



## An indecomposable $PD_3$ -complex : II

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**Abstract** We show that there are two homotopy types of  $PD_3$ -complexes with fundamental group  $S_3 *_{\mathbb{Z}/2\mathbb{Z}} S_3$ , and give explicit constructions for each, which differ only in the attachment of the top cell.

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In [3] we showed that  $\pi = S_3 *_{\mathbb{Z}/2\mathbb{Z}} S_3$  satisfies the criterion of [5] and thus is the fundamental group of a  $PD_3$ -complex. As  $\pi$  has infinitely many ends but is indecomposable, this illustrates a divergence from the known properties of 3-manifolds, and provides a counter-example to an old question of Wall [6]. In particular, the Sphere Theorem does not extend to all  $PD_3$ -complexes.

Here we shall give an explicit description of a finite  $PD_3$ -complex  $Y$  realizing this group. The construction is modelled on a similar construction for a  $PD_3$ -complex  $X$  with fundamental group  $S_3$ . In each case the cellular chain complex of the universal cover has the striking property that it is self-dual. In §2 we show a  $PD_3$ -complex with fundamental group  $\pi$  must be orientable, and we use Turaev's work to show there are two homotopy types of such  $PD_3$ -complexes. The 2-fold cover of  $Y$  is homotopy equivalent to  $L(3, 1) \# L(3, 1)$ , while a simple modification of our construction (suggested by the referee) gives a  $PD_3$ -complex with 2-fold cover homotopy equivalent to  $L(3, 1) \# -L(3, 1)$ . (This group was first suggested as a test case in [2].)

### 1 A finite complex with group $S_3 *_{\mathbb{Z}/2\mathbb{Z}} S_3$

Let  $G$  be a group and let  $\Gamma = \mathbb{Z}[G]$ ,  $\varepsilon : C_1 = \Gamma \rightarrow \mathbb{Z}$  and  $I(G) = \text{Ker}(\varepsilon)$  be the integral group ring, the augmentation homomorphism and the augmentation ideal, respectively. If  $M$  is a left  $\Gamma$ -module  $\overline{M}$  shall denote the conjugate right module, with  $G$ -action given by  $m.g = g^{-1}m$  for all  $g \in G$  and  $m \in M$ , and similarly  $\overline{N}$  shall denote the conjugate left module structure on a right  $\Gamma$ -module  $N$ . If  $C_*$  is a chain complex over  $\Gamma$  with an augmentation  $\varepsilon : C_0 \rightarrow \mathbb{Z}$

a *diagonal approximation* is a chain homomorphism  $\Delta : C_* \rightarrow C_* \otimes_{\mathbb{Z}} C_*$  (with diagonal  $G$ -action) such that  $(\varepsilon \otimes 1)\Delta = id_{C_*} = (1 \otimes \varepsilon)\Delta$ .

The cellular chain complex  $C_*(\tilde{K})$  for the universal covering space of a finite 2-complex  $K$  determined by a presentation for a group is isomorphic to the Fox-Lyndon complex of the presentation, via an isomorphism carrying generators corresponding to based lifts of cells of  $K$  to the standard generators.

The symmetric group  $S_3$  has a presentation  $\langle a, b \mid a^2, abab^{-2} \rangle$ . Let  $\pi = S_3 *_{\mathbb{Z}/2\mathbb{Z}} S_3$ , with presentation  $\langle a, b, c \mid r, s, t \rangle$ , where  $r = a^2$ ,  $s = abab^{-2}$  and  $t = acac^{-2}$ . The two obvious embeddings of  $S_3$  into  $\pi$  admit retractions, as  $\pi/\langle\langle b \rangle\rangle \cong \pi/\langle\langle c \rangle\rangle \cong S_3$ . Let  $A, B$  and  $C$  be the cyclic subgroups generated by the images of  $a, b$  and  $c$ , respectively. The inclusions of  $A$  into  $S_3$  and  $\pi$  induce isomorphisms on abelianization, while the commutator subgroups are  $S'_3 = B$  and  $\pi' = B * C$ . Thus these groups are semidirect products:  $S_3 \cong B \rtimes (\mathbb{Z}/2\mathbb{Z})$  and  $\pi \cong (B * C) \rtimes \mathbb{Z}/2\mathbb{Z}$ . In particular,  $\pi$  is virtually free, and so has infinitely many ends. However it follows easily from the Grushko-Neumann Theorem that  $\pi$  is indecomposable. (See [3]).

The above presentations determine finite 2-complexes  $K$  and  $L$ , with fundamental groups  $S_3$  and  $\pi$ , respectively. There are two obvious embeddings of  $K$  as a retract in  $L$ , with retractions  $r_b, r_c : L \rightarrow K$  given by collapsing the pair of cells  $\{c, t\}$  and  $\{b, s\}$ , respectively.

The chain complex  $C_*(\tilde{K})$  has the form

$$\mathbb{Z}[S_3]^2 \xrightarrow{\partial_2} \mathbb{Z}[S_3]^2 \xrightarrow{\partial_1} \mathbb{Z}[S_3],$$

where  $\partial_1(1, 0) = a - 1$ ,  $\partial_1(0, 1) = b - 1$ ,  $\partial_2(1, 0) = (a + 1, 0)$  and  $\partial_2(0, 1) = (b^2a + 1, a - b - 1)$ . The 2-chain  $\psi = (a - 1, -ba + a + b^2 - b)$  is a 2-cycle, and so determines an element of  $\pi_2(K) = H_2(\tilde{K}; \mathbb{Z})$ , by the Hurewicz Theorem. Let  $X = K \cup_{\psi} e^3$ , and let  $C_*$  be the cellular chain complex for the universal cover  $\tilde{X}$ . (Thus  $C_i = C_i(\tilde{K})$  for  $i \leq 2$  and  $C_3 \cong \mathbb{Z}[S_3]$ ). The dual cochain complex  $C^* = Hom_{\Gamma}(C_*, \mathbb{Z}[S_3])$  is a complex of right  $\mathbb{Z}[S_3]$ -modules.

We shall define new bases which display the structure of  $C_*$  to better advantage, as follows. Let  $e_1 = (1, 0)$  and  $e_2 = (-ba - b^2, 1)$  in  $C_1$  and  $f_1 = (1, 0)$  and  $f_2 = (0, -a)$  in  $C_2$ , and let  $g$  be the generator of  $C_3$  corresponding to the top cell. Then  $\partial_1 e_1 = a - 1$ ,  $\partial_1 e_2 = -b^2a + ba + b^2 - 1$ ,  $\partial_2 f_1 = (a + 1)e_1$ ,  $\partial_2 f_2 = (b^2a + a - 1)e_2$ , and  $\partial_3 g = \psi = (a - 1)f_1 + (-b^2a + ba + b - 1)f_2$ . The matrix for  $\partial_2$  with respect to the bases  $\{\tilde{e}_i\}$  and  $\{\tilde{f}_j\}$  is diagonal, and is hermitian with respect to the canonical involution of  $\mathbb{Z}[S_3]$ , while the matrix for  $\partial_3$  is the conjugate transpose for that of  $\partial_1$ . Hence the chain complex  $\overline{C^{3-*}}$

obtained by conjugating and reindexing the cochain complex  $C^*$  is isomorphic to  $C_*$ .

Let  $\beta = b^2 + b + 1$  and  $\nu = \sum_{s \in S_3} s = \beta(a + 1)$ .

**Lemma 1** *The complex  $X$  is a  $PD_3$ -complex with  $\tilde{X} \simeq S^3$ .*

**Proof** Since  $C_*$  is the cellular chain complex of a 1-connected cell complex  $H_0(C_*) \cong \mathbb{Z}$  and  $H_1(C_*) = 0$ . If  $\partial_2(rf_1 + sf_2) = 0$  then  $r(a + 1) = 0$  and  $s(b^2a + a - 1) = 0$ . Now the left annihilator ideals of  $a + 1$  and  $b^2a + a - 1$  in  $\mathbb{Z}[S_3]$  are principal left ideals, generated by  $a - 1$  and  $(b - 1)(ba - 1)$ , respectively. Hence  $r = p(a - 1)$  and  $s = q(b - 1)(ba - 1)$  for some  $p, q \in \mathbb{Z}[B]$ . A simple calculation gives  $\partial_3((p(ba + b + 1) + q(ba + b))g) = rf_1 + sf_2$  and so  $H_2(C_*) = 0$ .

If  $\partial_3hg = 0$  then  $h(a - 1) = 0$ , so  $h = h_1(a + 1)$  for some  $h_1 \in \mathbb{Z}[B]$ , and  $h(b^2a - ba - b + 1) = 0$ . Now  $h(b^2a - ba - b + 1) = h_1(1 - b)(a + b + 1)$ , so  $h_1(1 - b) = 0$ . Therefore  $h_1 = m\beta$  for some  $m \in \mathbb{Z}$ , so  $h = m\nu$  and  $H_3(C_*) = \mathbb{Z}[S_3]\nu g \cong \mathbb{Z}$ . Hence  $\tilde{X} \simeq S^3$ . Now  $H_3(X; \mathbb{Z}) = H_3(\mathbb{Z} \otimes_{\mathbb{Z}[S_3]} C_*) = \mathbb{Z}[1 \otimes g]$  and  $tr([1 \otimes g]) = \nu g$ , where  $tr : H_3(X; \mathbb{Z}) \rightarrow H_3(\tilde{X}; \mathbb{Z})$  is the transfer homomorphism. The homomorphisms from  $H^q(\overline{C^*})$  to  $H_{3-q}(C_*)$  determined by cap product with  $[X] = [1 \otimes g]$  may be identified with the Poincaré duality isomorphisms for  $\tilde{X}$ , and so  $X$  is a  $PD_3$ -complex.  $\square$

The verification that  $\tilde{X} \simeq S^3$  is essentially due to [4] and the fact that  $X$  is a  $PD_3$ -complex is due to [6]. The only novelty here is the diagonalization of  $\partial_2$ , which was a guiding feature in the study of  $\pi = S_3 *_Z S_3$ .

Let  $\Pi = \mathbb{Z}[\pi]$ . The cellular chain complex for the universal covering space  $\tilde{L}$  has the form

$$\Pi^3 \xrightarrow{\partial_2} \Pi^3 \xrightarrow{\partial_1} \Pi.$$

The differentials are given by  $\partial_1(1, 0, 0) = a - 1$ ,  $\partial_1(0, 1, 0) = b - 1$  and  $\partial_1(0, 0, 1) = c - 1$ ,  $\partial_2(1, 0, 0) = (a + 1, 0, 0)$ ,  $\partial_2(0, 1, 0) = (b^2a + 1, a - b - 1, 0)$  and  $\partial_2(0, 0, 1) = (c^2a + 1, 0, a - c - 1)$ . In particular,  $H_2(\tilde{L}; \mathbb{Z}) = \text{Ker}(\partial_2)$ .

Let  $\theta = (a - 1, -ba + a + b^2 - b, -ca + a + c^2 - c)$ . Then  $\partial_2(\theta) = 0$ , and so  $\theta$  determines an element of  $\pi_2(L) = H_2(\tilde{L}; \mathbb{Z})$ , by the Hurewicz Theorem. Let  $Y = L \cup_{\theta} e^3$  and let  $D_*$  be the cellular chain complex for the universal covering space  $\tilde{Y}$ .

Let $\tilde{e}_1 = (1, 0, 0)$ $\tilde{e}_2 = (-ba - b^2, 1, 0)$ $\tilde{e}_3 = (-ca - c^2, 0, 1)$ $\tilde{f}_1 = (1, 0, 0)$ $\tilde{f}_2 = (0, -a, 0)$ $\tilde{f}_3 = (0, 0, -a).$	Then $\partial_1 \tilde{e}_1 = a - 1$ $\partial_1 \tilde{e}_2 = ba - b^2a + b^2 - 1$ $\partial_1 \tilde{e}_3 = ca - c^2a + c^2 - 1$ $\partial_2 \tilde{f}_1 = (a + 1)\tilde{e}_1$ $\partial_2 \tilde{f}_2 = (b^2a + a - 1)\tilde{e}_2$ $\partial_2 \tilde{f}_3 = (c^2a + a - 1)\tilde{e}_3.$
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Moreover  $\theta = (a - 1)\tilde{f}_1 + (-b^2a + ba + b - 1)\tilde{f}_2 + (-c^2a + ca + c - 1)\tilde{f}_3$ . Let  $D^* = Hom_\Gamma(D_*, \Pi)$  be the cochain complex dual to  $D_*$ . Then it is easily seen that  $\overline{D^*} \cong D_{3-*}$ .

**Theorem 2** *The complex  $Y$  is a  $PD_3$ -complex.*

**Proof** Clearly  $H_0(D_*) \cong \mathbb{Z}$  and  $H_1(D_*) = 0$ . The argument of the first part of Lemma 1 extends immediately to show that the kernel of  $\partial_2$  is generated by  $(a - 1)\tilde{f}_1$ ,  $(b - 1)(ba - 1)\tilde{f}_2$  and  $(c - 1)(ca - 1)\tilde{f}_3$ . Hence these elements represent generators for  $H_2(D_*)$ . Let  $\tilde{g}$  be the generator for  $D_3$  corresponding to the top cell, so that  $\partial_3 \tilde{g} = \theta$ . Note that the image of  $g$  in  $\mathbb{Z} \otimes_\varepsilon D_3$  is a cycle, and represents a generator for  $H_3(Y; \mathbb{Z}) = H_3(\mathbb{Z} \otimes_\varepsilon D_*)$ . If  $h\theta = 0$  then (as in Lemma 1)  $h = h_1(a + 1)$  for some  $h_1 \in \mathbb{Z}[B * C]$  such that  $h_1(b - 1) = h_1(c - 1) = 0$ . It follows that  $h_1 = 0$ . Hence  $\partial_3$  is injective and so  $H_3(D_*) = 0$ .

Let  $\hat{1}$ ,  $\hat{e}^*$ ,  $\hat{f}^*$  and  $\hat{g}$  denote the bases of  $D^*$  dual to the above bases for  $D_*$ . Let  $\Delta$  be a diagonal approximation for  $D_*$  and suppose that  $\Delta(\tilde{g}) = \sum_{0 \leq q \leq 3} \sum_{i \in I(q)} x_i \otimes y_i$ , where  $x_i \in D_q$  and  $y_i \in D_{3-q}$ , for all  $i \in I(q)$  and  $0 \leq q \leq 3$ . Then  $\sum_{i \in I(3)} x_i = \tilde{g}$ . Let  $r_i = \hat{g}(x_i)$  for  $i \in I(3)$  and let  $\tilde{\xi}$  denote the image of  $\tilde{g}$  in  $H_3(Y; \mathbb{Z}) = \mathbb{Z} \otimes_\varepsilon D_3$ . Then  $\varepsilon(\hat{g} \cap \tilde{\xi}) = \varepsilon(\sum_{i \in I(3)} \overline{r_i} y_i) = \varepsilon(\sum_{i \in I(3)} \overline{r_i}) = \varepsilon(\overline{\hat{g}(\tilde{g})}) = 1$ , and so  $\hat{g} \cap \tilde{\xi}$  generates  $H_0(D_*)$ . Since  $H_1(D_*) = H_3(D_*) = H^0(\overline{D^*}) = H^2(\overline{D^*}) = 0$ ,  $-\cap \tilde{\xi}$  induces isomorphisms  $H^q(\overline{D^*}) \cong H_{3-q}(D_*)$  for all  $q \neq 1$ . The remaining case follows as in [5] from the facts that  $\overline{D^*} \cong D_{3-*}$  and  $\Delta$  is chain homotopic to  $\tau\Delta$ , where  $\tau : D_* \otimes D_* \rightarrow D_* \otimes D_*$  is the transposition defined by  $\tau(\alpha \otimes \omega) = (-1)^{pq} \omega \otimes \alpha$  for all  $\alpha \in D_p$  and  $\omega \in D_q$ . Thus  $Y$  is a  $PD_3$ -complex. □

Can the last step of this argument be made more explicit? The work of Handel [1] on diagonal approximations for dihedral groups may be adapted to give the following formulae for a diagonal approximation for the truncation to degrees  $\leq 2$  of  $D_*$  which is compatible with the above two embeddings of  $K$  as a retract in  $L$ :

$$\begin{aligned}
\Delta(1) &= 1 \otimes 1 \\
\Delta(\tilde{e}_1) &= \tilde{e}_1 \otimes a + 1 \otimes \tilde{e}_1, \\
\Delta(\tilde{e}_2) &= \tilde{e}_2 \otimes 1 - ba\tilde{e}_1 \otimes (b-1) - b^2\tilde{e}_1 \otimes (b^2a-1) - (ba-b) \otimes ba\tilde{e}_1 \\
&\quad - (b^2-b) \otimes b^2\tilde{e}_1 + b \otimes \tilde{e}_2, \\
\Delta(\tilde{e}_3) &= \tilde{e}_3 \otimes 1 - ca\tilde{e}_1 \otimes (c-1) - c^2\tilde{e}_1 \otimes (c^2a-1) - (ca-c) \otimes ca\tilde{e}_1 \\
&\quad - (c^2-c) \otimes c^2\tilde{e}_1 + c \otimes \tilde{e}_3, \\
\Delta(\tilde{f}_1) &= \tilde{f}_1 \otimes 1 + \tilde{e}_1 \otimes a\tilde{e}_1 + 1 \otimes \tilde{f}_1, \\
\Delta(\tilde{f}_2) &= \tilde{f}_2 \otimes a + (b^2+b)\tilde{f}_1 \otimes (a-ba) + (b^2a+b^2)\tilde{f}_2 \otimes (a-ba) \\
&\quad + ((ba+b^2-1)\tilde{e}_1 + \tilde{e}_2) \otimes ((b^2a)\tilde{e}_1 + ba\tilde{e}_2) \\
&\quad - ((b^2a+1)\tilde{e}_1 + ba\tilde{e}_2) \otimes ((ba+a+b^2+b)\tilde{e}_1 + (b^2a+a)\tilde{e}_2) \\
&\quad - ((a+b)\tilde{e}_1 + b^2a\tilde{e}_2) \otimes ((ba+b^2)\tilde{e}_1 + a\tilde{e}_2) - (a+1)\tilde{e}_1 \otimes \tilde{e}_1 \\
&\quad + (a-b) \otimes (b^2+b)\tilde{f}_1 + (a-b) \otimes (b^2a+b^2)\tilde{f}_2 + a \otimes \tilde{f}_2 \quad \text{and} \\
\Delta(\tilde{f}_3) &= \tilde{f}_3 \otimes a + (c^2+c)\tilde{f}_1 \otimes (a-ca) + (c^2a+c^2)\tilde{f}_3 \otimes (a-ca) \\
&\quad + ((ca+c^2-1)\tilde{e}_1 + \tilde{e}_3) \otimes ((c^2a)\tilde{e}_1 + ca\tilde{e}_3) \\
&\quad - ((c^2a+1)\tilde{e}_1 + ca\tilde{e}_3) \otimes ((ca+a+c^2+c)\tilde{e}_1 + (c^2a+a)\tilde{e}_3) \\
&\quad - ((a+c)\tilde{e}_1 + c^2a\tilde{e}_3) \otimes ((ca+c^2)\tilde{e}_1 + a\tilde{e}_3) - (a+1)\tilde{e}_1 \otimes \tilde{e}_1 \\
&\quad + (a-c) \otimes (c^2+c)\tilde{f}_1 + (a-c) \otimes (c^2a+c^2)\tilde{f}_3 + a \otimes \tilde{f}_3
\end{aligned}$$

These formulae were derived from the work of Handel by using the canonical involution of  $\mathbb{Z}[S_3]$  to switch right and left module structures and showing that  $C_*$  is a direct summand of a truncation of the Wall-Hamada resolution for  $S_3$ . (In Handel's notation  $a = y$ ,  $b = x$ ,  $e_1 = c_1^2$ ,  $e_2 = -c_1^1 - c_1^2(x + xy)$ ,  $f_1 = c_2^3$ ,  $f_2 = -c_2^1y + c_2^2x^2 - c_2^3y$  and  $g = -(c_3^1 + c_3^3)(x + y) - c_3^4y$ ). Handel's work also leads to a formula for  $\Delta(g)$ , but it is not clear what  $\Delta(\bar{g})$  should be.

## 2 Other $PD_3$ -complexes with this group

Having constructed one  $PD_3$ -complex with group  $\pi$  one may ask how many there are. Any such  $PD_3$ -complex must be orientable. For let  $w_1 : \pi \rightarrow \{\pm 1\}$  be a homomorphism and define an involution on  $\Gamma$  by  $\bar{g} = w_1(g)g^{-1}$ , for all  $g \in \pi$ . Let  $w = w_1(a)$  and  $R = \mathbb{Z}[\pi/\pi'] = \mathbb{Z}[a]/(a^2 - 1)$ . Let  $J = \text{Coker}(\bar{\partial}_2^{tr})$ ,

where  $\partial_2 : \Pi^3 \rightarrow \Pi^3$  is the presentation matrix for  $I(\pi)$  given in §1. Then  $R \otimes_{\Gamma} I(\pi) \cong R/(a+1) \oplus (R/(a+1, 3))^2$ , while  $R \otimes_{\Gamma} J \cong R/(a+w) \oplus (R/(a+w, 3))^2$ . If the pair  $(\pi, w_1)$  is realized by a  $PD_3$ -complex then  $I(\pi)$  and  $J$  are projective homotopy equivalent [5]. But then  $R \otimes_{\Gamma} I(\pi)$  and  $R \otimes_{\Gamma} J$  are projective homotopy equivalent  $R$ -modules, and so we must have  $w = 1$ .

If  $W$  is an oriented  $PD_3$ -complex with fundamental group  $G$  and  $c_W : W \rightarrow K(G, 1)$  is a classifying map let  $\mu(W) = c_{W*}[W] \in H_3(W; \mathbb{Z})$ . Two such  $PD_3$ -complexes  $W_1$  and  $W_2$  are homotopy equivalent if and only if  $\mu(W_1)$  and  $\mu(W_2)$  agree up to sign and the action of  $Out(G)$ . Turaev constructed an isomorphism  $\nu_C$  from  $H_3(G; \mathbb{Z})$  to a group  $[F^2(C), I(G)]$  of projective homotopy classes of module homomorphisms and showed that  $\mu \in H_3(G; \mathbb{Z})$  is the image of the orientation class of a  $PD_3$ -complex if and only if  $\nu_C(\mu)$  is the class of a self-homotopy equivalence [5].

When  $G = \pi = S_3 *_{Z/2Z} S_3$  we have  $F^2(C) \cong I(\pi)$ , and  $H_3(\pi; \mathbb{Z}) \cong H_3(\pi'; \mathbb{Z}) \oplus H_3(Z/2Z; \mathbb{Z}) \cong (Z/3Z)^2 \oplus (Z/2Z)$ . Let  $W'$  be the double cover of  $W$ , with fundamental group  $\pi' \cong (Z/3Z) * (Z/3Z)$ . Then  $W'$  is a connected sum, by Theorem 1 of [5], and so it is homotopy equivalent to one of the 3-manifolds  $L(3, 1) \# L(3, 1)$  and  $L(3, 1) \# -L(3, 1)$ . (These may be distinguished by the torsion linking forms on their first homology groups). In particular,  $\mu(W')$  has nonzero entries in each summand. Since  $\mu(W')$  is the image of  $\mu(W)$  under the transfer to  $H_3(\pi'; \mathbb{Z}) \cong (Z/3Z)^2$  the image of  $\mu(W)$  in each  $Z/3Z$ -summand must be nonzero. Let  $u \in H^1(W; \mathbb{F}_2)$  correspond to the abelianization homomorphism. Since  $\beta_2(W; \mathbb{F}_2) = \beta_1(W; \mathbb{F}_2) = 1 = \beta_2(\pi; \mathbb{F}_2)$  we have  $u^2 \neq 0$ , and so  $u^3 \neq 0$ , by Poincaré duality. It follows easily that the image of  $\mu(W)$  in the  $Z/2Z$ -summand must be nonzero also. (Note that  $W'$  is  $\mathbb{Z}_{(2)}$ -homology equivalent to  $S^3$  and so  $W$  is  $\mathbb{Z}_{(2)}[Z/2Z]$ -homology equivalent to  $RP^3$ ). Since reversing the orientation of  $W$  reverses that of  $W'$ , we may conclude that there are at most two distinct homotopy types of  $PD_3$ -complexes with fundamental group  $\pi$ , and that they may be detected by their double covers.

The retractions  $r_b$  and  $r_c$  of  $L$  onto  $K$  extend to maps  $r_b, r_c : Y \rightarrow X$ . These maps induce the same isomorphism  $H_3(Y; \mathbb{Z}) \cong H_3(X; \mathbb{Z})$ , and so their lifts to the double covers induce the same isomorphism  $H_3(Y'; \mathbb{Z}) \rightarrow H_3(X'; \mathbb{Z})$ . Hence  $Y' \simeq L(3, 1) \# L(3, 1)$ , rather than  $L(3, 1) \# -L(3, 1)$ . The referee has pointed out that if we use  $\xi = (a-1)\tilde{f}_1 + (-b^2a + ba + b - 1)\tilde{f}_2 - (-c^2a + ca + c - 1)\tilde{f}_3$  instead of  $\theta$  (changing only the sign of the final term) then  $Z = L \cup_{\xi} e^3$  is another  $PD_3$ -complex with  $\pi_1(Z) \cong \pi$ , and a similar argument shows that the double cover is now  $Z' \simeq L(3, 1) \# -L(3, 1)$ .

The question of whether every aspherical  $PD_3$ -complex is homotopy equivalent to a 3-manifold remains open. The recent article [7] gives a comprehensive

survey of Poincaré duality in dimension 3, emphasizing the role of the JSJ decomposition in relation to this question.

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## References

- [1] Handel, D. On products in the cohomology of the dihedral groups, Tôhoku Math. J. 45 (1993), 13-42. MathReview
- [2] Hillman, J.A. On 3-dimensional Poincaré duality complexes and 2-knot groups, Math. Proc. Cambridge Phil. Soc. 114 (1993), 215-218. MathReview
- [3] Hillman, J.A. An indecomposable  $PD_3$ -complex whose group has infinitely many ends, Math. Proc. Cambridge Phil. Soc., to appear (2005).
- [4] Swan, R.G. Periodic resolutions for finite groups. Ann. of Math. 72 (1960), 267-291. MathReview
- [5] Turaev, V.G. Three-dimensional Poincaré complexes: homotopy classification and splitting. (Russian) Mat. Sb. 180 (1989) 809–830. (Math. USSR-Sb. 67 (1990), 261-282.) MathReview
- [6] Wall, C.T.C. Poincaré complexes: I. Ann. of Math. 86 (1967), 213-245. MathReview
- [7] Wall, C.T.C. Poincaré duality in dimension 3, in *Proceedings of the Casson Fest (Arkansas and Texas 2003)* (edited by C.McA.Gordon and Y.Rieck), Geom. Topol. Monogr. 7 (2004), 1-26.

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