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Counting immersed surfaces in hyperbolic 3-manifolds

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Abstract We count the number of conjugacy classes of maximal, genus g, surface subroups in hyperbolic 3-manifold groups. For any closed hyperbolic 3-manifold, we show that there is an upper bound on this number which grows factorially with g. We also give a class of closed hyperbolic 3-manifolds for which there is a lower bound of the same type.

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1 Introduction

A major problem in the study of hyperbolic 3-manifolds is to determine the extent to which they contain useful surfaces. It is conjectured that every closed hyperbolic 3-manifold contains a number of immersed incompressible surfaces, each of which lifts to an embedding in a finite sheeted cover. The aim of this paper is to count the number in some appropriate sense.

We fix attention on a closed hyperbolic 3-manifold (or orbifold) M. Since we are interested in immersed surfaces up to homotopy, we shall be counting surface subgroups of $\pi_1 M$ only up to conjugacy. Moreover, given our topological motivation, it is natural not to distinguish between a given immersed surface and one which covers it, and this leads to different ways of counting.

We define $s(M, g)$ to be the number of conjugacy classes of *maximal* surface groups of genus at most g in $\pi_1(M)$. Two subgroups of $\pi_1 M$ are said to be *commensurable* if their intersection has finite index in both. Let $s_1(M, g)$ denote the number of classes of surface groups in $\pi_1 M$ under the equivalence relation generated by conjugacy and commensurability. Let $s_2(M, g)$ be the total number of conjugacy classes of surface subgroups in $\pi_1 M$ of genus at most g.

Thurston proved (Corollary 8.8.6 of [8]) that $s_2(M,g)$ is finite, but the proof does not give a practical upper bound. In [6], Soma gave an explicit upper bound, which is doubly exponential in the genus. We show:

Theorem 1.1 *Let M be a closed hyperbolic 3-manifold. Then there exists a constant $c_2 > 0$ such that, for large g,*

$$s_2(M,g) < e^{c_2 g \log g}.$$

We say that a totally geodesic immersion $f : S \to M$, is *transverse* if its image has a curve of transverse self-intersection; thus f is not a finite cover of an embedding.

Theorem 1.2 *Let M be a closed hyperbolic 3-manifold, and suppose there is a transverse, totally geodesic immersion $f : S \to M$, for some hyperbolic surface S. Then there exist constants $c_1, c_2 > 0$ such that, for large g,*

$$e^{c_1 g \log g} < s_1(M,g) \le s(M,g) \le s_2(g,M) < e^{c_2 g \log g}.$$

Theorem 1.2 may be viewed as a co-dimension one analogue of the Prime Geodesic Theorem for closed hyperbolic 3-manifolds (see [2]), which states that the number of conjugacy classes of maximal 1-manifold groups grows exponentially with the length of the corresponding geodesics. The 3-manifolds of Theorem 1.2, in this sense, grow faster in two dimensions than in one dimension. It is also interesting to count *finite-index* subgroups of $\pi_1 M-$ see [4]. For results on counting *totally geodesic* immersions of surfaces, see [5] and [3]. Anneke Bart and Brian Mangum have also obtained results on cutting and pasting immersed surfaces.

Organization In Section 2 we prove the upper bound. As in Thurston's finiteness proof, the first step is to homotop each immersed surface into pleated form. The rest of the proof relies on finding triangulations for the surfaces whose edges are sufficiently short, and counting the possible graphs which arise as 1-skeleta.

The proof of the lower bound is by an explicit construction of immersed incompressible surfaces, which is related to Thurston's bending deformation. We collect some preliminary material on surfaces and graphs in Section 3, and give the construction in Section 4. The main difficulty is to show that the resulting surface groups are inequivalent in the sense defined above. We do this by proving that they correspond to hyperbolic manifolds with non-isometric convex cores.

Section 5 is devoted to examples where M is a right-angled reflection orbifold. In this case, we are able to give more explicit, combinatorial proofs of both bounds. In particular, the proof of the upper bound avoids the usual theory of pleated surfaces, and as a corollary, we get an alternate proof of Thurston's finiteness theorem for these manifolds.

Theorem 1.3 *Let $P \subset \mathbb{H}^3$ be a compact, right-angled polyhedron, and let M be the associated reflection orbifold. Then for the constants of Theorem 1.2, we may take $c_1 = 1$ and $c_2 = 8c(P) + 1$, where $c(P)$ is the maximum number of edges in a face disk of P (see Section 5 for definitions).*

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2 An upper bound

In this section, we prove Theorem 1.1.

If G is a graph, let $\mathcal{V}(G)$ denote the vertex set of G, and let $\mathcal{E}(G)$ denote the edge set of G.

Lemma 2.1 *There is a constant $k = k(s)$ such that any hyperbolic surface S with $\mathrm{injrad}(S) \geq s$ contains an embedded graph G satisfying:*

(1) *Every edge $e \in \mathcal{E}(G)$ is a geodesic arc of length $< s$,*

(2) $|\mathcal{V}(G)| < kg$,

(3) *$degree(v) < k$ for all vertices $v \in \mathcal{V}(G)$, and*

(4) *$S - G$ is a disjoint union of open disks.*

Proof Let $\mathcal{B} = \{B_1, B_2, ..., B_n\}$ be a maximal collection of disjoint balls of radius $s/4$ in S, and let v_i denote the center of ball B_i. If $x, y \in S$ are points with $d(x,y) < s$, then there is a unique shortest arc in S connecting them, which we denote $e(x,y)$.

Let

$$\begin{aligned} \mathcal{S} &= \{(v_i, v_j) | int\, B(v_i, s/2) \cap int\, B(v_j, s/2) \neq \emptyset\}, \\ \mathcal{E} &= \{e(v_i, v_j) | (v_i, v_j) \in \mathcal{S}\} \end{aligned}$$

We define a subset G of S by:

$$G = \bigcup_{e \in \mathcal{E}} e,$$

and we give G the structure of a graph by declaring the vertex set of G to be:

$$\mathcal{V}(G) = \bigcup_{e,e' \in \mathcal{E}} e \cap e',$$

and then defining the edges of G to be closures of components of $G - \mathcal{V}(G)$.

Lemma 2.2 *There is a constant $k = k(s)$, (independent of $\mathcal{B}$) such that $|\mathcal{V}(G)| < kg$, and $degree(v) < k$ for all $v \in \mathcal{V}(G)$.*

Proof We have:

$$|\mathcal{B}| \leq \frac{Area(S)}{Area(B(x, s/4))} = \frac{4\pi(g-1)}{4\pi sinh^2(\frac{s}{8})},$$

so $|\mathcal{B}| < m_1 g$ for some m_1. If $B \subset S$ is a fixed $s/4$-ball, there are at most a fixed number– call it m_2– of disjoint $s/4$-balls which will fit within an s-neighborhood of B. So we have

$$|\mathcal{E}| = |\mathcal{S}| < m_2 |\mathcal{B}| < m_1 m_2 g.$$

Let $e \in \mathcal{E}$. Since length$(e) < s$, there are at most a fixed number– call it m_3– of disjoint $s/4$-balls in S whose centers can fit within an s-neighborhood of e. Therefore, e can intersect at most $\binom{m_3}{2}$ different $e(v_i, v_j)$'s. Since no pair of geodesic arcs of length less than s can intersect more than once in S, we have

$$\begin{aligned} |\mathcal{V}(G)| &= |\bigcup_{e,e' \in \mathcal{E}} e \cap e'|, \\ &\leq \binom{m_3}{2} |\mathcal{E}| \\ &< m_1 m_2 \binom{m_3}{2} g. \end{aligned}$$

The fact that e can intersect at most $\binom{m_3}{2}$ other edges of $\mathcal{E}$ also implies

$$deg(v) < 2\binom{m_3}{2} + 2,$$

for all $v \in \mathcal{V}(G)$. Therefore we may take $k = Max\{m_1 m_2 \binom{m_3}{2}, 2\binom{m_3}{2} + 2\}$. □

Lemma 2.3 *The surface $S - G$ is a disjoint union of open disks.*

Proof Let $\mathcal{C} = \{B(v_1, s/2), ..., B(v_n, s/2)\}$. Since $\mathcal{B}$ is maximal, $\mathcal{C}$ covers S, and therefore it is enough to show that every ball $B(v_i, s/2) \in \mathcal{C}$ is covered by (the closures of) simply connected components of $S - G$.

The maximality of $\mathcal{C}$ implies that $\mathcal{C} - B(v_i, s/2)$ covers $\partial B(v_i, s/2)$. Let $B_{i1}, ..., B_{im} \in \mathcal{C} - B(v_i, s/2)$ be balls which cover $\partial B(v_i, s/2)$, arranged such that $B_{ij} \cap B_{i(j+1)} \neq \emptyset$ for all j. Then each triple $(v_i, v_{ij}, v_{i(j+1)})$ is the vertex set of a geodesic triangle, T_j, contained in an s-ball (embedded, since $\mathrm{injrad}(S) \geq s$). Thus each T_j bounds a disk D_j, which is the union of (closures of) components of $S - G$. The ball $B(v_i, s/2)$ is covered by $\{D_1, ..., D_m\}$. □

This concludes the proof of Lemma 2.1. □

We now return to the proof of the upper bound. By [8], given any π_1-injective immersion $f : S \to M$, we can find a hyperbolic structure on S, and a homotopy of f so that it is pleated with respect to this structure. Let $s = \mathrm{injrad}(M)$; since f is pleated it takes closed loops in S to closed loops of equal or shorter length in M, and therefore $\mathrm{injrad}(S) \geq s$. There is then a graph $G \subset S$, with the properties stated in Lemma 2.1.

Suppose $f' : S' \to M$ is another π_1-injective, pleated map of a genus g surface, with a graph $G' \subset S'$ given by Lemma 2.1. With slight abuse of terminology, we say that f and f' are *homotopic* if there is a map $g : S' \to S$ such that fg is homotopic to f'.

Suppose there is a bijection $h : \mathcal{V}(G) \to \mathcal{V}(G')$, and label the vertices $\mathcal{V}(G) = \{v_1, ..., v_n\}$, and $\mathcal{V}(G') = \{v'_1, ..., v'_n\}$, where $v'_i = hv_i$. The map h induces a graph $h(G)$ on $\mathcal{V}(G')$ by the rule that v'_i and v'_j are adjacent in $h(G)$ if and only if v_i and v_j are adjacent in G.

Lemma 2.4 *Suppose that $d(f'v'_i, fv_i) < s/4$ for all i, and that $h(G) = G'$. Then f and f' are homotopic.*

Proof Let δ_i be a segment of length less than $s/4$ connecting fv_i and $f'v'_i$. Sliding the v'_i's along the δ_i's, we may homotope $f'|_{G'}$ to a map $g : G' \to M$, so that $g(v'_i) = f(v_i)$ and so that gG' is contained in $f'G' \bigcup_i \delta_i$. If E' is an edge of G', then its image under g is given by:

$$g(E') = f'(E') \cup \delta_i \cup \delta_j, \text{ (for some i,j)},$$

which has total length less than s. Since $\mathrm{injrad}(M) = s$, any two segments of M with the same endpoints which have length less than s are homotopic, fixing endpoints. Therefore, the map $f'|_{G'}$ is homotopic to $fh|_{G'}$. Since the complementary regions of G and G' are disks, the map h can be extended to a map $h: S' \to S$, such that f' and fh are homotopic. □

Let $\mathcal{V} = \{v_1, ..., v_k\}$ be a fixed set of vertices, and let $\mathcal{G}(\mathcal{V})$ be the set of all graphs on $\mathcal{V}$. Note that the relation of equality between graphs on $\mathcal{V}$ is stronger than the relation of isomorphism between graphs on $\mathcal{V}$.

Let

$$\mathcal{G}(\mathcal{V}, n) = \{G \in \mathcal{G}(\mathcal{V}) | \text{ each vertex of } G \text{ has degree at most } n\}$$

A computation shows that

$$|\mathcal{G}(\mathcal{V}, n)| \leq |\mathcal{V}|^{n|\mathcal{V}|}. \tag{1}$$

Let $k_1 = k(s)$ be the constant provided by Lemma 2.1. Let $\mathcal{C}$ be a collection of balls of radius $s/4$ which covers M, and let $k_2 \geq |\mathcal{C}|$. Suppose we have a collection $\mathcal{S}$ of more than $(k_1 g)(k_1 g)^{k_2}(k_1 g)^{k_1^2 g}$ pleated maps $f_i : S_i \to M$, where S_i is a hyperbolic surface of genus g. Associated to each $f_i \in \mathcal{S}$ is a graph $G_i \subset S_i$ satisfying the properties stated in Lemma 2.1. Since $|\mathcal{V}(G_i)| < k_1 g$, there is a subset $\mathcal{S}_1 \subset \mathcal{S}$ of $(k_1 g)^{k_2}(k_1 g)^{k_1^2 g}$ pleated maps whose graphs all have the same number of vertices. Since there are at most $(k_1 g)^{k_2}$ ways to map $\mathcal{V}(G_i)$ to $\mathcal{C}$, then there is a subset $\mathcal{S}_2 \subset \mathcal{S}_1$ of $(k_1 g)^{k_1^2 g}$ pleated maps, such that, if we fix a map $f \in \mathcal{S}_2$ with associated graph G, then for any $f_i \in \mathcal{S}_2$, there are bijections $h_i : \mathcal{V}(G_i) \to \mathcal{V}(G)$, such that, for all $v \in \mathcal{V}(G_i), d(fh_i(v), f_i(v)) \leq s/4$. By Equation 1, there is a pair of distinct maps $f_i, f_j \in \mathcal{S}_2$ such that $h_i(G_i) = h_j(G_j)$. Therefore, by Lemma 2.4, f_i is homotopic to f_j. This concludes the proof of the upper bound.

3 Properties of graphs and surfaces

By a *metric graph*, G, we mean a graph with a metric space structure, determined by the following procedure: we assign a length to each edge, e, and give e a path metric induced from the interval $[0, \text{length}(e)]$. The path metric on the edges then determines a path metric on G. The *standard* metric on G is obtained by setting the length of each edge to be one. A map $f : X \to Y$ between metric spaces is a *(k,c)-quasi-isometry* $(k, c > 1)$ if $\frac{d(fx_1, fx_2)}{k} - c < d(x_1, x_2) < kd(fx_1, fx_2) + c$ for all $x_1, x_2 \in X$, and $d(y, f(X)) < c$ for all $y \in Y$.

Lemma 3.1 *Let G and G' be metric graphs, with no vertices of degree 2, and with all edges longer than $u = 6k^3(k^2+3)c$. Then if G and G' are (k,c)-quasi-isometric, they are isomorphic.*

Proof Let $f : G \to G'$ be a (k,c)-quasi-isometry. Then, there exist numbers k', c' and $s > 0$, and a map $g : G' \to G$, which is a (k', c')-quasi-isometry, such that $d(gfx, x) < s$, and $d(fgy, y) < s$ for all $x \in G$ and $y \in G'$. In fact, we may take $k' = k$, $c' = 3kc$, and $s = kc$.

Let x be a vertex of G of degree $n > 2$, and let $e_1, ..., e_n$ be distinct edges incident to x. Since f is a (k,c)-quasi-isometry, any point of fe_i is within distance c of another point of fe_i, and since length$(e_i) > k(k^2+2)c$, then Diameter$(fe_i) > k(k^2+2)c/k - c = (k^2+1)c$. Thus, for each i, there are points $x_i \in e_i$ such that

$$(k^2+1)c < d(fx_i, fx) < (k^2+1)c + c.$$

Then

$$\begin{aligned} d(x_i, x) &> d(fx_i, fx)/k - \frac{c}{k} \\ &> (k^2+1)c/k - \frac{c}{k} \\ &= kc. \end{aligned}$$

Since edges of G all have length greater than $2d(x_i, x)$, then

$$d(x_i, x_j) = d(x_i, x) + d(x_j, x) > 2kc,$$

and so
$$d(fx_i, fx_j) > (2kc)/k - c > c.$$

So, in the $(k^2+2)c$-neighborhood of fx, there are n points fx_i, such that $|d(fx_i, fx) - d(fx_j, x)| < c$, and $d(fx_i, fx_j) > c$ for all $i \neq j$. This implies that fx is within distance $(k^2+2)c$ of a vertex of G'.

If x is a degree 1 vertex, then we claim that $y = fx$ is within distance $k'(k'^2 + 2)c'$ of a degree 1 vertex of G'. For suppose not. Then there is an interval e of length $2k'(k'^2+2)c'$ isometrically embedded in G', such that fx is the midpoint of e. Let e_1 and e_2 be sub-intervals of e with $e_1 \cap e_2 = \{fx\}$, and length$(e_i) = k'(k'^2+2)c'$. Then Diameter$(ge_i) > [k'(k'^2+2)c']/k' - c' = (k'^2+1)c'$, so for both i's there exists a point $y_i \in e_i$ such that

$$(k'^2+1)c' < d(gy_i, gy) < (k'^2+1)c' + c'.$$

We have:

$$\begin{aligned} d(y_i, y) &> \frac{d(gy_i, gy)}{k'} - \frac{c'}{k'} \\ &> \frac{(k'^2+1)c'}{k'} - \frac{c'}{k'} \\ &= k'c' \end{aligned}$$

Since all edges of G' have length greater than $d(y_i, y)$, then

$$d(y_1, y_2) = d(y_1, y) + d(y_2, y) > 2k'c',$$

and
$$d(gy_1, gy_2) > (2k'c')/k' - c' = c'.$$

However, we have $d(gy, x) = d(gfx, x) < s = kc$, so gy and x are contained in the same edge. Since x has degree one, and

$$\begin{aligned} d(gy_i, x) &> d(gy_i, gy) - d(gy, x) \\ &> (k'^2+1)c' - s \\ &= (k^2+1)(3kc) - kc \\ &> kc = s \\ &> d(gy, x), \end{aligned}$$

then $d(gy_i, x) = d(gy_i, gy) + d(gy, x)$. Then

$$\begin{aligned} |d(gy_1, x) - d(gy_2, x)| &= |d(gy_1, gy) + d(gy, x) - (d(gy_2, gy) + d(gy, x))| \\ &= |d(gy_1, gy) - d(gy_2, y)| \\ &< c'. \end{aligned}$$

Since $d(gy_i, x) = d(gy_i, gy) + d(gy, x) < (k'^2+2)c' + s = (k^2+2)(3kc) + kc$, then gy_i and x are contained in the same edge. Therefore $d(gy_1, gy_2) = |d(gy_1, x) - d(gy_2, x)| < c'$, for a contradiction. Therefore, fx is within distance $k'(k'^2+2)c'$ of a degree 1 vertex of G'.

Let

$$\begin{aligned} t &= Max((k^2+2)c, k'(k'^2+2)c') \\ &= Max((k^2+2)c, k(k^2+2)3kc) \\ &= 3k^2(k^2+2)c \\ &< u/2 \end{aligned}$$

We have shown $f(\mathcal{V}(G)) \subset N_t(\mathcal{V}(G'))$. Since $t < u/2$, there is an induced map $f^* : \mathcal{V}(G) \to \mathcal{V}(G')$, defined by $f^*v = N_{u/2}(fv) \cap \mathcal{V}(G')$. Similarly, if

$t' = Max((k'^2+2)c, k(k^2+2)c)$, then $t' < u/2$, $g(\mathcal{V}(G')) \subset N_{t'}(\mathcal{V}(G))$, and there is an induced map $g^* : \mathcal{V}(G') \to \mathcal{V}(G)$.

We have:

$$\begin{aligned} d(gf^*v, v) &\leq d(v, gfv) + d(gfv, gf^*v) \\ &\leq s + k'd(fv, f^*v) + c' \\ &\leq s + k(3k^2(k^2+2)c) + 3kc \\ &< u/2 \end{aligned}$$

Therefore $g^*f^*v = v$, and similarly $f^*g^*v = v$ for all $v \in \mathcal{V}(G')$. So the maps f^* and g^* are inverses to each other, and hence bijections.

Furthermore, if v_1, v_2 are adjacent vertices in G, with an edge e connecting them, then $f(e)$ is a (k,c)-quasi-geodesic segment, intersecting $N_t(f^*v_1 \cup f^*v_2)$, but disjoint from $N_t(\mathcal{V}(G') - \{f^*v_1, f^*v_2\})$, and this implies that f^*v_1 and f^*v_2 are adjacent in G'. A similar statement holds for g^*, and so the map f^* induces an isomorphism from G to G'. □

Let G be a graph, with a non-separating basepoint p contained in the interior of an edge. Let $N(p)$ be a regular open neighborhood of p in G, and let $G/p = G - N(p)$. Then G/p has the structure of a graph, with $\mathcal{V}(G/p) = \mathcal{V}(G) \cup \{v_1, v_2\}$, where v_1 and v_2 are degree one vertices. Say that the pairs (G,p) and (G',p') are *inequivalent* if the universal cover of G/p is not isomorphic to the universal cover of G'/p'.

Lemma 3.2 *Let G be a bouquet of two circles, with basepoint p in the interior of an edge. Then there is a constant $c > 0$ such that, for all n, the number of inequivalent covers $(\tilde{G}, \tilde{p})$ of (G,p) of degree n is at least $(cn)!$.*

Proof Let a, b be the edges of G, and assume $p \in b$. Any graph $\tilde{G}$ whose vertices all have degree four is a cover of G, and we may specify the covering map by directing the edges of $\tilde{G}$, and labeling each of them either a or b. Let σ be an order two permutation on $\{1, ..., n\}$. We consider $4n$-fold covers $\pi : (\tilde{G}, \tilde{p}) \to (G,p)$ constructed as follows (see Figure 1):

(1) Begin with $4n$ vertices $v_1, ..., v_{4n}$, and, for all i, connect v_i and v_{i+1} with an edge labeled a.

(2) If either

    (a) i is even

    (b) $i \geq 2n$, or

(c) $i = 2j - 1$, with $j \leq n$, and $\sigma j = j$,

then attach a 1-cycle labeled b to v_i.

(3) If $\sigma(i) = j$, and $i \neq j$, then attach a 2-cycle with edges labeled b to v_{2i-1}, v_{2j-1}.

(4) Choose a basepoint, $\tilde{p}$, in the 1-cycle which is adjacent to v_0 and labeled b.

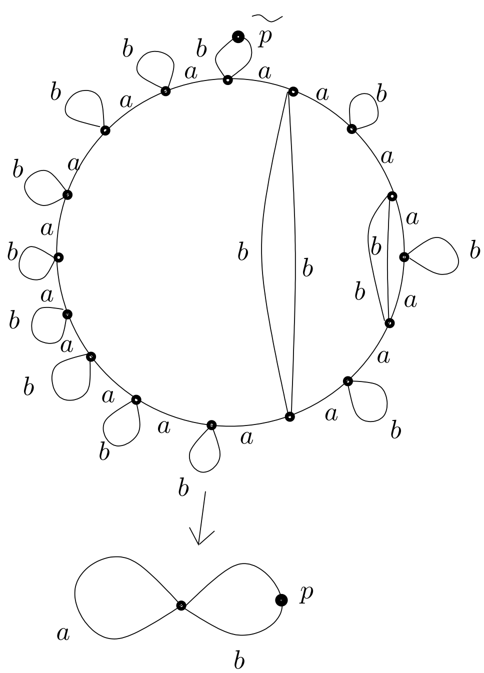


Figure 1: A cover of G, with $2n = 16$ and $\sigma = (1,2)(3,4)$

Let $\rho : T \to \tilde{G}/\tilde{p}$ be the universal cover. We give T the standard path metric (see above). Let $\mathcal{E}_1$ be the set of edges of T whose endpoints are equidistant from ∂T. Note that $\mathcal{E}_1$ is preserved under any isomorphism of T, and that $\mathcal{E}_1$ contains ρ^{-1} of any 1-cycle. Let $\mathcal{E}_2$ be the set of edges in $\mathcal{E}(G) - \mathcal{E}_1$ which are adjacent to at least two elements of $\mathcal{E}_1$. Then $\mathcal{E}_2$ is also isomorphism-invariant. Since no edge labeled a has endpoints which are equidistant from $\partial(\tilde{G}/\tilde{p})$, then $\rho^{-1}\pi^{-1}a \cap \mathcal{E}_1 = \emptyset$, and since every edge labeled a in $\tilde{G}$ is adjacent to a 1-cycle, then $\rho^{-1}\pi^{-1}a \subset \mathcal{E}_2$. Also, any element of $\rho^{-1}\pi^{-1}b - \mathcal{E}_1$ can be adjacent to (and distinct from) at most one element of $\mathcal{E}_1$, so $\rho^{-1}\pi^{-1}b \cap \mathcal{E}_2 = \emptyset$. Therefore $\mathcal{E}_2 = \rho^{-1}\pi^{-1}a$.

Let $d_a : \mathcal{V}(T) \to \mathbb{Z}$ be the function defined by setting $d_a(x)$ to be the distance from x to ∂T along paths whose edges (except for the final edge) are all labeled a. Let $\mathcal{V}_i$ be the set of vertices of T with $d_a(x) = i$. Since $\mathcal{E}_2 = \rho^{-1}\pi^{-1}a$ is isomorphism-invariant, then $\mathcal{V}_i$ is isomorphism invariant. The transposition (ij) occurs in the cycle decomposition of σ if and only if there is an edge between $\mathcal{V}_{2i}$ and $\mathcal{V}_{2j}$, and thus the permutation σ is determined by the isomorphism type of T. Since the number of such permutations grows as a factorial in n, the number of inequivalent covers $(\tilde{G}, \tilde{p})$ also grows as a factorial in n. □

Lemma 3.3 *Let S be a hyperbolic surface, let L be a non-separating collection of disjoint loops in S, and let u be an arbitrary function of two variables. Then there is a finite cover $\tilde{S}$ of S, and a bouquet of two circles G such that*

(1) *there is a (k,c)-quasi-isometry $g : \tilde{S} \to G$,*

(2) *both edges of G have length greater than $u(k,c)$, and*

(3) *$g(\tilde{L})$ is a point, for some 1-1 lift $\tilde{L}$ of L.*

Proof After passing to a preliminary cover, we may assume that $S - L$ has positive genus. Then there is a map of S onto a bouquet of two circles, $g_1 : S \to G_1 = a \cup b$, such that g_1 induces a surjection on fundamental groups, and $g_1(L)$ is a point. We give G_1 the standard metric (so a and b have length 1), and since S and G_1 are compact, g_1 is a (k, c_1)-quasi-isometry for some k, c_1.

Let $c = c_1 + 2$, and let $s > u(k,c)$. Let $\pi : \tilde{G}_1 \to G_1$ be a $2s$-fold cover of G_1, where $\rho^{-1}a$ is a single cycle, and $\pi^{-1}b$ is a set of $2s-2$ cycles, as indicated in Figure 2. There is an induced cover $\tilde{S}$ of S, to which L lifts. Let $\tilde{b}$ be the component of $\pi^{-1}b$ which maps 2 to 1 onto b, and let G be the graph $\pi^{-1}a$, with the vertices of $\tilde{b}$ identified to a point. Then G is homeomorphic to a bouquet of two circles, and there is a $(1,2)$-quasi-isometry $g_2 : \tilde{G}_1 \to G$. Thus $g_2 g_1 : \tilde{S} \to G$ is a (k,c)-quasi-isometry, the edges of G all have length at least $s > u(k,c)$, and there are 1-1 lifts $\tilde{L}$ of L such that $g_2 g_1 \tilde{L}$ is a point. □

Let S be a closed hyperbolic surface, with a non-separating collection of loops L in S, and let $(\tilde{S}, \tilde{L})$ be a finite cover of S, with a 1-1 lift $\tilde{L}$ of L. Let $\tilde{S}/\tilde{L}$ be the metric completion of $\tilde{S} - \tilde{L}$, which is a hyperbolic surface with geodesic boundary, and let $P \subset \mathbb{H}^2$ be the universal cover of $\tilde{S}/\tilde{L}$. For another such cover $(\tilde{S}', \tilde{L}')$, we define analogous objects $\pi', \tilde{S}'$ and P'. We say that the pairs $(\tilde{S}, \tilde{L})$ and $(\tilde{S}', \tilde{L}')$ are *inequivalent* if P is not isometric to P'.

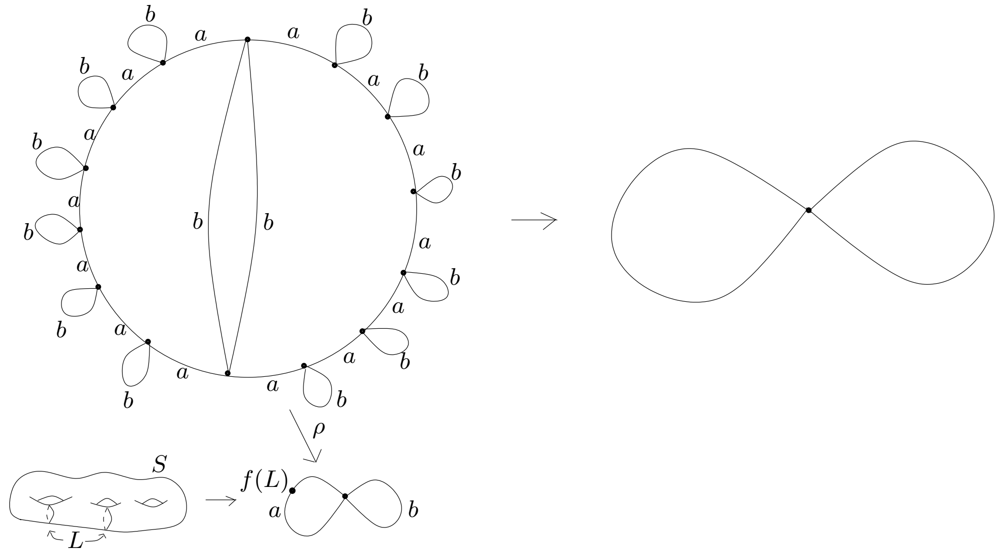


Figure 2: There is a cover of S which maps onto a bouquet of circles, with long edges.

Lemma 3.4 *Let (S, L) be as above. Then the number of inequivalent covers $(\tilde{S}, \tilde{L})$ of degree n is at least $(cn)!$ for some constant $c > 0$.*

Proof We replace S with the cover given by Lemma 3.3. We are free to do this, since the effect on covering degrees is linear. We have a (k, c)-quasi-isometry $g : S \to G$, where G is a bouquet of two circles, whose lengths are at least $u(k, c)$, where u is a function of k and c which we will determine later.

Let $(\tilde{G}, \tilde{p})$, and $(\tilde{G}', \tilde{p}')$ be inequivalent covers of G, with induced covers $(\tilde{S}, \tilde{L})$ and $(\tilde{S}', \tilde{L}')$ of S. Let P (resp. P') be the universal cover of $\tilde{S}/\tilde{L}$ (resp. $\tilde{S}'/\tilde{L}'$), and let T (resp. T') be the universal cover of $\tilde{G}/\tilde{p}$ (resp. $\tilde{G}'/\tilde{p}'$). There is a (k, c)-quasi-isometry $h : P \to T$ (resp. $h' : P' \to T'$) obtained by lifting g to $\mathbb{H}^2$ and then restricting the domain. Let $h'^* : T' \to P'$ be a quasi-inverse of h'. If $(\tilde{S}, \tilde{L})$ and $(\tilde{S}', \tilde{L}')$ are equivalent, then there is an isometry $i : P' \to P$. Then the map $hih'^* : T' \to T$ is a (k', c')-quasi-isometry where k' and c' depend only on k and c, and not on our choices of covers. By Lemma 3.3, we may assume that the edges of G, and hence also of T and T', are longer than the constant $u(k', c')$ given by Lemma 3.1. Then T and T' are isomorphic, contradicting the assumption that $(\tilde{G}, \tilde{p})$ and $(\tilde{G}', \tilde{p}')$ are inequivalent. So $(\widetilde{S}, \widetilde{L})$ and $(\widetilde{S}', \widetilde{L}')$ are inequivalent, and the result now follows from Lemma 3.2. □

4 Constructing immersed surfaces by gluing convex manifolds

The material in this section is based on Thurston's bending deformation, and the treatment of this which is given in [1]. We begin with a lemma about gluing convex manifolds. For a space X, we let $\mathring{X}$ denote the interior of X.

Lemma 4.1 *Suppose that X and X' are hyperbolic n-manifolds with convex boundary, and that U, U' are compact n-submanifolds, with an isometry $f : U \to U'$ such that*

(1) $f(\partial U \cap \mathring{X}) \subset \partial X'$, $f^{-1}(\partial U' \cap \mathring{X}') \subset \partial X$, *and*

(2) $\partial X \cap f^{-1}\partial X'$ *is a non-empty subsurface of* ∂X. *Then the identification space $Y = X \cup_{U=fU} X'$ is a hyperbolic manifold with convex boundary.*

Proof Condition 1 guarantees that the identification space Y is a manifold, with $\partial Y = (\partial X - U) \cup (\partial X' - U') \cup (\partial X \cap f^{-1}\partial X')$. Condition 2 guarantees that $\partial X - U$ (resp. $\partial X' - U'$) can be extended to an open set $V \subset \partial Y$ (resp. $V' \subset \partial Y$), such that Y is locally convex at all points in V. Then V, V' and $interior(\partial X \cap f^{-1}\partial X')$ define an open covering of ∂Y by convex sets, and so Y is locally convex at all points in ∂Y. □

If W is a hyperbolic 3-manifold with convex boundary, let Core(W) denote the convex core of W. This is the unique smallest convex sub-manifold of W which carries $\pi_1 W$. For any set U in W, let $N_\epsilon(U) = \{x \in U | d(x, U) \leq \epsilon\}$.

Suppose S is an orientable (possibly disconnected) hyperbolic surface with geodesic boundary, that $\ell_{ij} \subset \partial S$ ($i = 1, 2,\ j = 1, ..., n$) are distinct boundary components, with orientations induced from S, and that length(ℓ_{1j}) = length(ℓ_{2j}) for all j. Let L be the disjoint union of the ℓ_{ij}'s, and let $S(L)$ be the surface formed by identifying each ℓ_{1j} and ℓ_{2j} (preserving orientations) to a single loop, which we call ℓ_j; we view $S(L)$ as a metric space, with path metric inherited from the S_i's. For each j, let θ_j be angle between 0 and π.

Lemma 4.2 *There is a number $r = r(\theta_1, ..., \theta_n) > 0$ such that, if L has a collar neighborhood of radius r in S, then there is a hyperbolic 3-manifold W, with convex boundary, such that:*

(a) $W = \text{Core}(W)$,

(b) $W \cong S(L) \times I$, *and there is an isometric embedding $i : S(L) \to W$, such that W retracts onto $i(S(L))$,*

(c) *there is a regular neighborhood* $N(\ell_j)$ *of each* ℓ_i *in* $S(L)$ *so that* $i(N(\ell_j))$ *is the union of a pair of geodesic annuli meeting at an angle* θ_j,

(d) *for some* $\epsilon > 0$, *the set* $\mathcal{A} = \{i(\ell_1), ..., i(\ell_n)\}$ *has the following geometric characterization:*
$\mathcal{A} = \{$ *infinite or closed geodesics* γ *in* $W | d(\gamma, F) > \epsilon$ *for some component* F *of* $\partial W\}$.

Proof We will assume that $n = 1$, the proof in the general case being similar. To reduce notation, let $\ell_i = \ell_{i1}$, $\theta = \theta_1$, and $r = r_1$.

The outline of the construction is as follows. We first construct a hyperbolic solid torus V, which is a convex thickening of two annuli. We then construct a convex hyperbolic 3-manifold Z, by gluing together $S(L) \times I - N(\ell_1)$, $S(L) \times I - N(\ell_2)$ and V. Then finally we let $W = \text{Core}(Z)$.

To construct the hyperbolic solid torus V, let $\epsilon > 0$, let $\tilde{\ell}$ be a geodesic in $\mathbb{H}^3$, let Q_1 and Q_2 be geodesic half-planes in $\mathbb{H}^3$, with $\partial Q_i = \tilde{\ell}$, and let the angle of intersection of Q_1 and Q_2 be θ. Let $Q = Q_1 \cup Q_2$, let $B = N_\epsilon(Q)$, and let H_Q be the convex component of $\mathbb{H}^3 - Q$. For $p \in \partial N_\epsilon(Q_i)$, let P_p be a hyperbolic plane such that $P_p \cap N_\epsilon(Q_i) = p$.

Claim *There is an* $r = r(\theta)$ *such that, if* $p \in \partial N_\epsilon(Q_i)$ *and* $d(p, \tilde{\ell}) > r$, *then* $P_p \cap B = \{p\}$.

Proof of Claim There are two components of $\partial N_\epsilon Q_i - N_\epsilon(\tilde{\ell})$; one is contained in H_Q, which we denote $\partial^- N_\epsilon(Q_i)$, and the other is denoted $\partial^+ N_\epsilon(Q_i)$.

Without loss of generality, let $p \in \partial N_\epsilon(Q_1)$. Suppose $p \in \partial^+ N_\epsilon(Q_1)$. Let P_1 be the geodesic plane containing Q_1, and let H_1 be the half-space bounded by P_1 and containing p. If $r > \epsilon$, then $P_p \cap N_\epsilon Q_1 = P_p \cap N_\epsilon(P_1) = \{p\}$, and so $P_p \subset H_1$. Since $N_\epsilon(Q_2) \cap H_1 \subset N_\epsilon(\tilde{\ell})$, and since $d(p, \tilde{\ell}) > \epsilon$, then $P_p \cap N_\epsilon(Q_2) = \emptyset$. Therefore $P_p \cap B = \{p\}$.

Suppose $p \in \partial^- N_\epsilon(Q_1)$. Let $Q^\perp$ be a hyperbolic plane which is orthogonal to $\tilde{\ell}$, and which contains p. Let $\alpha_i = Q_i \cap Q^\perp$, and let β be a geodesic ray bisecting the α_i's. Let $\delta(t)$ be a geodesic intersecting β orthogonally at a distance t from the endpoint of β (see Figure 3).

Let $\partial^\pm N_\epsilon(\alpha_i) = \partial^\pm N_\epsilon Q_i \cap Q^\perp$. If $\delta(t) \cap B \neq \emptyset$, then by symmetry, $\delta(t)$ makes the same angle with both $\partial^- N_\epsilon(\alpha_1)$ and $\partial^- N_\epsilon(\alpha_2)$; call this angle $\psi(t)$. For small t, we have $\psi(t) > 0$, and for large enough t, we have $\delta(t) \cap N_\epsilon(\alpha_i) = \emptyset$, so there must be some t_0 for which $\psi(t_0) = 0$, and there is a corresponding geodesic $\delta(t_0)$ which is tangent to $B \cap Q^\perp$ at two points. There is then a hyperbolic plane, $Q(t_0)$, orthogonal to $Q^\perp$, which contains $\delta(t_0)$, and which is tangent to B at two points.

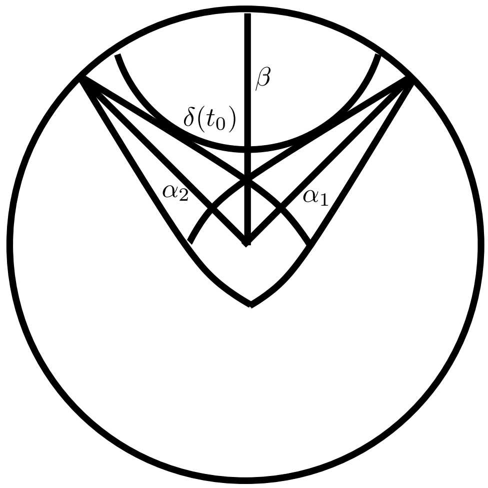


Figure 3: A convex thickening

Let $r = d(\tilde{\ell}, \delta(t_0) \cap B)$, and suppose $d(p, \tilde{\ell}) > r$. Then $\delta(t_0)$ separates p from $N_\epsilon(\alpha_2)$ in $Q^\perp - B$. Since the geodesic $P_p \cap Q^\perp$ can intersect $\delta(t_0)$ at most once, we must have $P_p \cap Q^\perp \cap N_\epsilon(\alpha_2) = \emptyset$, so $P_p \cap Q^\perp \cap B = \{p\}$. Let $H^\pm$ be the half-spaces in $\mathbb{H}^3$ bounded by $Q^\perp$. By symmetry, if $P_p \cap N_\epsilon(Q_2) \neq \emptyset$, then $P_p \cap H^+ \cap N_\epsilon(Q_2)$ and $P_p \cap H^- \cap N_\epsilon(Q_2)$ are both non-empty, and therefore, by convexity, $P_p \cap Q^\perp \cap N_\epsilon(Q_2) \neq \emptyset$, which is a contradiction. Thus we conclude that $P_p \cap N_\epsilon(Q_2) = \emptyset$, so $P_p \cap B = \{p\}$, proving the claim. □

Let Hull(B) denote the convex hull of B in $\mathbb{H}^3$, which is the intersection of all closed half-spaces containing B. Let τ be a hyperbolic isometry (i.e. with real trace) in $\mathrm{Isom}^+(\mathbb{H}^3)$ such that $\tilde{\ell}/\tau$ is isometric to ℓ. Since τ is hyperbolic, it preserves both Q_1 and Q_2, and hence also Hull(B). Let $s > r(\theta)$, and let $V = [\mathrm{Hull}(B) \cap N_s(\tilde{\ell})]/\tau$.

Let $S \times [-\epsilon, \epsilon]$ be a radius ϵ thickening of S (more precisely, embed $\pi_1 S$ as a Fuchsian subgroup of $\mathrm{Isom}(\mathbb{H}^3)$, embed $\tilde{S}$ in $\mathbb{H}^3$ isometrically as a $\pi_1 S_i$-equivariant subset, and then define $S \times [-\epsilon, \epsilon] = N_\epsilon(\tilde{S})/\pi_1 S$). Let $U_i = (V - N_r(\tilde{\ell})) \cap (N_\epsilon(Q_i)/\tau)$, and let $U_i' = S \times [-\epsilon, \epsilon] \cap (N_s(\ell_i) - N_r(\ell_i))$.

By the claim, $\mathrm{Hull}(B) - N_r(\tilde{\ell}) = B - N_r(\tilde{\ell})$, which implies that U_i is isometric to U_i'. By Lemma 4.1, we may form a hyperbolic manifold, Z, with convex boundary, from $(S \times [-\epsilon, \epsilon]) - (N_r(\ell_1) \cup N_r(\ell_2)) \amalg V$ by gluing each U_i to U_i' via isometry.

Since Z has convex boundary, there is a unique smallest convex submanifold, $W = \mathrm{Core}(Z) \subset Z$, which is homotopy equivalent to Z. By the construction of Z, we have an isometric embedding $i : S(L) \to Z$, and $i(\ell)$ has a neighborhood in Z which is a union of geodesic annuli meeting at an angle θ. Since $i(S)$ is totally geodesic in Z, with geodesic boundary, we have that $i(S) \subset W$, and so $i(S(L)) \subset W$.

Since $\mathrm{Core}(\pi_1 W)$ is compact, then $\pi_1 W$ is a quasi-Fuchsian surface group, and the only possibilities are that $W \cong S(L) \times I$ or that $W \cong S(L)$. Since $i(S(L))$ is not totally geodesic, $W \ncong S(L)$, so $W \cong S(L) \times I$. Thus W satisfies properties a, b and c of the lemma.

We now prove that W satisfies property d. The surface $i(S(L))$ is an embedded copy of $S(L)$ in W, which separates W into two components, W^+ and W^-. Let W^- be the component which is convex in a neighborhood of ℓ, and let $F = W^- \cap \partial W$.

The distance from ℓ to F may be approximated to arbitrary precision by lengths of geodesic segments β which are perpendicular to ℓ. Let β be such a segment, and let $\tilde{\beta}$ and $\tilde{\ell}$ be lifts to $\tilde{W} \subset \mathbb{H}^3$. Let A_1, A_2 be radius r, totally geodesic annuli in $S(L)$, with $A_1 \cap A_2 = \ell$, and let $\tilde{A}_i \subset \tilde{W}$ be a lift of $i(A_i)$ so that $\tilde{A}_1 \cap \tilde{A}_2 = \tilde{\ell}$. Let $Q^\perp$ be a geodesic plane containing $\tilde{\beta}$ and perpendicular to $\tilde{\ell}$, and let $q = Q^\perp \cap \tilde{\ell}$. Then $Q^\perp \cap \tilde{A}_i = \alpha_i$ is a geodesic segment of length at least r, and α_1 and α_2 meet at the point q in an angle θ. Let T be the triangle with sides α_1 and α_2, and let δ be the third side of T. By convexity, $T \subset \tilde{W}$, and so $\mathrm{length}(\beta) \geq d(q, \delta)$. Let T' be a triangle with two ideal vertices and regular vertex q, with angle θ, and let $s(\theta)$ be the distance from q to the opposite side of T'. Then as r becomes large, $d(q, \delta)$ approaches $s(\theta)$. Therefore $d(\ell, F) \approx \mathrm{length}(\beta) \approx s(\theta)$, and these approximations can be made arbitrarily close by increasing r. Therefore, after possibly increasing r and decreasing ϵ, we have that $d(\ell, F) > \epsilon$.

Let γ be some other geodesic in W. Then γ also represents a geodesic in Z. Since V contains a unique closed geodesic, and no infinite geodesics, then $\gamma \cap (Z - V) \neq \emptyset$. However, $(Z - V) \cong (S(L) \times [-\epsilon, \epsilon]) - N_r(\ell)$, and so γ is within ϵ of both boundary components of W. □

Next, some more notation. Suppose $f : S \to M$ is a totally geodesic immersion, and suppose $\ell \subset f(S)$ is a double curve, along which the angle of self-intersection is θ. Let $\ell_1, \ell_2 \subset S$ be two components of $f^{-1}\ell$, and let $L = \ell_1 \cup \ell_2$.

Suppose that ℓ_i has a collar neighborhood of radius $r = r(\theta)$, where r is the constant in Lemma 4.2. Let $N(\ell_i)$ be an embedded collar neighborhood of ℓ_i, and let $\partial N(\ell_i) = \{\ell_{i1}, \ell_{i2}\}$. Further suppose, for convenience, that L is non-separating. Let S^- be a compact surface homeomorphic to $S - intN(L)$. The hyperbolic structure on S induces a hyperbolic structure on S^- for which the components of $\partial S^- = \{\ell_{11}, \ell_{12}, \ell_{21}, \ell_{22}\}$ are all geodesics with the same length that ℓ has in S. Also, the orientation on S_i induces an orientation on ℓ_{ij}. Let $S(L)$ be the surface obtained from S^- by identifying ℓ_{1j} and ℓ_{2j}, so that their orientations agree, and let $W = W(L, \theta) \cong S(L) \times I$ be the hyperbolic 3-manifold given by Lemma 4.2. The immersion $f : S \to M$ induces an immersion $f : S^- \to M$, which induces an immersion $g : S(L) \to M$, and extends to an immersion $g : W \to M$.

Lemma 4.3 *If ℓ_1 and ℓ_2 have collar neighborhoods of radius $r(\theta)$ in S, then the immersion $g : S(L) \to M$ is π_1-injective.*

Proof Since W has convex boundary, the map $g : W \to M$ is π_1-injective, so $g : S(L) \to M$ is π_1-injective also. □

Lemma 4.3 provides a way of constructing a great many π_1-injective immersions. For suppose $\tilde{S}$ is a cover of S, and suppose that ℓ has two (1-1) lifts, $\tilde{\ell}_1$ and $\tilde{\ell}_2$ in $\tilde{S}$. We say that the pair $(\tilde{S}, \tilde{L} = \tilde{\ell}_1 \cup \tilde{\ell}_2)$ is *admissable* if the $\tilde{\ell}_i$'s have collar neighborhoods of radius r in $\tilde{S}$. If $(\tilde{S}, \tilde{L})$ is admissable, then by Lemma 4.3, the immersion $g : \tilde{S}(\tilde{L}) \to M$ is π_1-injective. Our goal is to show that different choices of admissable pairs will often result in inequivalent immersions; counting the number of choices for such pairs will then prove Theorem 1.2.

Suppose, then, that $(\tilde{S}, \tilde{L})$ and $(\tilde{S}', \tilde{L}')$ are admissable pairs. Let W be the convex thickening of $\tilde{S}(\tilde{L})$ provided by Lemma 4.2, with map $i : \tilde{S}(\tilde{L}) \to W$, and let $\ell = i(\tilde{L}) \subset W$. Let P be the universal cover of $\widetilde{S}^-$.

We let C be the universal cover of W; note that C is isometric to the convex hull of the limit set of $g_* \pi_1 \tilde{S}(\tilde{L})$. We also define analogous objects W', i', ℓ', π', $\tilde{S}'^-$, P' and C' for $(\tilde{S}', \tilde{L}')$.

Recall from the previous section that the pairs $(\tilde{S}, \tilde{L})$ and $(\tilde{S}', \tilde{L}')$ are *inequivalent* if P and P' are not isometric.

Lemma 4.4 *If $(\tilde{S}, \tilde{L})$ and $(\tilde{S}', \tilde{L}')$ are admissable, inequivalent pairs, then C and C' are not isometric.*

Proof Suppose $\phi : C \to C'$ is an isometry. Identify P with a component of the pre-image of $\widetilde{S}^-$ in C. By Property d of Lemma 4.2, $\phi(\pi^{-1}\ell) = \pi'^{-1}\ell'$, so $\phi(P)$ is a totally geodesic surface in C', bounded by a subset of $\pi'^{-1}\ell'$. We claim that $\phi(P)$ is isometric to P'.

To see this, let A be a separating, geodesic annulus embedded in W', containing ℓ'. Since $\phi(P) \cap \partial C = \emptyset$, each component of $\pi'^{-1}A \cap \phi(P)$ is an infinite geodesic; since each component of $\pi'^{-1}A$ contains a unique infinite geodesic, then each component of $\pi'^{-1}A \cap \phi(P)$ is a lift of ℓ'. Let Q' be the closure of a component of $\phi(P) - \pi'^{-1}\ell'$, and let δ be a geodesic arc connecting two boundary components of Q'. Since $\phi(P) \cap \pi'^{-1}\ell' = \phi(P) \cap \pi'^{-1}A$, then $\pi'\mathring{\delta} \subset W' - A$, and since $W' - A$ retracts onto the totally geodesic surface $i(\tilde{S}' - \ell')$, and $\pi'\delta$ is a geodesic, then $\pi'\delta \subset i(\tilde{S}'^-)$. Therefore we may fix a copy of P' in C' which contains at least two boundary components of Q'. This implies that P' and Q' are contained in the same geodesic plane H in C'. Both P' and Q' are the closure of the unique component of $H - \pi'^{-1}\ell'$ which contains $\mathring{\delta}$, and therefore $P' = Q'$.

Finally, if R' is any other component of $\phi(P) - \pi'^{-1}(\ell')$, then $\pi' P' = \pi' Q' = \pi' R' = \tilde{S}'^-$, which forces Q' and R' to meet at an angle θ, contradicting the fact that $\phi(P)$ is totally geodesic. Therefore $\phi(P) = Q' = P'$. □

Proof (Of Theorem 1.2) We have already proved the upper bound, and we now prove the lower bound. We are given a totally geodesic immersion of a hyperbolic surface $f : S \to M$, with a loop of transverse intersection ℓ, at an angle θ. Let $r = r(\theta)$ be the constant given by Lemma 4.2. Let ℓ_1 and ℓ_2 be components of $f^{-1}\ell$, and let $L = \{\ell_1 \cup \ell_2\}$. Since $\pi_1 S$ is LERF (by [7]), there is a finite cover $\pi : \tilde{S} \to S$ in which L lifts to a non-separating curve with collar neighborhood of radius r. Replacing the immersion $f : S \to M$ with the immersion $f\pi : \tilde{S} \to M$, we may assume that L is non-separating, with a collar neighborhood of radius at least r. Then any further cover $(\tilde{S}, \tilde{L})$ of (S, L) (where $\tilde{L}$ is a 1-1 lift of L) is admissable.

By Lemma 3.4, the number of admissable, inequivalent covers $(\tilde{S}, \tilde{L})$ of (S, L) grows factorially in the covering degree of $\tilde{S}$, which is proportional to the genus of the resulting immersed surfaces. Since the pairs are admissable, the corresponding immersions are π_1-injective by Lemma 4.3. Since the pairs are inequivalent, Lemma 4.4 implies that the corresponding subgroups of $\pi_1 M$ are inequivalent under the relation generated by commensurability and conjugacy. Theorem 1.2 follows. □

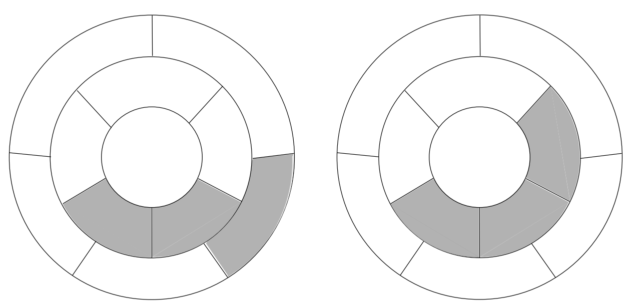
Remark The proof actually shows that the surface subgroups are inequivalent under the relation generated by commensurability and conjugacy *in* $PSL_2(C)$.

5 Example: Reflection orbifolds

Now we consider the special case of right-angled reflection orbifolds. These always have transversely immersed, totally geodesic surfaces (corresponding to any pair of adjacent faces) and so Theorem 1.2 applies. However, in this case there are combinatorial proofs which give more explicit information.

We first give a combinatorial construction of immersed incompressible surfaces in right-angled reflection orbifolds. This is a well-known construction, and the treatment here is similar to that of [1]. We will need the following definitions and notation.

Let P be a right-angled polyhedron, let $\Gamma(P)$ be the group generated by reflections in the faces of P, and let $\Gamma^+(P)$ denote the subgroup of orientation-preserving elements in P. We define a *face disk* of P to be a connected and simply connected union of faces of P, and a *face loop* on P to be a collection of n faces F_i such that F_i is adjacent to F_j if and only if $|i-j| = 1 \pmod{n}$. Let D be a face disk of P. We say that D satifies the *convexity condition* if the faces of P transverse to D form a face loop (see Figure 4).



(a) (b)

Figure 4: (a) A face disk satisfying the convexity condition (b) A face disk which does not satisfy the convexity condition

If F is a face in P, we let $\rho_F : \mathbb{H}^3 \to \mathbb{H}^3$ denote reflection in the plane containing F. If X and Y are spaces, and ρ is a homeomorphism from a subset of X to a subset of Y, let $X \cup_\rho Y$ denote X glued to Y via the gluing map ρ.

The incompressibility results of this section are essentially an elaboration of the following elementary fact: given a collection of planes in $\mathbb{H}^3$, each meeting the next at right angles, and such that non-consecutive planes are disjoint, then the group generated by their reflections is isomorphic to the obvious 2-dimensional reflection group.

Lemma 5.1 *Suppose that P_1 and P_2 are right-angled polyhedra in $\mathbb{H}^3$, that $D_i \subset P_i$ are face disks satisfying the convexity condition, and that $F_i \subset P_i$ are faces transverse to D_i, with $D_i \cap F_i = e_{i1}, ..., e_{im}$. Suppose there is an orientation-reversing isometry $\rho : F_1 \to F_2$ such that $\rho(F_1, e_{11}, ..., e_{1m}) = (F_2, e_{21}, ..., e_{2m})$. Then $D_1 \cup_\rho D_2$ satisfies the convexity condition in $P_1 \cup_\rho P_2$.*

Proof Since P_i are right-angled polyhedra, $P_1 \cup_\rho P_2$ is a right-angled polyhedron also.

The lemma then follows from the easily verified fact that the amalgamation of two face loops along a common face is again a face loop. □

Since D abuts the reflection planes of P in right angles, it defines a closed 2-orbifold $\bar{D}$ in $\mathbb{H}^3/\Gamma(P)$ by making the edges of D into reflectors. The image of $\pi_1\bar{D}$ in Γ is generated by the reflections in the faces of P which D interesects transversely. The following result about the incompressibility of $\pi_1\bar{D}$ is a reformulation of a well-known property of Coxeter groups.

Proposition 5.2 *If D satisfies the convexity condition, then the fundamental group of $\bar{D}$ injects into $\Gamma(P)$.*

Proof We fix a copy of D and P in $\mathbb{H}^3$. It will suffice to show that the union of D with its translates under $\pi_1(\bar{D})$ forms an embedded disk.

Since $\bar{D}$ is a right-angled reflection orbifold, it may be developed in $\mathbb{H}^3$ by successively doubling (P, D) across the faces of P which D intersects transversely.

Let F be a face of P which D intersects transversely, and let $P_2 = P \cup \rho_F P$. Then D develops in P_2 as $D \cup \rho_F D$. By Lemma 5.1, $D \cup \rho_F D$ satisfies the convexity condition in $P \cup \rho_F P$. In particular, $D \cup \rho_F D$ is a disk.

Iterating this process, we obtain after each doubling an embedded disk. It follows that the union of D with all its translates is embedded and simply connected. □

Suppose that $\widehat{P} \subset \mathbb{H}^3$ is a right-angled polyhedron which is made up of copies of P, and that D is a face disk in $\widehat{P}$ satisfying the convexity condition. By Theorem 5.2, the reflection orbifold $\bar{D}$ is incompressible in $\mathbb{H}^3/\Gamma(\widehat{P})$, so under the covering map $\pi : \mathbb{H}^3/\Gamma(\widehat{P}) \to \mathbb{H}^3/\Gamma(P)$, it projects to an immersed incompressible 2-orbifold $\pi\bar{D}$. As we did in Section 3, we can cut and paste $\pi\bar{D}$, to create many different immersions of 2-orbifolds in $\mathbb{H}^3/\Gamma(P)$.

Let $e_1, e_2 \subset \partial D$ be edges of $\widehat{P}$, let F_i be the faces adjacent to e_i in D, and let F_i' be the face adjacent to e_i in $\widehat{P} - D$. Suppose there is an orientation-preserving element $\gamma \in \Gamma(P)$ such that $\gamma(F_1, e_1) = (F_2, e_2)$. We "cut" $\bar{D}$ by first making e_1, e_2 into boundary edges (not reflector edges); call this orbifold $\bar{D}'$. We then "paste" by forming the quotient orbifold $\bar{D}'/\rho_{F_2'}\gamma$. Since $\rho_{F_2}\gamma$ takes e_1 to e_2, this is a closed 2-orbifold, and since $\rho_{F_2}\gamma \in \Gamma(P)$, there is a natural map from $\bar{D}'/\rho_{F_2}\gamma$ into $\mathbb{H}^3/\Gamma(P)$.

We repeat this procedure at different edges of D, and encode the collection of gluings by an order 2 permutation σ, defined on the edges of D. Let D_σ denote the immersed, closed 2-orbifold in $\mathbb{H}^3/\Gamma(P)$ obtained by cutting and pasting $\pi\bar{D}$ according to the permutation σ, and let $f_\sigma : D_\sigma \to \mathbb{H}^3/\Gamma(P)$ be the associated immersion. We define the *orbifold degree* of a vertex v of D_σ to be the degree of v in the universal cover of D_σ.

Proposition 5.3 *If D satisfies the convexity condition, and if each vertex of D_σ has (orbifold) degree 4, then the immersion $f_\sigma : D_\sigma \to \mathbb{H}^3/\Gamma(P)$ is π_1-injective.*

Proof The proof is similar to the proof of Theorem 5.2. We fix a copy of $\widehat{P}$ and D in $\mathbb{H}^3$ and show that the translates of D under $\pi_1(D_\sigma)$ form an embedded disk.

The translates of D under $\pi_1(D_\sigma)$ may be obtained by successively doubling $\widehat{P}$ via the various symmetries in $\Gamma(P)$ which correspond to the permutation σ. After the first doubling, then, D is glued to ρD along some edge, via some $\rho \in \Gamma(P)$. The faces D and ρD satisfy the convexity condition, in $\widehat{P}$ and $\rho\widehat{P}$ respectively, so by Lemma 5.1, their union satisfies the convexity condition in $\widehat{P} \cup \rho\widehat{P}$.

Doubling again along a face adjacent to the initial doubling face, we get four copies of D around a vertex, which by the vertex condition will glue appropriately. Again, by Lemma 5.1, this union will be a face disk satisfying the convexity condition.

Iterating this process, we obtain after each doubling an embedded disk, and therefore the union of D with all its translates is an embedded disk. □

We now give an explicit construction of a family of inequivalent immersed surfaces in a right-angled reflection orbifold.

We are given a right-angled polyhedron P. Compact Coxeter polyhedra in $\mathbb{H}^3$ have vertices of degree three, and no faces with fewer than five edges; an Euler characteristic computation then shows that every Coxeter polyhedron has a face with exactly five edges.

Let F_1 be a face of P with five edges. We paste together n copies of F_1, and attach one additional adjacent face F_2, as indicated in Figure 5; the resulting complex, which we call D, is a face disk for a right-angled Coxeter polyhedron $\widehat{P}$ made up of n copies of P. We cut and paste the reflection orbifold $\bar{D}$ along alternating edges of D, as shown in Figure 5. Associated to any order 2 permutation, σ, of these edges there is an edge pairing, and a resulting 2-orbifold D_σ, with an immersion $f_\sigma : D_\sigma \to \mathbb{H}^3/\Gamma(P)$.

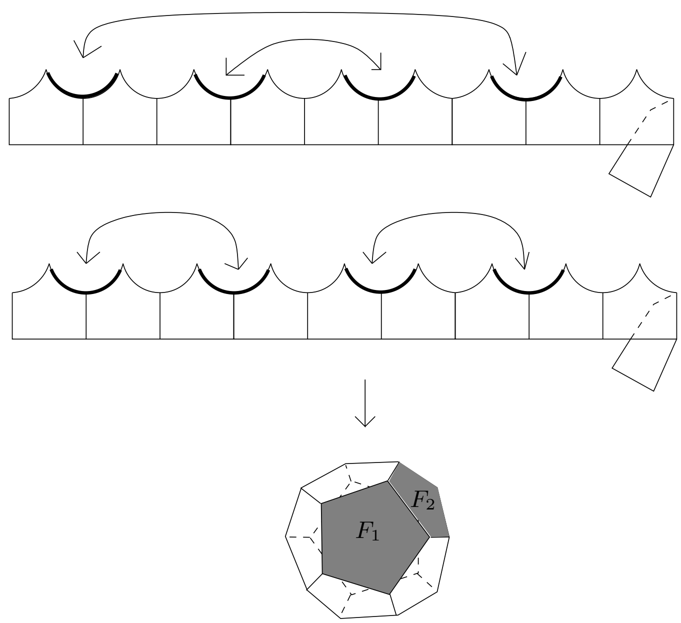


Figure 5: Different gluings will result in inequivalent surfaces

Lemma 5.4 *The immersion $f_\sigma : D_\sigma \to \mathbb{H}^3/\Gamma(P)$ is π_1-injective.*

Proof The union of two adjacent faces of a compact Coxeter polyhedron always satisfies the convexity condition, since such a polyhedron cannot have face loops of length less than five. Therefore D satisfies the convexity condition.

Also, since σ has order two, the vertices of D_σ all have orbifold degree four, and so the lemma follows from Proposition 5.3. ◻

The resulting immersions are in fact inequivalent. The key point is that we have control over the limit sets.

Let $\tilde{P}$ be a fixed lift of P to $\mathbb{H}^3$, and let $\tilde{D} \subset \tilde{P}$ be a fixed lift of D, which determines a fixed map $f_{\sigma *}\pi_1 D_\sigma \to \Gamma(P)$. Let $\Lambda(f_{\sigma *}\pi_1 D_\sigma)$ denote the limit set of $f_{\sigma *}\pi_1 D_\sigma$ in $\mathbb{H}^3$. Let $\mathcal{H} = \{H_1, ..., H_n\} \subset \mathbb{H}^3$ denote the half-spaces which intersect $\tilde{P}$ in faces orthogonal to $\tilde{D}$; we label these so that $H_i \cap H_{i+1} \neq \emptyset$, and $H_i \cap H_j = \emptyset$ if $|i-j| \geq 2 \pmod n$. Then we have:

Lemma 5.5

(1) $\Lambda(f_{\sigma *}\pi_1 D_\sigma) \cap H_i \neq \emptyset$ *for all* i.

(2) $\Lambda(f_{\sigma *}\pi_1 D_\sigma) \subset \bigcup_{i=1}^n H_i$.

Proof Each H_i is stabilized by $< \rho_{H_{i-1}}, \rho_{H_{i+1}} >$. Since H_{i-1} and H_{i+1} are non-adjacent, this group is infinite, and so it has a limit point in H_i, proving 1.

Since every point in $\Lambda(f_{\sigma *}\pi_1 D_\sigma)$ is a limit of a sequence of points in $\tilde{D}_\sigma$, then to prove 2, it is enough to show that $\tilde{D}_\sigma \subset \bigcup_{i=1}^n H_i \cup \tilde{D}$.

As in the proof of Lemma 5.3, $\tilde{D}_\sigma$ may be constructed by successive doublings. More precisely, there is a sequence of Coxeter polyhedra $\tilde{P} = P_1, P_2, ...$, and face disks $D_i \subset P_i$, such that $(P_{i+1}, D_{i+1}) = (P_i, D_i) \cup (\gamma_i P_i, \gamma_i D_i)$, for some $\gamma_i \in f_{\sigma *}\pi_1 D_\sigma$, and $\tilde{D}_\sigma = \bigcup_i D_i$. Let $\mathcal{Q}_i$ be the set of hyperbolic planes of P_i which are transverse to D_i.

Claim For every i, if $Q \in \mathcal{Q}_i$,then $Q \subset \bigcup_{H_j \in \mathcal{H}} H_j$.

Proof of claim The statement is true for $i = 1$, and suppose it is true for $i = n$. Let J_n be the plane fixed by γ_n, and suppose without loss of generality that $J_n \subset H_1$. All of the planes in $\mathcal{Q}_{n+1} - \mathcal{Q}_n$ correspond to faces of $\rho_n P_n$ which are not adjacent to the face containing J_n. Since $\gamma_n P_n$ is a Coxeter polyhedron, these planes are disjoint from J_n, and therefore J_n separates every plane in $\mathcal{Q}_{n+1} - \mathcal{Q}_n$ from $\tilde{D}$. Therefore every plane in $\mathcal{Q}_{n+1} - \mathcal{Q}_n$ is contained in H_1, and so every plane in $\mathcal{Q}_{n+1}$ is contained in some H_j. The claim follows by induction.

From the claim it follows that $D_j \subset \bigcup_i H_i$ for all j, and so $\tilde{D}_\sigma \subset \bigcup_i H_i$, as required. ◻

Let σ and σ' be distinct order two permutations. We choose arbitrary lifts $\tilde{D}$ and $\tilde{D}'$ of D in $\mathbb{H}^3$, and these determine maps $f_{\sigma *} : \pi_1 D_\sigma \to \Gamma(P)$ and $f_{\sigma' *} : \pi_1 D_{\sigma'} \to \Gamma(P)$. Let $\Gamma = f_{\sigma *}\pi_1 D_\sigma$, and $\Gamma' = f_{\sigma' *}\pi_1 D_{\sigma'}$.

Proposition 5.6 *No conjugate of* Γ *is commensurable with* Γ' *in* $\Gamma(P)$*.*

Before proving this, we need one further lemma. Let $A, B \subset \mathbb{H}^3$, and let $\mathcal{Q}$ be the collection of all the geodesic planes in $\mathbb{H}^3$ which project to ∂P. Define the *combinatorial distance* $d_P(A, B)$ to be the number of planes in $\mathcal{Q}$ which separate A and B. Let $\mathcal{H} = \{H_1, ..., H_k\}$ be the half spaces perpendicular to $\partial\tilde{D}$. For each H_i, there is an element $\gamma_i \in \Gamma$ such that $\gamma_i \tilde{D} \cap \tilde{D} \subset \partial H_i$.

Lemma 5.7 *Suppose* $A \subset \mathbb{H}^3$*, and that* $H_i \supset A$*. Then* $d_P(\gamma_i \tilde{D}, A) < d_P(\tilde{D}, A)$*.*

Proof Otherwise there is a plane $Q \in \mathcal{Q}$ which separates $\gamma_i \tilde{D}$ from A but does not separate $\tilde{D}$ from A. Then Q must separate a subset of $\tilde{D}$ from $\gamma_i \tilde{D}$, and so $Q \cap \mathring{\tilde{D}} \neq \emptyset$. Also, $Q \neq \partial H_i$, and so there is a j such that $Q, \partial H_i$, and ∂H_j all intersect, which forces Q to be adjacent to H_i, contradicting the assumption that $Q \cap \mathring{\tilde{D}} \neq \emptyset$. □

Proof of Proposition 5.6 Since Γ and Γ' were chosen only up to an arbitrary conjugation (depending on the choice of lifts $\tilde{D}$ and $\tilde{D}'$), it is enough to show that Γ is not commensurable with Γ', and for this, it is enough to show that $\Lambda(\Gamma) \neq \Lambda(\Gamma')$.

Suppose that $\Lambda(\Gamma) = \Lambda(\Gamma')$, and suppose also that $\tilde{D} \neq \tilde{D}'$. Let $\mathcal{H} = \{H_1, ..., H_k\}$ and $\mathcal{H}' = \{H_1', ..., H_k'\}$ be the half spaces perpendicular to $\partial\tilde{D}$ and $\partial\tilde{D}'$, let $\gamma_i \in \Gamma$ such that $\gamma_i \tilde{D} \cap \tilde{D} \subset \partial H_i$, and define γ_i' similarly. Then Lemma 5.5 implies that every H_i' is contained in some H_j, and in particular $\tilde{D}' \subset H_j$ for some j. By Lemma 5.7, $d_P(\gamma_j \tilde{D}, \tilde{D}') < d_P(\tilde{D}, \tilde{D}')$, and we may replace $\tilde{D}$ by $\gamma_j \tilde{D}$, without affecting $\Lambda(\Gamma)$. Eventually, we have $d_P(\tilde{D}, \tilde{D}') = 0$, so $\tilde{D}$ and $\tilde{D}'$ intersect in some lift of P.

Recall $\tilde{D}$ and $\tilde{D}'$ both consist of n lifts of F_1 and a single lift of F_2. If $\tilde{D}$ and $\tilde{D}'$ overlap on some lift of F_1 and differ on the lift of F_2, then there is some H_j (adjacent to the lift of F_2) which is disjoint from every H_i', contradicting Lemma 5.5. So $\tilde{D}$ and $\tilde{D}'$ contain the same lift of F_2, and it then follows that $\tilde{D}$ and $\tilde{D}'$ co-incide.

We now have $\tilde{D} = \tilde{D}'$, but by assumption, the gluings σ and σ' disagree on some edge, which lifts to some H_i. Then $\mathcal{F} = \tilde{D} \cup \gamma_i \tilde{D}$ and $\mathcal{F}' = \tilde{D} \cup \gamma_i' \tilde{D}$ are face disks in a Coxeter polyhedron $\widehat{P}$ (made of $2n$ copies of $\tilde{P}$) which satisfy the convexity condition. Let $\mathcal{H}$ and $\mathcal{H}'$ be the sets of half-spaces transverse to $\mathcal{F}$ and $\mathcal{F}'$. Since σ and σ' disagree, there is an $H_i \in \mathcal{H}$ which is disjoint from every $H_j \in \mathcal{H}'$. Lemma 5.5 then shows that $\Lambda(\Gamma) \neq \Lambda(\Gamma')$, which is a contradiction. □

The lower bound given in Theorem 1.3 can be computed from Proposition 5.6 as follows: first we must modify the construction of D_σ to produce closed, non-singular surfaces, instead of 2-orbifolds. This can be done by starting with 4 copies of F_1 and $4n$ copies of F_2, which can be glued into a closed surface. There $4((n-1)/2) = 2n-2$ edges (corresponding to the dark edges of Figure 5) along which we may cut and paste via an arbitrary transposition σ, to get an immersion of a closed surface S_σ. By the same method as the proof of Proposition 5.6, it can be shown that different choices of σ result in inequivalent, π_1-injective immersions.

The Euler characteristic of S_σ is $-n-c$, for some fixed constant c depending on the number of sides of F_2, and there are at least $(n-1)!$ choices of transpositions σ resulting in inequivalent immersions. We have $g = (n+c)/2+1$, so $n-1 = 2g-c-3$, and so $s_1(M,g) > (2g-c-3)! > e^{g \log g}$, for large g.

Finally, we give a combinatorial proof of the upper bound in this setting. The first step is to show that every incompressible immersed surface in a reflection orbifold can be homotoped to a standard form.

We view $\mathbb{H}^3/\Gamma(P)$ as an orbifold, with an underlying polyhedral complex determined by P.

Lemma 5.8 *Let S be a closed surface of positive genus, and let $\Gamma \subset \Gamma(P)$, with $\Gamma \cong \pi_1 S$. Then there is a polyhedral 2-complex K on S, and a immersion $f : S \to \mathbb{H}^3/\Gamma(P)$, which is cellular with respect to K and P, such that $f_*(\pi_1 S) = \Gamma$, and such that the universal cover of S lifts to an embedded plane in $\mathbb{H}^3$.*

Proof There is a unique cover $\pi : M \to \mathbb{H}^3/\Gamma(P)$ such that $Image(\pi_*) = \Gamma$. The pre-image of P in M defines a polyhedral complex $K(P)$ on M. By Scott's compact core theorem, M has a compact core M^-, such that $\pi_1(M^-) \cong \Gamma$, and Waldhausen's theorems, $M^- \cong S \times I$. Let $K(P)^-$ denote the polyhedra of $K(P)$ which intersect M^-. There is an embedding $i : S \to \partial K(P^-)$, and the polyhedral structure of $\partial K(P^-)$ pulls back to a polyhedral structure K on S. Then $\pi i : S \to \mathbb{H}^3/\Gamma(P)$ is the desired immersion. □

Let $\tilde{P}$ be a lift of P to $\mathbb{H}^3$. A sub-complex $\mathcal{F}$ of the 2-skeleton of K is a *face disk* if $f|_{\mathcal{F}}$ lifts to a face disk of $\tilde{P}$. If v is a vertex of $\partial\mathcal{F}$, $deg_K(v)$ denotes the degree of v in K, whereas $deg_{\mathcal{F}}(v)$ denotes the degree of v in $\mathcal{F}$. Let $\mathcal{V}(\mathcal{F}, i) = \{v \in \partial\mathcal{F} | deg_{\mathcal{F}}(v) = i\}$.

We require the following general fact about compact Coxeter polyhedra in $\mathbb{H}^3$.

Lemma 5.9 *Let $\mathcal{F}$ be a face disk of P. If $|\mathcal{V}(\mathcal{F}, 2)| \leq 4$, then $\mathcal{F}$ contains more than half of the faces of P.*

Proof Let $\mathcal{F}' = P - \mathcal{F}$. If $|\mathcal{V}(\mathcal{F}, 2)| = 0$, then $\mathcal{F}'$ is a single face. If $|\mathcal{V}(\mathcal{F}, 2)| = 1$, then $|\mathcal{V}(\mathcal{F}', 3)| = 1$, which implies the existence of an edge in P which is adjacent to only one face, which is impossible. If $|\mathcal{V}(\mathcal{F}, 2)| = 2$, then $\mathcal{F}'$ consists of exactly two faces of P. If $|\mathcal{V}(\mathcal{F}, 2)| = 3$, then $\mathcal{F}'$ consists of three faces meeting at a common vertex. If $|\mathcal{V}(\mathcal{F}, 2)| = 4$, then, adjacent to its boundary $\mathcal{F}'$ contains faces $F_1, ..., F_4$, with $F_i \cap F_{i+1} \neq \emptyset$ (mod 4). Since P is a right-angled, compact Coxeter polyhedron, it contains no face loops of length three or four, and no vertices of degree four, and so this is impossible.

We conclude that $\mathcal{F}'$ contains at most three faces. Since a compact, right-angled Coxeter polyhedron in $\mathbb{H}^3$ must contain at least twelve faces, the result follows. □

A face disk $\mathcal{F}$ of K is *special* if

(1) $deg_K(v) \geq 4$ for all $v \in \partial\mathcal{F}$, and

(2) $|\mathcal{V}(\mathcal{F}, 2)| \geq 5$.

Lemma 5.10 *There is a choice of f and K as in Lemma 5.8, such that K has a partition into special face disks.*

We say that the resulting immersed surface S is in *standard form.* For a complex K, let $K^{(i)}$ denote the i-skeleton, and let $|K^{(i)}|$ denote the cardinality of $K^{(i)}$.

Proof Let f and K be as given by Lemma 5.8. If F_1, F_2 are in $K^{(2)}$, say that $F_1 \equiv F_2$ if there is a vertex $v \in F_1 \cap F_2$ with $deg_K(v) = 3$. Let $\mathcal{F}_1, ..., \mathcal{F}_n$ be the equivalence classes of $\equiv$. Every $\mathcal{F}_i$ lifts to a single copy of P in $\mathbb{H}^3$, and so $\mathcal{F}_i$ is a face disk for all i.

Suppose $|\mathcal{V}(\mathcal{F}_i, 2)| \leq 4$ for some i. Then we enlarge $\mathcal{F}_i$ to a maximal face disk, by adding as many faces of $K^{(2)}$ as possible. Let $\mathcal{F}_i'$ denote $\partial P - f(\mathcal{F}_i)$. Then $f|_{\mathcal{F}}$ can be homotoped rel. boundary, and the polyhedral structure on S can be changed, so that $f(\mathcal{F}_i) = \mathcal{F}_i'$, and f remains a cellular immersion. Also, when lifted to the universal cover, this homotopy is supported on a single copy of P, and one may check that the lift of f remains an embedding. We then obtain new equivalence classes, and repeat the process. By Lemma 5.9, every step reduces $|K^{(2)}|$, so the process must eventually terminate in a special complex. □

Suppose now that $f : K \to \mathbb{H}^3/\Gamma(P)$ is as given by Lemma 5.10. Let K' be the corresponding complex of special face disks.

Lemma 5.11 *If v is a vertex of K with $deg_K(v) \leq 5$, then $deg_{K'}(v) \geq 4$.*

Proof Suppose $deg_{K'}(v) \leq 3$. Let $\mathcal{F}_i$, $i = 1, ..., n \leq 3$ be the face disks of K' which are adjacent to v. Lift f to an embedding $\tilde{f} : \tilde{S} \to \mathbb{H}^3$, choose a lift $\tilde{v}$ of v, and let $\tilde{\mathcal{F}}_i$ be the lift of $\mathcal{F}_i$ adjacent to v. Then $\tilde{f}\tilde{\mathcal{F}}_i$ is a face disk of some lift of P. Since P is a right-angled polyhedron, then in a neighborhood of $\tilde{v}$, the tesellation of $\mathbb{H}^3$ is equivalent to the coordinate planes of $\mathbb{R}^3$. Since $\tilde{f}\tilde{\mathcal{F}}_i$ is a face disk, then in a neighborhood of $\tilde{v}$, $\tilde{f}\tilde{\mathcal{F}}_i$ is contained in a single octant. Since $n \leq 3$ and $\bigcup_i \tilde{f}\tilde{\mathcal{F}}_i$ is neighborhood of $\tilde{f}(\tilde{v})$, then either $deg_K(v) \geq 6$ or $\bigcup_i \tilde{f}\tilde{\mathcal{F}}_i$ is contained in a single octant. However in the latter case $deg_K(v) = 3$, so the face disks $\mathcal{F}_i$ are not special, for a contradiciton. □

Lemma 5.12 *There is a fixed constant $c > 0$ such that, if $f : S \to \mathbb{H}^3/\Gamma(P)$ is a cellular immersion with respect to a polyhedral structure K on S, and if f is in standard form, then $genus(S) \geq c|K^{(2)}|$.*

Proof Since f is in standard form, the 2-skeleton of K has a partition into special face disks. Let K' be the corresponding complex of S, and let $\mathcal{F} \in K'^{(2)}$.

Let $v \in \mathcal{V}(\mathcal{F}, 2)$, and suppose $deg_{K'}(v) \leq 3$. Since $deg_{\mathcal{F}}(v) = 2$, then we have $deg_K(v) < 2 * deg_{K'}(v) \leq 6$. Therefore by Lemma 5.11 $deg_{K'}(v) \geq 4$, for a contradiction. Therefore every $v \in \mathcal{V}(\mathcal{F}, 2)$ has $deg_{K'}(v) \geq 4$. Since $\mathcal{F}$ is special, $|\mathcal{V}(\mathcal{F}, 2)| \geq 5$, and so

(*) each $\mathcal{F}$ has at least five vertices v with $deg_{K'}(v) \geq 4$.

Let V, E and F represent the number of vertices, edges and faces in the complex K'. For $\mathcal{F} \in K'^{(2)}$, let $V(\mathcal{F}, n)$ be the number of vertices which are distinct in $\mathcal{F}$ (not necessarily in K') and which have degree n in K'. We have:

$$\begin{aligned}
\chi(S) &= V - E + F \\
&= V + (-1/2)(\sum_{v \in K'^{(0)}} deg(v)) + F \\
&= -1/2(\sum_{v \in K'^{(0)}} deg(v) - 2) + F \\
&= -1/2\left(\sum_{\mathcal{F}\in K'^{(2)}} 1\frac{V(\mathcal{F},3)}{3} + \sum_{\mathcal{F}\in K'^{(2)}} 2\frac{V(\mathcal{F},4)}{4} + \sum_{\mathcal{F}\in K'^{(2)}} 3\frac{V(\mathcal{F},5)}{5} + ...\right) + F \\
&\leq -1/2\left(\sum_{\mathcal{F}\in K'^{(2)}} \frac{1}{2}[V(\mathcal{F},4) + V(\mathcal{F},5) + ...]\right) + F \\
&\leq -1/2\left(\frac{1}{2}5F\right) + F, \text{ by (*)} \\
&= -\frac{1}{4}F
\end{aligned}$$

Since each face disk $\mathcal{F}$ can contain at most $|P^{(2)}|$ faces of P, then $|K^{(2)}| \leq |P^{(2)}||K'^{(2)}|$, and the lemma follows. □

Combining Lemma 5.12 with Lemma 5.8, we get the following, which may be of some independent interest:

Proposition 5.13 *For P and M as in Theorem 1.3, there is a constant c such that every immersed incompressible genus g surface in M can be homotoped to a union of at most cg faces of P.*

We now resume the proof of the upper bound.

By Lemma 5.12, the total number of maximal face disks is linear in the genus, and so the number of edges on the boundary of these faces is linear in the genus also. The number of choices for the face-disks is then exponential in g, and the number of ways of gluing the face disks along the boundary edges is factorial in g, resulting in a factorial upper bound for the total number of immersions.

We now sketch the proof of the explicit bound given in Theorem 1.3. Let K' be a special face-disk complex for S. The above analysis gives $s_2(M, g) \leq$

$e^{c_1 g}(|K'^{(1)}|)!$, for some constant c_1. The proof of Lemma 5.12 shows that $\chi(S) \leq -\frac{1}{4}|K'(2)|$, so $|K'^{(2)}| \leq 8g - 8$. Let $c(P)$ be the maximum number of edges in a face disk of P. Then $|K'^{(1)}| \leq c(P)|K'^{(2)}|$, and so $s_2(M,g) \leq e^{c_1 g}((8g-8)c(P))!$, and so for large g, $s_2(M,g) \leq e^{(8c(P)+1)g\log(g)}$.

6 Further Questions

Theorem 1.2 prompts the question:

Question 1 Does the conclusion of Theorem 1.2 hold for every closed, hyperbolic 3-manifold?

Of course, even proving that $s(g,M)$ is not constantly zero on any M is very difficult.

It is noteworthy that, in the examples of Theorem 1.2, the factorial growth was achieved entirely by quasi-Fuchsian surfaces. If Thurston's virtual fibering conjecture is true, there should always exist in addition a certain number of geometrically infinite immersions. However, the general constructions of π_1-injective immersions which are currently known seem to produce geometrically infinite examples only sporadically.

Question 2 Let M be a closed hyperbolic 3-manifold, and let $i(g)$ denote the number of conjugacy classes of maximal, geometrically infinite surface groups of genus at most g in $\pi_1 M$. Does $i(g)/s(g) \to 0$ as $g \to \infty$? Does $i(g)$ have sub-factorial (or even polynomial) growth?

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