ISSN 1472-2739 (on-line) 1472-2747 (printed) **369**

*Algebraic & Geometric Topology*
Volume 5 (2005) 369–378
Published: 30 April 2005

# All integral slopes can be Seifert fibered slopes for hyperbolic knots

Kimihiko Motegi
Hyun-Jong Song

**Abstract** Which slopes can or cannot appear as Seifert fibered slopes for hyperbolic knots in the 3-sphere $S^3$? It is conjectured that if $r$-surgery on a hyperbolic knot in $S^3$ yields a Seifert fiber space, then $r$ is an integer. We show that for each integer $n \in \mathbb{Z}$, there exists a tunnel number one, hyperbolic knot $K_n$ in $S^3$ such that $n$-surgery on $K_n$ produces a small Seifert fiber space.

**AMS Classification** 57M25, 57M50

**Keywords** Dehn surgery, hyperbolic knot, Seifert fiber space, surgery slopes

*This paper is dedicated to Donald M. Davis on the occasion of his 60th birthday.*

## 1 Introduction

Let $K$ be a knot in the 3-sphere $S^3$ with a tubular neighborhood $N(K)$. Then the set of *slopes* for $K$ (i.e., $\partial N(K)$-isotopy classes of simple loops on $\partial N(K)$) is identified with $\mathbb{Q} \cup \{\infty\}$ using preferred meridian-longitude pair so that a meridian corresponds to $\infty$. A slope $\gamma$ is said to be *integral* if a representative of $\gamma$ intersects a meridian exactly once, in other words, $\gamma$ corresponds to an integer under the above identification. In the following, we denote by $(K;\gamma)$ the 3-manifold obtained from $S^3$ by Dehn surgery on a knot $K$ with slope $\gamma$, i.e., by attaching a solid torus to $S^3 - \mathrm{int}N(K)$ in such a way that $\gamma$ bounds a meridian disk of the filled solid torus. If $\gamma$ corresponds to $r \in \mathbb{Q} \cup \{\infty\}$, then we identify $\gamma$ and $r$ and write $(K;r)$ for $(K;\gamma)$.

We denote by $\mathcal{L}$ the *set of lens slopes* $\{r \in \mathbb{Q} \mid \exists$ hyperbolic knot $K \subset S^3$ such that $(K;r)$ is a lens space$\}$, where $S^3$ and $S^2 \times S^1$ are also considered as lens spaces. Then the cyclic surgery theorem [7] implies that $\mathcal{L} \subset \mathbb{Z}$. A result of Gabai [10, Corollary 8.3] shows that $0 \notin \mathcal{L}$, a result of Gordon and Luecke [14] shows that $\pm 1 \notin \mathcal{L}$. In [19] Kronheimer and Mrowka prove that $\pm 2 \notin \mathcal{L}$. Furthermore, a result of Kronheimer, Mrowka, Ozsváth and Szabó [20] implies that $\pm 3, \pm 4 \notin \mathcal{L}$. Besides, Berge [4, Table of Lens Spaces] suggests that if $n \in \mathcal{L}$, then $|n| \geq 18$ and not every integer $n$ with $|n| \geq 18$ appears in $\mathcal{L}$. Fintushel and Stern [9] had shown that 18-surgery on the $(-2, 3, 7)$ pretzel knot yields a lens space.

*Which slope* (*rational number*) *can or cannot appear in the set of Seifert fibered slopes* $\mathcal{S} = \{r \in \mathbb{Q} \mid \exists$ *hyperbolic knot* $K \subset S^3$ *such that* $(K; r)$ *is Seifert fibered*$\}$*?* It is conjectured that $\mathcal{S} \subset \mathbb{Z}$ [12].

The purpose of this paper is to prove:

**Theorem 1.1** *For each integer* $n \in \mathbb{Z}$*, there exists a tunnel number one, hyperbolic knot* $K_n$ *in* $S^3$ *such that* $(K_n; n)$ *is a small Seifert fiber space* (*i.e., a Seifert fiber space over* $S^2$ *with exactly three exceptional fibers*)*.*

**Remark** Since $K_n$ has tunnel number one, it is embedded in a genus two Heegaard surface of $S^3$ and strongly invertible [26, Lemma 5]. See [22, Question 3.1].

Theorem 1.1, together with the previous known results, shows:

**Corollary 1.2** $\mathcal{L} \subsetneq \mathbb{Z} \subset \mathcal{S}$.

**Remarks**

(1) For the *set of reducing slopes* $\mathcal{R} = \{r \in \mathbb{Q} \mid \exists$ hyperbolic knot $K \subset S^3$ such that $(K; r)$ is reducible$\}$, Gordon and Luecke [13] have shown that $\mathcal{R} \subset \mathbb{Z}$. In fact, the cabling conjecture [11] asserts that $\mathcal{R} = \emptyset$.

(2) For the *set of toroidal slopes* $\mathcal{T} = \{r \in \mathbb{Q} \mid \exists$ hyperbolic knot $K \subset S^3$ such that $(K; r)$ is toroidal$\}$, Gordon and Luecke [15] have shown that $\mathcal{T} \subset \mathbb{Z}/2$ (integers or half integers). In [28], Teragaito shows that $\mathbb{Z} \subset \mathcal{T}$ and conjectures that $\mathcal{T} \subsetneq \mathbb{Z}/2$.

**Acknowledgements** We would like to thank the referee for careful reading and useful comments.
The first author was partially supported by Grant-in-Aid for Scientific Research (No. 15540095), The Ministry of Education, Culture, Sports, Science and Technology, Japan.

## 2 Hyperbolic knots with Seifert fibered surgeries

Our construction is based on an example of a longitudinal Seifert fibered surgery given in [17].

Let $k \cup c$ be a 2-bridge link given in Figure 1, and let $K_n$ be a knot obtained from $k$ by $\frac{1}{-n+4}$-surgery along $c$.

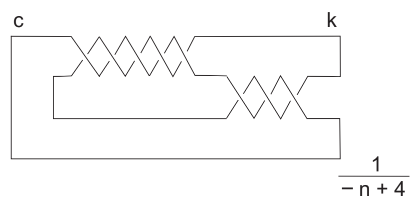


Figure 1: $K_n$

We shall say that a Seifert fiber space is of *type* $S^2(n_1, n_2, n_3)$ if it has a Seifert fibration over $S^2$ with three exceptional fibers of indices $n_1, n_2$ and $n_3$ $(n_i \geq 2)$. Since $K_4$ is unknotted, $(K_4; 4)$ is a lens space $L(4,1)$. For the other $n$'s, we have:

**Lemma 2.1** *$(K_n; n)$ is a small Seifert fiber space of type $S^2(3, 5, |4n-15|)$ for any integer $n \neq 4$.*

**Proof** Since the linking number of $k$ and $c$ is one (with suitable orientations), $(K_n; n)$ has surgery descriptions as in Figure 2.

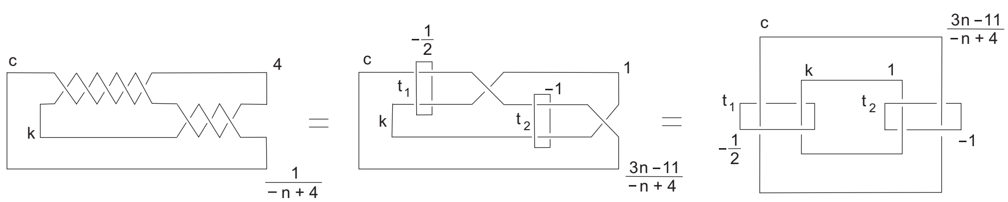


Figure 2: Surgery descriptions of $(K_n; n)$

Let us take the quotient by the strong inversion of $S^3$ with an axis $L$ as shown in Figure 3.

Then we obtain a branch knot $b'$ which is the image of the axis $L$. The Montesinos trick ([25], [6]) shows that $-\frac{1}{2}, -1, \frac{3n-11}{-n+4}$ and 1-surgery on $t_1, t_2, c$ and

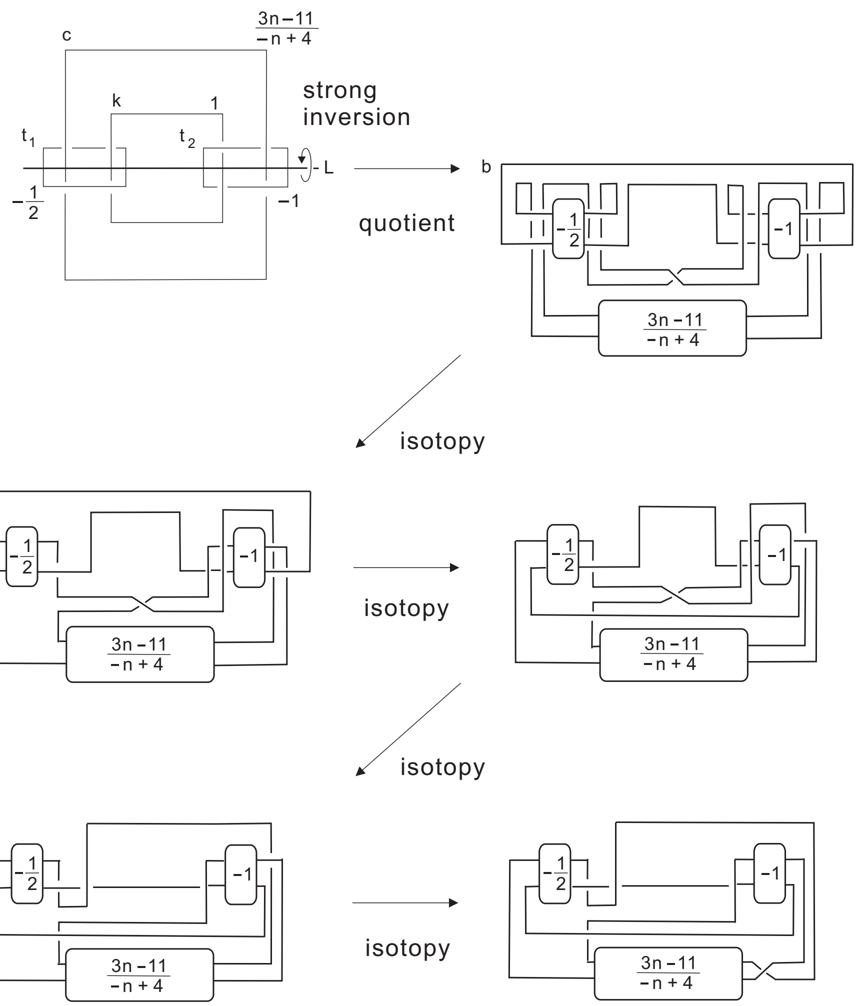


Figure 3

$k$ in the upstairs correspond to $-\frac{1}{2}, -1, \frac{3n-11}{-n+4}$ and 1-untangle surgery on $b'$ in the downstairs, where an $r$*-untangle surgery* is a replacement of $\frac{1}{0}$-untangle by $r$-untangle. (We adopt Bleiler's convention [5] on the parametrization of rational tangles.) These untangle surgeries convert $b'$ into a link $b$ (Figure 3).

Following the sequence of isotopies in Figures 3 and 4, we obtain a Montesinos link $M(\frac{2}{5}, -\frac{2}{3}, \frac{n-4}{4n-15})$.

Since $(K_n; n)$ is the double branched cover of $S^3$ branched over the Montesinos

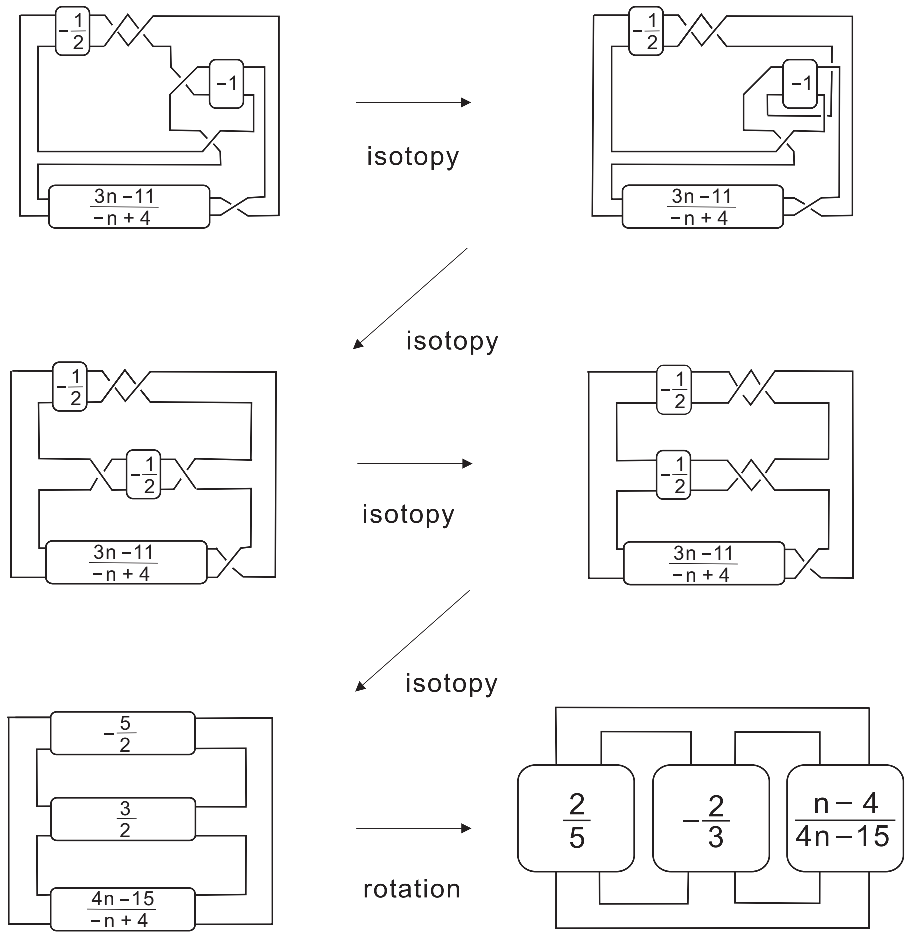


Figure 4: Continued from Figure 3

link $M(\frac{2}{5}, -\frac{2}{3}, \frac{n-4}{4n-15})$, $(K_n; n)$ is a Seifert fiber space of type $S^2(3, 5, |4n-15|)$ as desired. □

**Lemma 2.2** *The knot $K_n$ is hyperbolic if $n \neq 3, 4, 5$.*

**Proof** Note that the 2-bridge link given in Figure 1 is not a $(2, p)$-torus link, and hence by [23] it is a hyperbolic link. If $n \neq 3, 4, 5$, then $|-n+4| > 1$ and it follows from [1, Theorem 1] (also [3, Theorem 1.2]) that $K_n$ is a hyperbolic knot. See also [16, Corollary A.2], [24, Theorem 1.2] and [2, Theorem 1.1]. □

**Remark** It follows from [21], [18] that $K_n$ is a nontrivial knot except when $n = 4$. An experiment using Weeks' computer program "SnapPea" [31] suggests that $K_3$ and $K_5$ are hyperbolic, but we will not use this experimental results.

**Lemma 2.3** *The knot $K_n$ has tunnel number one for any integer $n \neq 4$.*

**Proof** Since the link $k \cup c$ is a two-bridge link, the tunnel number of $k \cup c$ is one with unknotting tunnel $\tau$; A regular neighborhood $N(k \cup c \cup \tau)$ is a genus two handlebody and $S^3 - \mathrm{int}N(k \cup c \cup \tau)$ is also a genus two handlebody, see Figure 5.

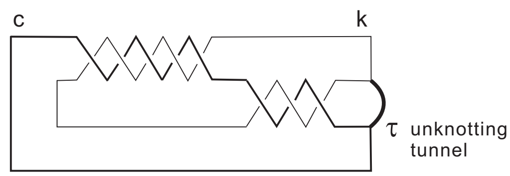


Figure 5

Then the general fact below (in which $k \cup c$ is not necessarily a two-bridge link) shows that the tunnel number of $K_n$ is less than or equal to one. Since our knot $K_n$ $(n \neq 4)$ is knotted in $S^3$, the tunnel number of $K_n$ is one. □

**Claim 2.4** *Let $k \cup c$ be a two component link in $S^3$ which has tunnel number one. Assume that $c$ is unknotted in $S^3$. Then every knot obtained from $k$ by twisting along $c$ has tunnel number at most one.*

**Proof** Let $\tau$ be an unknotting tunnel and $V$ a regular neighborhood of $k \cup c \cup \tau$ in $S^3$; $V$ is a genus two handlebody. Since $\tau$ is an unknotting tunnel for $k \cup c$, by definition, $W = S^3 - \mathrm{int}V$ is also a genus two handlebody. Take a small tubular neighborhood $N(c) \subset \mathrm{int}V$ and perform $-\frac{1}{n}$-surgery on $c$ using $N(c)$. Then we obtain a knot $k_n$ as the image of $k$ and obtain a genus two handlebody $V(c; -\frac{1}{n})$. Note that $V(c; -\frac{1}{n})$ and $W$ define a genus two Heegaard splitting of $S^3$, see Figure 6, where $c_n^*$ denotes the core of the filled solid torus.

Then it is easy to see that an arc $\tau_n$ given by Figure 6 is an unknotting tunnel for $k_n$ as desired. □

Now we are ready to prove Theorem 1.1. Lemmas 2.1, 2.2 and 2.3 show that our knots $K_n$ enjoy the required properties, except for $n = 3, 4, 5$. To prove

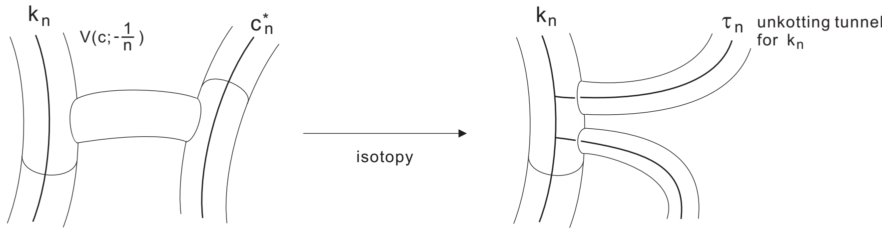


Figure 6

Theorem 1.1, we find hyperbolic knots $K'_n$ so that $(K'_n; n)$ is Seifert fibered for $n = 3, 4, 5$ (instead of showing that $K_3$, $K_5$ are hyperbolic). As the simplest way, let $K'_3$, $K'_4$ and $K'_5$ be the mirror image of $K_{-3}$, $K_{-4}$ and $K_{-5}$, respectively. Since $K_{-3}$, $K_{-4}$ and $K_{-5}$ are tunnel number one, hyperbolic knots by Lemmas 2.2 and 2.3, their mirror images $K'_3$, $K'_4$ and $K'_5$ are also tunnel number one, hyperbolic knots. It is easy to observe that $(K'_3; 3)$ (resp. $(K'_4; 4)$, $(K'_5; 5)$) is the mirror image of $(K_{-3}; -3)$ (resp. $(K_{-4}; -4)$, $(K_{-5}; -5)$). By Lemma 2.1, $(K_{-3}; -3)$, $(K_{-4}; -4)$ and $(K_{-5}; -5)$ are Seifert fibered, and hence $(K'_3; 3)$, $(K'_4; 4)$ and $(K'_5; 5)$ are also Seifert fibered. Putting $K_n$ as $K'_n$ for $n = 3, 4, 5$, we finish a proof of Theorem 1.1. □

## 3 Identifying exceptional fibers

In [24], Miyazaki and Motegi conjectured that if $K$ admits a Seifert fibered surgery, then there is a trivial knot $c \subset S^3$ disjoint from $K$ which becomes a Seifert fiber in the resulting Seifert fiber space, and verified the conjecture for several Seifert fibered surgeries [24, Section 6], see also [8]. Furthermore, computer experiments via "SnapPea" [31] suggest that such a knot $c$ is realized by a short closed geodesic in the hyperbolic manifold $S^3 - K$, for details see [24, Section 9], [27].

In this section, we verify the conjecture for Seifert fibered surgeries given in Theorem 1.1.

Recall that $K_n$ is obtained from $k$ by $\frac{1}{-n+4}$-surgery on the trivial knot $c$ (i.e., $(n-4)$-twist along $c$), see Figure 1. Denote by $c_n$ the core of the filled solid torus. Then $K_n \cup c_n$ is a link in $S^3$ such that $c_n$ is a trivial knot.

**Lemma 3.1** *After $n$-surgery on $K_n$, $c_n$ becomes an exceptional fiber of index $|4n-15|$ in the resulting Seifert fiber space $(K_n; n)$.*

**Proof** Following the sequences given by Figures 3 and 4, we have a Montesinos link with three arcs $\gamma$, $\tau_1$ and $\tau_2$ as in Figure 7, where $n = 1$ in the final Montesinos link, and $\gamma$, $\tau_1$, $\tau_2$ and $\kappa$ are the images of $c$, $t_1$, $t_2$ and $k$, respectively.

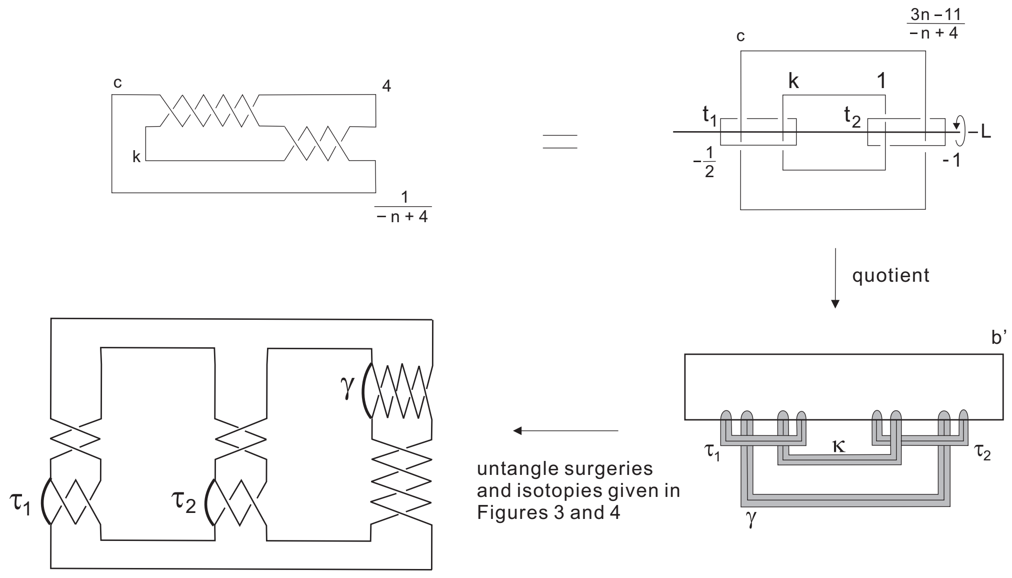


Figure 7: Positions of exceptional fibers

From Figure 7 we recognize that $t_1, t_2$ and $c$ become exceptional fibers of indices $5$, $3$ and $|4n - 15|$, respectively in $(K_n; n)$. □

For $n \neq 3, 4, 5$, $c_n$ becomes an exceptional fiber of index $|4n - 15|$, which is the unique maximal index, in $(K_n; n)$. Experiments via "SnapPea" [31] suggest that $c_n$ is a shortest closed geodesic in $S^3 - K_n$ $(n \neq 3, 4, 5)$. For sufficiently large $|n|$, hyperbolic Dehn surgery theorem [29], [30] shows that $c_n$ is the unique shortest closed geodesic in $S^3 - K_n$.

Let us assume that $n = 3, 4, 5$. Then we have put $K_n$ as the mirror image of $K_{-n}$ in the proof of Theorem 1.1. Let $k' \cup c'$ be the mirror image of the link $k \cup c$. Then $K_n$ is obtained also from $k'$ by $\frac{1}{-n-4}$-surgery on $c'$ (i.e., $(n+4)$-twist along $c'$); we denote the core of the filled solid torus by $c'_n$. Note that there is an orientation reversing diffeomorphism from $(K_{-n}; -n)$ to $(K_n; n)$ sending $c_{-n}$ (regarded as a fiber in $(K_{-n}; -n)$) to $c'_n$ (regarded as a fiber in $(K_n; n)$). Thus the above observation implies that $c'_n$ becomes an exceptional fiber of index $|4n + 15|$, which is the unique maximal index, in $(K_n; n)$ $(n = 3, 4, 5)$.

## References


[1] **M Aït Nouh**, **D Matignon**, **K Motegi**; Twisted unknots, C. R. Acad. Sci. Paris, Ser. I **337** (2003), 321–326.

[2] **M Aït Nouh**, **D Matignon**, **K Motegi**; Obtaining graph knots by twisting unknots, Topology Appl. **146–147** (2005), 105–121.

[3] **M Aït Nouh**, **D Matignon**, **K Motegi**; Geometric types of twisted knots, preprint.

[4] **J Berge**; Some knots with surgeries yielding lens spaces, unpublished manuscript.

[5] **S A Bleiler**; Knots prime on many strings, Trans. Amer. Math. Soc. **282** (1984), 385–401.

[6] **S A Bleiler**; Prime tangles and composite knots, in: Knot theory and manifolds (Vancouver, B.C. 1983), Lect. Notes in Math. vol. **1144**, Springer-Verlag, 1985, pp. 1–13.

[7] **M Culler**, **C McA Gordon**, **J Luecke**, **P B Shalen**; Dehn surgery on knots, Ann. Math. **125** (1987), 237–300.

[8] **M Eudave-Muñoz**; On hyperbolic knots with Seifert fibered Dehn surgeries, Topology Appl. **121** (2002), 119–141.

[9] **R Fintushel**, **R J Stern**; Constructing lens spaces by surgery on knots, Math. Z. **175** (1980), 33–51.

[10] **D Gabai**; Foliations and the topology of 3-manifolds. III, J. Differential Geom. **26** (1987), 479–536.

[11] **F González-Acuña**, **H Short**; Knot surgery and primeness, Math. Proc. Cambridge Philos. Soc. **99** (1986), 89–102.

[12] **C McA Gordon**; Dehn Filling; a survey, Proc. Mini Semester in Knot Theory, Banach Center, Warsaw, Poland, 1995.

[13] **C McA Gordon**, **J Luecke**; Only integral surgeries can yield reducible manifolds, Math. Proc. Cambridge Philos. Soc. **102** (1987), 97–101.

[14] **C McA Gordon**, **J Luecke**; Knots are determined by their complements, J. Amer. Math. Soc. **2** (1989), 371–415.

[15] **C McA Gordon**, **J Luecke**; Dehn surgeries on knots creating essential tori, I, Comm. Anal. Geom. **4** (1995), 597–644.

[16] **C McA Gordon**, **J Luecke**; Non-integral toroidal Dehn surgeries, Comm. Anal. Geom. **12** (2004), 417–485.

[17] **K Ichihara**, **K Motegi**, **H-J Song**; Longitudinal Seifert fibered surgeries on hyperbolic knots, preprint.

[18] **M Kouno**, **K Motegi**, **T Shibuya**; Twisting and knot types, J. Math. Soc. Japan 44 (1992), 199–216.

[19] **P Kronheimer**, **T Mrowka**; Dehn surgery, the fundamental group and SU(2). `arXiv:math.GT/0312322`

[20] **P Kronheimer**, **T Mrowka**, **P Ozsváth**, **Z Szabó**; Monopoles and lens space surgeries. `arXiv:math.GT/0310164`

[21] **Y Mathieu**, Unknotting, knotting by twists on disks and property (P) for knots in $S^3$, Knots 90 (ed. by Kawauchi), Proc. 1990 Osaka Conf. on Knot Theory and Related Topics, de Gruyter, 1992, 93–102

[22] **T Mattman**, **K Miyazaki**, **K Motegi**; Seifert fibered surgeries which do not arise from primitive/Seifert-fibered constructions, Trans. Amer. Math. Soc. (to appear).

[23] **W Menasco**; Closed incompressible surfaces in alternating knot and link complements, Topology **23** (1984), 37–44.

[24] **K Miyazaki**, **K Motegi**; Seifert fibered manifolds and Dehn surgery III, Comm. Anal. Geom. **7** (1999), 551–582.

[25] **J M Montesinos**; Surgery on links and double branched coverings of $S^3$, in: Knots, groups, and 3-manifolds (Papers dedicated to the memory of R H Fox), Ann. Math. Studies **84** (1975), 227–260.

[26] **K Morimoto**; There are knots whose tunnel numbers go down under connected sum, Proc. Amer. Math. Soc. **123** (1995), 3527–3532.

[27] **K Motegi**; An experimental study of Seifert fibered Dehn surgery via SnapPea, in: Proceedings of the Winter Workshop of Topology/Workshop of Topology and Computer (Sendai, 2002/Nara, 2001), Interdisc. Inform. Sci. **9** (2003), 95–125.

[28] **M Teragaito**; Toroidal surgeries on hyperbolic knots, Proc. Amer. Math. Soc. **130** (2002), 2803–2808.

[29] **W P Thurston**; The geometry and topology of 3-manifolds, Lecture notes, Princeton University, 1979. `http://www.msri.org/publications/books/gt3m/`

[30] **W P Thurston**; Three dimensional manifolds, Kleinian groups and hyperbolic geometry, Bull. Amer. Math. Soc. 6 (1982), 357–381.

[31] **J Weeks**; SnapPea: a computer program for creating and studying hyperbolic 3-manifolds, freely available from `http://geometrygames.org/SnapPea/`

*Department of Mathematics, Nihon University*
*Tokyo 156-8550, Japan*
and
*Division of Mathematical Sciences, Pukyong National University*
*599-1 Daeyondong, Namgu, Pusan 608-737, Korea*

Email: `motegi@math.chs.nihon-u.ac.jp` and `hjsong@pknu.ac.kr`

Received: 10 March 2005 Revised: 25 March 2005