in $\tilde{X}$ is the limit of periodic flat planes. Indeed this holds for any compact 2-complex X whose universal cover is isomorphic to the product of two trees [8].

4 Polygons of groups

4.1 The algebraic angle versus the geometric angle

Since the elements c and y have axes which intersect perpendicularly in a plane in $\tilde{X}$, the natural geometric angle between the subgroups $\langle c \rangle$ and $\langle y \rangle$

of $\pi_1 X$ is $\frac{\pi}{2}$. However, the algebraic Gersten-Stallings angle (see [7]) between these subgroups is $\frac{\pi}{3}$. To see this, we must show that there is no non-trivial relation of the form $c^k y^l c^m y^n = 1$.

Since $\tilde{X}$ is isomorphic to the cartesian product $\tilde{V} \times \tilde{H}$, of two trees and c and y correspond to distinct factors, it follows that the only relations that must be checked are rectangular (i.e., $|k| = |m|$ and $|l| = |n|$). However, these are easily ruled out by the anti-torus and the fact that X is nonpositively curved.

4.2 Square of groups and triangle of groups

The complex Z can be thought of in a natural way as a $K(\pi, 1)$ for a negatively curved *square of groups* (see [7, 5, 2]) with cyclic edge groups and trivial face group.

Because the algebraic angle between $\langle c \rangle$ and $\langle y \rangle$ in $\pi_1 X$ is $\frac{\pi}{3}$, it is tempting to form an analogous nonpositively curved triangle of groups D. The face group of D is trivial, the edge groups of D are cyclic, the vertex groups of D are isomorphic to $\pi_1 X$, and each edge group of D is embedded on one (clockwise) side as $\langle c \rangle$ and on the other (counter-clockwise) side as $\langle y \rangle$. This can be done so that the resulting triangle of groups D has $\mathbb{Z}_3$ symmetry. The tension between the algebraic and geometric angles should endow $\pi_1 D$ with some interesting properties. For instance, I suspect that $\pi_1 D$ fails to be the fundamental group of a compact nonpositively curved space, but it fails for reasons different from the usual types of problems.

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