ISSN 1472-2739 (on-line) 1472-2747 (printed) **577**

Algebraic & Geometric Topology
Volume 5 (2005) 577–613
Published: 30 June 2005

Surfaces in the complex projective plane and their mapping class groups

Susumu Hirose

Abstract An orientation preserving diffeomorphism over a surface embedded in a 4-manifold is called extendable, if this diffeomorphism is a restriction of an orientation preserving diffeomorphism on this 4-manifold. In this paper, we investigate conditions for extendability of diffeomorphisms over surfaces in the complex projective plane.

AMS Classification 57Q45; 57N05, 20F38

Keywords Knotted surface, plane curve, mapping class group, spin mapping class group

Dedicated to Professor Yukio Matsumoto for his 60th birthday

1 Introduction

There are deformations of embedded surfaces in 4-manifolds which induce isotopically non-trivial diffeomorphisms on surfaces. We introduce two typical examples.

For the first example, we consider a deformation of an annulus embedded in $S^3 \times [-1,1]$ so that, under this deformation, the boundary of this annulus is fixed. Let $S^1 \times [0,1]$ be an annulus embedded in $S^3 \times \{0\} \subset S^3 \times [-1,1]$, and $t\colon S^3 \times [-1,1] \to [-1,1]$ a projection to the second factor. We deform $S^1 \times [0,1]$ as in Figure 1. First, we isotope $S^1 \times [0,1]$ in S^3 from (1) to (3). Next, we isotope $S^1 \times [0,1]$ so that outside of the annulus A of (3) $t = 0$, and inside $t > 0$. Then we isotope $S^1 \times [0,1]$ inside A so that, when we push A down to $S^3 \times \{0\}$, $S^1 \times [0,1]$ is as in (4). Finally, we isotope $S^1 \times [0,1]$ in S^3 from (4) to (6). The composition of these deformations induce a square of Dehn twist about the core circle $S^1 \times \{\frac{1}{2}\}$ of $S^1 \times [0,1]$.

For the second example, we consider a deformation of a non-singular plane curve of degree 3. A torus is defined as a quotient of the complex plane by a lattice $\mathbb{Z} + \mathbb{Z}\sqrt{-1}$. We embed this torus into the complex projective plane

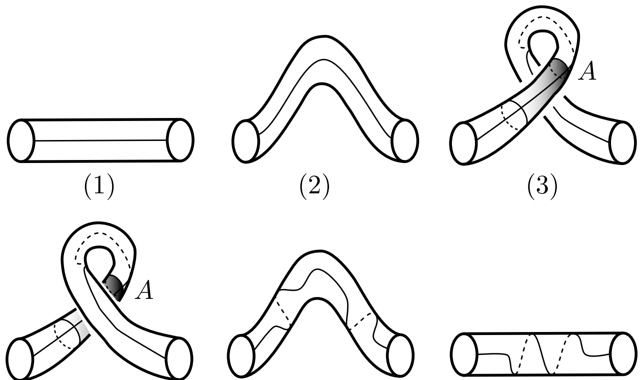


Figure 1

by using the Weierstrass $\wp$ function associated to this lattice, then the image of this embedding is a non-singular plane curve of degree 3. We deform this lattice, $\mathbb{Z} + \mathbb{Z}(\sqrt{-1} + t)$, where $0 \leq t \leq 1$ is a parameter of this deformation. Then the embedding is deformed isotopically and, finally (when $t = 1$), brought back to the original position. This deformation induces a Dehn twist on the non-singular plane curve of degree 3.

In this paper, we investigate a topological meaning of the above phenomena.

We settle a general formulation. Let M be a simply connected compact oriented smooth 4-manifold (possibly with boundary) and F be a compact oriented smooth 2-manifold (possibly with boundary) embedded in M. We call the pair (M, F) a *knotted surface*. In particular, if F is characteristic, that is to say, $F \cdot X \equiv X \cdot X \mod 2$ for any $X \in H_2(M, \mathbb{Z})$, then we call this pair (M, F) a *knotted characteristic surface*. An orientation preserving diffeomorphism ψ over F is *extendable* if there is an orientation preserving diffeomorphism Ψ over M such that $\Psi|_F = \psi$. In general, for an oriented manifold A and its submanifold B, we denote

$$\mathrm{Diff}_+(A, \mathrm{fix}\, B) = \left\{ \psi \;\middle|\; \begin{array}{l} \text{an orientation preserving diffeomorphism over } A \\ \text{such that } \psi|_B = id_B \end{array} \right\}.$$

If $B = \phi$, we denote this group by $\mathrm{Diff}_+(M)$. The group $\pi_0(\mathrm{Diff}_+(F, \mathrm{fix}\, \partial F))$ is called the *mapping class group* of F and denoted by $\mathcal{M}_F$. If F is a closed oriented surface of genus g, this group is denoted by $\mathcal{M}_g$. We define

$$\mathcal{E}(M, F) = \{\psi \in \mathcal{M}_F \mid \psi \text{ is extendable } \}.$$

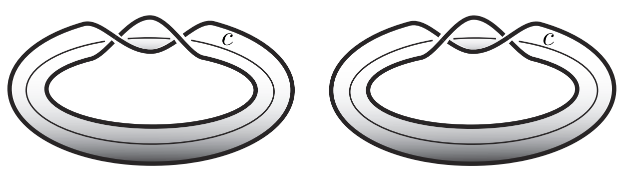
This is a subgroup of $\mathcal{M}_F$ and is a central object of this paper.

In the case where $M = S^4$, there are several works on this group. Let (S^4, Σ_g) be the genus g trivial knotted surface in S^4. When $g = 1$, Montesinos [19] investigated $\mathcal{E}(S^4, \Sigma_1)$, and when $g \geq 2$, the author [11] investigated $\mathcal{E}(S^4, \Sigma_g)$. Let (S^3, k) be a knot in S^3 and $(S^4, S(k))$ (resp. $(S^4, \tilde{S}(k))$) the spun (resp. the twisted spun) of (S^3, k). When (S^3, k) is a torus knot, Iwase [13] investigated $\mathcal{E}(S^4, S(k))$ and $\mathcal{E}(S^4, \tilde{S}(k))$, and when (S^3, k) is an arbitrary knot, the author [10] investigated these groups.

In this paper, we investigate the case where M is the complex projective plane $\mathbb{CP}^2$. In §3, we treat the case where $(\mathbb{CP}^2, \Sigma_g)$ is a standard embedding of Σ_g. In §4, we treat the case where $(\mathbb{CP}^2, F)$ is a non-singular plane curve. From §5 to the end of this paper, we treat the case where $(\mathbb{CP}^2, F)$ is a connected sum of a non-singular plane curve of degree 3 and a trivial embedding.

2 Preliminary: A Hopf band on the boundary of the 4-ball

A link L in S^3 is called a *fibered link* if there is a map $\phi\colon S^3 \setminus L \to S^1$ which is a fiber bundle projection. For each $t \in S^1$, $\phi^{-1}(t) = F$, which does not depend on t, is called the *fiber* of L. Since ϕ is a bundle projection, $S^3 \setminus L$ is diffeomorphic to the quotient of $F \times [0,1]$ by an equivalence $(x, 0) \sim (h(x), 1)$ where h is a diffeomorphism over F and called the *monodromy* of L.



Positive Hopf band Negative Hopf band

Figure 2

A *Hopf band* is an annulus embedded in S^3 as in Figure 2. In this picture, there are two types of Hopf bands. In this note, we treat both types of Hopf bands. The boundary of a Hopf band is called a *Hopf link*. The Hopf link is a fibered link whose fiber is the Hopf band and whose monodromy is a Dehn twist about the core circle of the Hopf band. Let B be a Hopf band in S^3 which is a boundary of a 4-ball D^4. We push the interior of B into the interior of D^4 and let B' be the annulus obtained by this deformation and let c be the core circle of B'.

Proposition 2.1 *The Dehn twist T_c about c is extendable, i.e. there is an element $T \in \mathrm{Diff}_+(D^4, \mathrm{fix}\,\partial D^4)$ such that $T|_{B'} = T_c$.*

Proof Since ∂B is a fibered link, whose fiber is B and whose monodromy is T_c, there is an orientation preserving diffeomorphism ψ of S^3 such that $\psi|_B = T_c$, and there is an isotopy ψ_t $(t \in [0,1])$ with $\psi_0 = id_{S^3}$ and $\psi_1 = \psi$, which is defined by shifting fibers. Let $N(\partial D^4)$ be the regular neighborhood of ∂D^4 in D^4. We parametrize $N(\partial D^4) = S^3 \times [0,2]$ so that $S^3 \times \{0\} = \partial D^4$ and $B' = \partial B \times [0,1] \cup B \times \{1\}$. Let T be a diffeomorphism defined as follows

$$T|_{N(\partial D^4)}(x,t) = \begin{cases} (\psi_t(x), t) & 0 \le t \le 1 \\ (\psi_{2-t}(x), t) & 1 \le t \le 2 \end{cases}$$

$$T|_{D^4 \setminus N(\partial D^4)} = id.$$

This is the diffeomorphism which we need. □

Remark 2.2 Let (S^4, Σ_g) be the genus g surface standardly embedded in S^4. In [11], the author showed that $T_{c_4} T_{c_3} T_{c_4}^{-1} \in \mathcal{E}(S^4, \Sigma_g)$ by using Montesinos' result [19, Theorem 5.3] (c_3 and c_4 are as in Figure 7) . We show this fact by using Proposition 2.1. The 4-sphere S^4 is constructed from two 4-balls D^4_+, D^4_- with attaching along the boundary $S^3 = \partial D^4_+ = \partial D^4_-$. We parametrize the regular neighborhood $N(\partial D^4_+) = S^3 \times [0,2]$ in D^4_+ so that $\partial D^4_+ = S^3 \times \{0\}$. The regular neighborhood N of $T_{c_4}(c_3)$ in Σ_g is a Hopf band in $S^3 \subset S^4$. We push the interior of N into the interior of D^4_+, then we get an annulus N' properly embedded in D^4_+. We may assume, by the above parametrization of $N(\partial D^4_+)$, $N' \cap S^3 \times \{t\} = \partial N \times \{t\}$ for $0 \le t < 2$ and $N' \cap S^3 \times \{2\} = N \times \{2\}$. We denote $D^4_+ \setminus S^3 \times [0,1)$ by D'. By applying Proposition 2.1 to $(D', N' \cap D')$, we show that there is an element $T \in \mathrm{Diff}_+(D', \mathrm{fix}\,\partial D')$ such that $T|_{N' \cap D'} = T_{c_4} T_{c_3} T_{c_4}^{-1}$. Therefore, we see $T_{c_4} T_{c_3} T_{c_4}^{-1} \in \mathcal{E}(S^4, \Sigma_g)$.

3 Surfaces standardly embedded in the complex projective plane

For the free action of $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ on $\mathbb{C}^3 \setminus \{(0,0,0)\}$ defined by $\lambda(z_0, z_1, z_2) = (\lambda z_0, \lambda z_1, \lambda z_2)$, we take the quotient $\mathbb{CP}^2 = (\mathbb{C}^3 \setminus \{(0,0,0)\})/\mathbb{C}^*$. This space $\mathbb{CP}^2$ is a closed oriented 4-manifold and called the *complex projective plane*. This 4-manifold $\mathbb{CP}^2$ is constructed from D^4 by attaching a 2-handle h^2 along the frame 1 trivial knot K_0 in ∂D^4, and attaching a 4-handle h^4. A *3-dimensional handlebody* H_g is an oriented 3-manifold which is constructed from a 3-ball with attaching g 1-handles. Any image of embeddings of H_g into $\mathbb{CP}^2$ are isotopic each other. Therefore, $(\mathbb{CP}^2, \partial H_g)$ is unique. A surface *standardly embedded* in $\mathbb{CP}^2$ is $(\mathbb{CP}^2, \partial H_g)$. We obtain:

Theorem 3.1 *For any* g, $\mathcal{E}(\mathbb{CP}^2, \partial H_g) = \mathcal{M}_g$.

Proof Let D^4 be the 4-ball used to construct $\mathbb{CP}^2$ and $N(\partial D^4)$ be the regular neighborhood of ∂D^4 in D^4. We parametrize $N(\partial D^4) = S^3 \times [0, -1]$, so that $S^3 \times \{0\} = \partial D^4$ and, for $-1 \leq t < 0$, $S^3 \times \{t\}$ is in the interior of D^4. Since the image of embedding of H_g in $\mathbb{CP}^2$ is unique up to isotopy, we assume that $H_g \subset S^3 \times \{-1\}$ and that each simple closed curve c on H_g which corresponds to Lickorish generator of mapping class group $\mathcal{M}_g$ is a trivial knot in $S^3 \times \{-1\}$. The regular neighborhood $N(c)$ of c on ∂H_g is an annulus trivially embedded in $S^3 \times \{-1\}$. At first, we deform H_g in $S^3 \times \{-1\}$ so that, if we forget the second factor $[0, -1]$, $c \cup K_0$ becomes a Hopf link in S^3. We push $N(c)$ into $\partial(D^4 \cup h^2)$, then $N(c)$ becomes a Hopf band in ∂h^4. By applying Proposition 2.1, we see that T_c is extendable in h^4, and so in $\mathbb{CP}^2$. □

4 Non-singular plane curves

We review here the topological description of non-singular plane curves by Akbulut and Kirby [1] (see also [6, 6.2.7]).

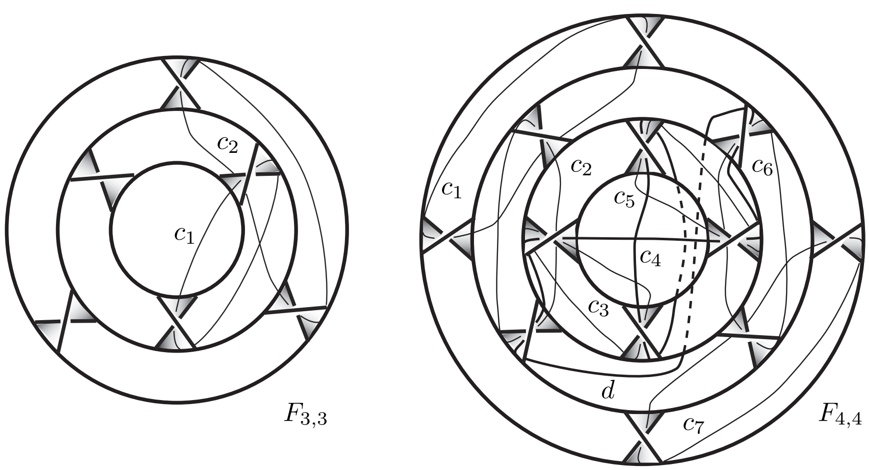


Figure 3

An (m,n)-*torus link* $T_{m,n}$ is an oriented link in $S^3 = \partial D^4$ consisting of $\gcd(m,n)$ oriented circles in the boundary of the tubular neighborhood U of the trivial knot, representing $m\mu + n\lambda$ in $H_1(\partial U;\mathbb{Z})$, where $\mu =$ [the meridian of the trivial knot] and $\lambda =$ [the longitude of the trivial knot]. There is a canonical Seifert surface $F_{m,n}$ for $T_{m,n}$, consisting of n-disks connected by $m(n-1)$ twisted bands as in Figure 3. As K_0, we take a trivial knot given by pushing $T_{1,0}$ into the complement of U (see the left hand side of Figure 4). From here, we consider only the case where $m = n = d$. As shown in the right hand side of Figure 4, $T_{d,d}$ becomes d components trivial link in $\partial(D^4 \cup h^2)$. Let D_d be disjoint 2-disks in $\partial(D^4 \cup h^2)$ which bound this trivial link.

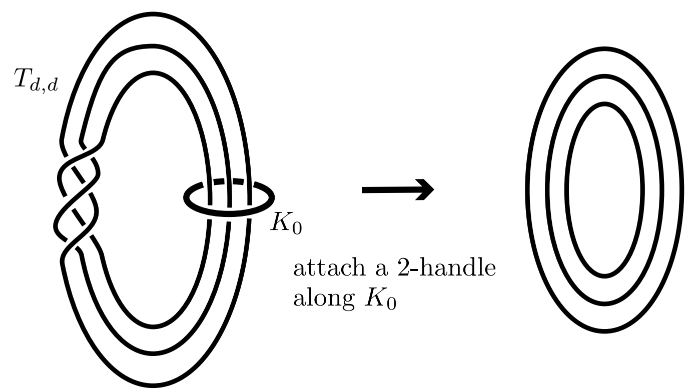


Figure 4

Let K_d be a non-singular plane curve of degree d, then K_d is a genus $\frac{(d-1)(d-2)}{2}$ closed oriented surface embedded in $\mathbb{CP}^2$. We remark that K_d is unique up to isotopy, $K_d = \{[X : Y : Z] \in \mathbb{CP}^2 | X^d + Y^d + Z^d = 0\}$ and $[K_d] = d[\mathbb{CP}^1] \in H_2(\mathbb{CP}^2;\mathbb{Z})$. Akbulut and Kirby showed:

Proposition 4.1 $K_d = F_{d,d} \cup D_d$.

Thus we obtain:

Theorem 4.2 *When* $d = 3, 4$, $\mathcal{E}(\mathbb{CP}^2, K_d) = \mathcal{M}_{g_d}$, *where* $g_d = \frac{(d-1)(d-2)}{2}$.

Proof When $d = 3$, K_3 is homeomorphic to a 2-dimensional torus T^2. In $F_{3,3}$ (see Figure 3), each regular neighborhood of c_1 and c_2 is a Hopf band. Therefore, by Proposition 2.1, T_{c_1} and T_{c_2} are elements of $\mathcal{E}(\mathbb{CP}^2, K_3)$. On the other hand, T_{c_1} and T_{c_2} generate $\mathcal{M}_1$. Hence, $\mathcal{E}(\mathbb{CP}^2, K_3) = \mathcal{M}_1$.

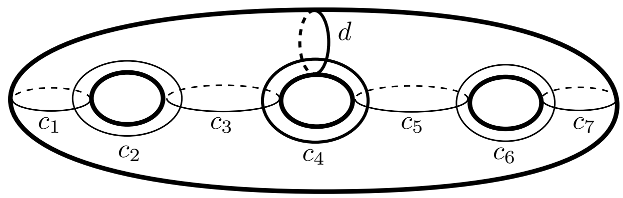


Figure 5

When $d = 4$, we do the same as the above case. We remark that the Dehn twists about the simple closed curves in Figure 5 corresponding to the simple closed curves in $F_{4,4}$ (see Figure 3) with the same symbols generate the mapping class group of genus 3 surface [16]. □

When $d \geq 5$, $\mathcal{E}(\mathbb{CP}^2, K_d)$ is unknown. It is, however, not the case that $\mathcal{E}(\mathbb{CP}^2, K_d) = \mathcal{M}_{g_d}$, because, when d is odd, K_d is a characteristic surface, so the Rokhlin quadratic form on $H_1(K_d;\mathbb{Z}_2)$ is well-defined (we review the definition of the Rokhlin quadratic form in the next section). By the definition of the Rokhlin quadratic form, if a diffeomorphism on K_d is extendable to $\mathbb{CP}^2$, this diffeomorphism should preserve this form. Hence:

Theorem 4.3 *When d is an odd integer greater than or equal to 5, $\mathcal{E}(\mathbb{CP}^2, K_d)$ is a proper subgroup of $\mathcal{M}_{g_d}$, where $g_d = \frac{(d-1)(d-2)}{2}$.*

5 Connected sum of the non-singular plane curve of degree 3 and trivial knotted surface

We define knotted surfaces investigated from here to the end of this paper. The images of any embeddings of a 3-dimensional handlebody H_g into S^4 are isotopic each other. We call this Σ_g-knot $(S^4, \partial H_g)$ *a trivial Σ_g-knot*, and this is denoted by (S^4, Σ_g). Let $(\mathbb{CP}^2, K_3)$ be a nonsingular cubic plane curve. We define connected sum of $(\mathbb{CP}^2, K_3)$ and (S^4, Σ_{g-1}) following the construction by Boyle [3] as follows. We choose points p and q on K_3 and Σ_{g-1} respectively, and find small 4-balls B_1 and B_2 centered at p and q such that the pairs $(B_1, B_1 \cap K_3)$ and $(B_2, B_2 \cap \Sigma_{g-1})$ are equivalent to the standard pair (B^4, B^2). Now we glue the pairs $(S^4 \setminus \mathrm{int}(B_1), K_3 \setminus \mathrm{int}(B_1))$ and $(\mathbb{CP}^2 \setminus \mathrm{int}(B_2), \Sigma_{g-1} \setminus \mathrm{int}(B_2))$ together by an orientation-reversing diffeomorphism $f : \partial B_1 \to \partial B_2$ such that $f(\partial B_1 \cap K_3) = \partial B_2 \cap \Sigma_{g-1}$. Since the connected sum of $\mathbb{CP}^2$ and S^4 is diffeomorphic to $\mathbb{CP}^2$, we get a surface in $\mathbb{CP}^2$ and denote this characteristic knotted surface by $(\mathbb{CP}^2, K_3\#\Sigma_{g-1})$. From here to the end of this paper, we investigate on the group $\mathcal{E}(\mathbb{CP}^2, K_3\#\Sigma_{g-1})$.

For a knotted characteristic surface (M, F), where M is a simply connected smooth closed oriented 4-manifold, we define a quadratic form (*the Rokhlin quadratic form*) $q_F : H_1(F;\mathbb{Z}_2) \to \mathbb{Z}_2$: Let P be a compact surface embedded in M, with its boundary contained in F, normal to F along its boundary, and its interior is transverse to F. Let P' be a surface transverse to P obtained by sliding P parallel to itself over F. Define $q_F([\partial P]) = \#(\mathrm{int}P \cap (P' \cup F)) \bmod 2$. This is a well-defined quadratic form with respect to the $\mathbb{Z}_2$-homology intersection form $(,)_2$ on F, i.e. for each pair of elements x, y of $H_1(F;\mathbb{Z}_2)$, $q_F(x+y) = q_F(x) + q_F(y) + (x,y)_2$. By the definition of the Rokhlin quadratic from q_F, if $\psi \in \mathrm{Diff}_+(F)$ is extendable, then ψ preserves q_F, that is to say, $q_F(\psi_*(x)) = q_F(x)$ for any $x \in H_1(F;\mathbb{Z}_2)$. We will show,

Theorem 5.1 *For any $g \geq 2$,*

$$\mathcal{E}(\mathbb{CP}^2, K_3\#\Sigma_{g-1}) = \left\{ \psi \in \mathcal{M}_g \;\middle|\; \begin{array}{c} q_{K_3\#\Sigma_{g-1}}(\psi_*(x)) = q_{K_3\#\Sigma_{g-1}}(x) \\ \text{for any } x \in H_1(K_3\#\Sigma_{g-1};\mathbb{Z}_2) \end{array} \right\}.$$

In §6, we investigate on a system of generators for the right hand side group in the equation of Theorem 5.1. In §7, we show that each element of this system of generators is extendable.

6 A finite set of generators for the odd spin mapping class group

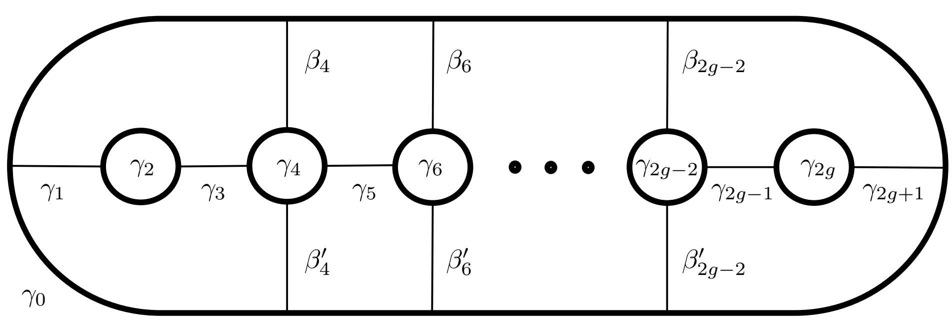


Figure 6

We settle some notations. Let P_g be a planar surface constructed from a 2-disk by removing g copies of disjoint 2-disks. As indicated in Figure 6, we denote the boundary components of P_g by $\gamma_0, \gamma_2, \ldots, \gamma_{2g}$, and denote some properly embedded arcs of P_g by $\gamma_1, \gamma_3, \ldots, \gamma_{2g+1}$, $\beta_4, \ldots, \beta_{2g-2}$ and $\beta'_4, \ldots, \beta'_{2g-2}$. On $\partial(P_g \times [-1,1]) = \Sigma_g$, we define $c_{2i-1} = \partial(\gamma_{2i-1} \times [-1,1])$ $(1 \leq i \leq g+1)$, $b_{2j} = \partial(\beta_{2j} \times [-1,1])$, $b'_{2j} = \partial(\beta'_{2j} \times [-1,1])$ $(2 \leq j \leq g-1)$, and $c_{2k} = \gamma_{2k} \times \{0\}$ $(1 \leq k \leq g)$. In Figures 7 and 8, these circles are illustrated and some of them are oriented.

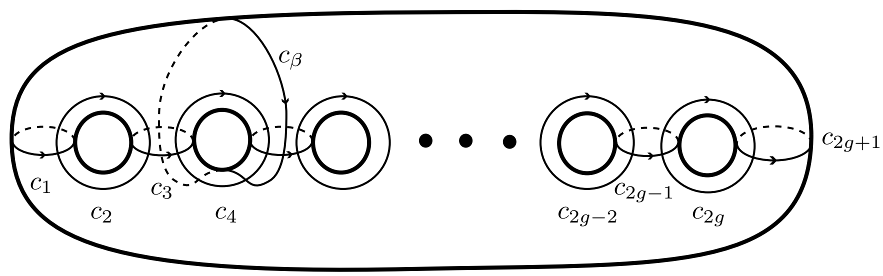


Figure 7

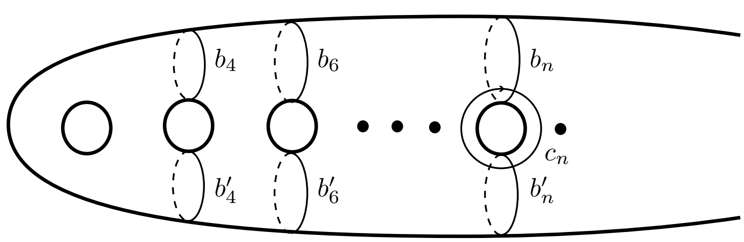


Figure 8

We set a basis of $H_1(\Sigma_g;\mathbb{Z})$ as in Figure 9, where $x_1 = [c_1$ with opposite orientation $]$, $x_i = [b_{2i}$ with proper orientation $]$ $(2 \leq i \leq g-1)$, $x_g = [c_{2g+1}]$, and $y_i = [c_{2i}$ with opposite orientation $]$.

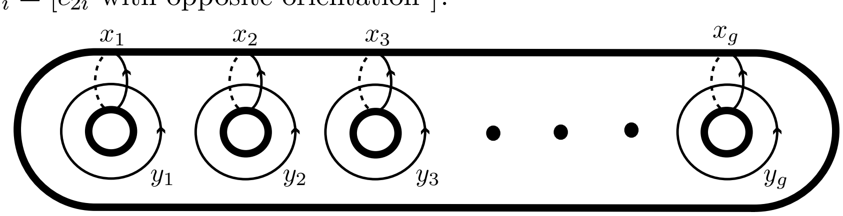


Figure 9

A map $q : H_1(\Sigma_g;\mathbb{Z}_2) \to \mathbb{Z}_2$ is called a *quadratic form* with respect to the $\mathbb{Z}_2$-homology intersection form $(,)_2$ on Σ_g (for short, $\mathbb{Z}_2$-*quadratic form* on Σ_g) if $q(x+y) = q(x) + q(y) + (x,y)_2$, for each pair of elements x, y of $H_1(\Sigma_g;\mathbb{Z}_2)$. For the basis $\{x_1, y_1, \ldots, x_g, y_g\}$ introduced above, we define $Arf(q) = \sum_{i=1}^{g} q(x_i)q(y_i)$. We call a $\mathbb{Z}_2$-quadratic form q *even* quadratic from (resp. *odd* quadratic form) if $Arf(q) = 0$ (resp. $Arf(q) = 1$). We define

$$\mathcal{SP}_g[q] = \{\psi \in \mathcal{M}_g \mid q(\psi_*(x)) = q(x) \text{ for any } x \in H_1(\Sigma_g;\mathbb{Z}_2)\}.$$

As is shown in [21], for two $\mathbb{Z}_2$-quadratic forms q, q' on Σ_g, if $Arf(q) = Arf(q')$, then there is an element $\psi' \in \mathcal{M}_g$ so that $q(\psi'_*(x)) = q'(x)$ for any $x \in H_1(\Sigma_g;\mathbb{Z}_2)$. Therefore, if $Arf(q) = Arf(q')$, then $\mathcal{SP}_g[q]$ and $\mathcal{SP}_g[q']$ are conjugate in $\mathcal{M}_g$. By the definition of $\mathbb{Z}_2$-quadratic from, values of a quadratic form is completely determined by its value for the basis of $H_1(\Sigma_g;\mathbb{Z}_2)$. Let q_0 and q_1 be $\mathbb{Z}_2$-quadratic forms so that $q_0(x_i) = q_0(y_i) = 0$ for $1 \leq i \leq g$, $q_1(x_1) = q_1(y_1) = 1$ and $q_1(x_j) = q_1(y_j) = 0$ for $2 \leq j \leq g$. Then q_0 is an even quadratic form and q_1 an odd quadratic from. If q is even, then $\mathcal{SP}_g[q]$ is conjugate to $\mathcal{SP}_g[q_0]$ in $\mathcal{M}_g$, on the other hand, if q is odd, then $\mathcal{SP}_g[q]$ is conjugate to $\mathcal{SP}_g[q_1]$ in $\mathcal{M}_g$. Hence, for the sake of getting some information about groups $\mathcal{SP}_g[q]$, it suffices to consider only on $\mathcal{SP}_g[q_0]$ and $\mathcal{SP}_g[q_1]$. The group $\mathcal{SP}_g[q_0]$ is called the *even spin mapping class group*, and the group $\mathcal{SP}_g[q_1]$ is called the *odd spin mapping class group*. The spin mapping class group is defined by Harer [8], [9]. In [11], we get a system of generators for $\mathcal{SP}_g[q_0]$. In this section, we will obtain a system of generators for $\mathcal{SP}_g[q_1]$.

Let M be a simply connected smooth closed oriented 4-manifold, (M,F) a knotted characteristic surface and q_F the Rokhlin quadratic form for (M,F). Rokhlin [20] showed (see also [17] and [5]),

$$Arf(q_F) \equiv \frac{\sigma(M) - F\cdot F}{8} \mod 2,$$

where $\sigma(M)$ is the signature of M. By the above formula, we can see $q_{K_3\#\Sigma_{g-1}}$ is an odd quadratic form. Hence, we get a system of generators for $\mathcal{SP}_g[q_{K_3\#\Sigma_{g-1}}]$ from that for $\mathcal{SP}_g[q_1]$.

We introduce some notations used for describing a system of generators for $\mathcal{SP}_g[q_1]$. For a simple closed curve a on Σ_g, T_a denotes the Dehn twist about a. The order of composition of maps is the functional one: T_bT_a means we apply T_a first, then T_b. For elements a, b and c of a group, we write $\overline{c} = c^{-1}$, and $a * b = ab\overline{a}$. We define some elements of $\mathcal{M}_g$ as follows:

$$C_i = T_{c_i},\ B_i = T_{b_i},\ B'_i = T_{b'_i},$$
$$X_i = C_{i+1}C_i\overline{C_{i+1}}\ ,\ X_i^* = \overline{C_{i+1}}\,C_iC_{i+1}\ \ (4 \le i \le 2g),$$
$$Y_{2j} = C_{2j}B_{2j}\overline{C_{2j}}\ ,\ Y_{2j}^* = \overline{C_{2j}}\,B_{2j}C_{2j}\ \ (2 \le j \le g-1),$$
$$D_i = C_i^2\ \ (1 \le i \le 2g+1),$$
$$DB_{2j} = B_{2j}^2\ \ (2 \le j \le g-1),$$
$$T_1 = B_4C_5C_7 \cdots C_{2g+1}.$$

When $g \ge 3$, G_g denotes the subgroup of $\mathcal{M}_g$ generated by C_1, C_2, C_3, X_i $(4 \le i \le 2g)$, Y_{2j} $(2 \le j \le g-1)$, D_i $(1 \le i \le 2g+1)$, DB_{2j} $(2 \le j \le g-1)$, and T_1. It is clear that X_i^* and Y_{2j}^* are elements of G_g. When $g = 2$, the subgroup of $\mathcal{M}_2$ generated by C_1, C_2, C_3, X_4, and D_j $(1 \le j \le 5)$ is denoted by G_2. For two simple closed curves l and m on Σ_g, l and m are called G_g*-equivalent* (denoted by $l \underset{G_g}{\sim} m$) if there is an element ϕ of G_g such that $\phi(l) = m$.

We show that $G_g = \mathcal{SP}_g[q_1]$. That is to say, we show,

Theorem 6.1 *If $g = 2$, $\mathcal{SP}_2[q_1]$ is generated by C_1, C_2, C_3, X_4, and D_j $(1 \le j \le 5)$. If $g \ge 3$, $\mathcal{SP}_g[q_1]$ is generated by C_1, C_2, C_3, X_i $(4 \le i \le 2g)$, Y_{2j} $(2 \le j \le g-1)$, D_k $(1 \le k \le 2g+1)$, DB_{2l} $(2 \le l \le g-1)$, and T_1.*

We prove Theorem 6.1 by using the same method as in the proof of Theorem 3.1 in [11]. By an easy calculation, we can check that each generator of G_g is an element of $\mathcal{SP}_g[q_1]$, therefore, $G_g \subset \mathcal{SP}_g[q_1]$. Hence, we should show $\mathcal{SP}_g[q_1] \subset G_g$. In the case where $g = 2$, we use the Reidemeister-Schreier method to show $\mathcal{SP}_g[q_1] \subset G_2$ (§6.4). In the case where $g \ge 3$, we use other method to show $\mathcal{SP}_g[q_1] \subset G_g$. Here, we present this method in outline.

The integral symplectic group is denoted by $\mathrm{Sp}(2g, \mathbb{Z})$ and the $\mathbb{Z}_2$ symplectic group by $\mathrm{Sp}(2g, \mathbb{Z}_2)$. The generators of these groups are known (on $\mathrm{Sp}(2g, \mathbb{Z})$ see for example [12], on $\mathrm{Sp}(2g, \mathbb{Z}_2)$ see for example [7, Chap.3]), and these generators are induced by the action of $\mathcal{M}_g$ on $H_1(\Sigma_g, \mathbb{Z})$ or $H_1(\Sigma_g, \mathbb{Z}_2)$. Therefore, the homomorphism $\Phi\colon\ \mathcal{M}_g \to \mathrm{Sp}(2g, \mathbb{Z})$, defined by the action of $\mathcal{M}_g$ on $H_1(\Sigma_g, \mathbb{Z})$, is a surjection, and $\Psi\colon\ \mathrm{Sp}(2g, \mathbb{Z}) \to \mathrm{Sp}(2g, \mathbb{Z}_2)$, defined by changing the coefficient from $\mathbb{Z}$ to $\mathbb{Z}_2$, is a surjection. In §6.1, we show $\ker\Phi \subset G_g$. In §6.2, we introduce a finite system of generators for $\ker\Psi$, and, for each generator, we show that one of its inverse by Φ is an element of G_g. Hence, we conclude $\ker\Psi\circ\Phi \subset G_g$. In §6.3, we introduce a finite system of generators for $\Psi \circ \Phi(\mathcal{SP}_g[q_1])$, and, for each generator, we show that one of its inverse by $\Psi \circ \Phi$ is an element of G_g. As a consequence, we show $\mathcal{SP}_g[q_1] \subset G_g$.

6.1 Step 1 for the case where $g \geq 3$

There is a natural surjection $\Phi\colon \mathcal{M}_g \to \mathrm{Sp}(2g, \mathbb{Z})$ defined by the action of $\mathcal{M}_g$ on $H_1(\Sigma_g; \mathbb{Z})$. The kernel of Φ is denoted by $\mathcal{I}_g$ and called *the Torelli group*. In this subsection, we prove the following lemma:

Lemma 6.2 *The Torelli group $\mathcal{I}_g$ is a subgroup of G_g.*

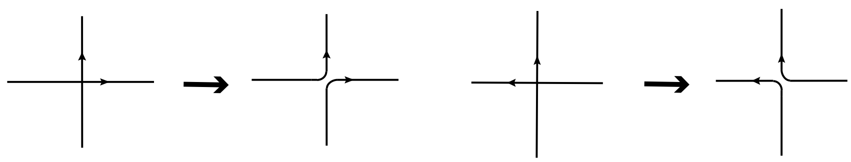


Figure 10

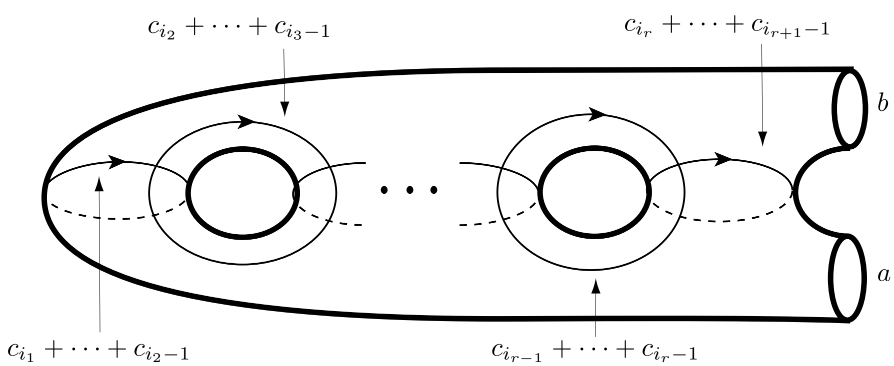


Figure 11

Johnson [14] showed that, when g is larger than or equal to 3, $\mathcal{I}_g$ is finitely generated. We review his result. For oriented simple closed curves shown in Figure 7, we refer to $(c_1, c_2, \ldots, c_{2g+1})$ and $(c_\beta, c_5, \ldots, c_{2g+1})$ as *chains*. For oriented simple closed curves d and e which intersect transversely in one point, we construct an oriented simple closed curve $d+e$ from $d \cup e$ as follows: choose a disk neighborhood of the intersection point and in it make a replacement as indicated in Figure 10. For a consecutive subset $\{c_i, c_{i+1}, \ldots, c_j\}$ of a chain, let $c_i + \cdots + c_j$ be the oriented simple closed curve constructed by repeated applications of the above operations. Let $(i_1, \ldots, i_{r+1})$ be a subsequence of

$(1,2,\ldots,2g+2)$ (resp. $(\beta,5,\ldots,2g+2)$). We construct the union of circles $\mathcal{C} = (c_{i_1}+\cdots+c_{i_2-1})\cup(c_{i_2}+\cdots+c_{i_3-1})\cup\cdots\cup(c_{i_r}+\cdots+c_{i_{r+1}-1})$. If r is odd, a regular neighborhood of $\mathcal{C}$ is homeomorphic to the compact surface indicated in Figure 11 whose boundaries are a and b. Let $\phi = T_bT_a^{-1}$, then ϕ is an element of $\mathcal{I}_g$. We denote ϕ by $[i_1,\ldots,i_{r+1}]$, and call this *the odd subchain map* of $(c_1,c_2,\ldots,c_{2g+1})$ (resp. $(c_\beta,c_5,\ldots,c_{2g+1})$) with *length* $r+1$. Johnson [14] showed the following theorem:

Theorem 6.3 [14, Main Theorem] *For $g\geq 3$, the odd subchain maps of the two chains $(c_1,c_2,\ldots,c_{2g+1})$ and $(c_\beta,c_5,\ldots,c_{2g+1})$ generate $\mathcal{I}_g$.*

We use the following results by Johnson [14].

Lemma 6.4 [14] *(a) C_j commutes with $[i_1,i_2,\cdots]$ if and only if j and $j+1$ are either both contained in or are disjoint from the i's.*
(b) If $i\neq j+1$, then $\overline{C_j} * [\cdots,j,i,\cdots] = [\cdots,j+1,i,\cdots]$.
(c) If $k\neq j$, then $C_j * [\cdots,k,j+1,\cdots] = [\cdots,k,j,\cdots]$.
(d) $[1,2,3,4][1,2,5,6,\ldots,2n]B_4 * [3,4,5,\ldots,2n] = [5,6,\ldots,2n][1,2,3,4,\ldots,2n]$, where $3\leq n\leq g$.

Remark 6.5 Johnson showed (d) only in the case where $n=g$. But we can apply the proof of Lemma 10 of [14] for the case where $3\leq n<g$, since we can regard each surfaces in Figure 18 of [14] as a surface of genus n which is a submanifold of Σ_g.

We prove that any odd subchain map of $(c_1,c_2,c_3,\ldots,c_{2g+1})$ or $(c_\beta,c_5,c_6,\ldots,c_{2g})$ is a product of elements of G_g. The following lemma shows that any odd subchain map of $(c_\beta,c_5,c_6,\ldots,c_{2g})$ is a product of an odd subchain map of $(c_1,c_2,c_3,\ldots,c_{2g+1})$ and elements of G_g.

Lemma 6.6 *For any odd subchain map h of $(c_\beta,c_5,c_6,\ldots,c_{2g+1})$, there is an element g of G_g such that $g*h$ is an odd subchain map of $(c_1,c_2,c_3,\ldots,c_{2g+1})$.*

Proof If there is not β in the sequence which define h, then h is an odd subchain map of $(c_1,c_2,c_3,\ldots,c_{2g+1})$. Hence, it suffices to treat the case where the sequence defining h includes β. If $g = C_{2g+1}^{\epsilon_{2g+1}}\cdots C_7^{\epsilon_7}C_5^{\epsilon_5}B^{-1}$ $(\epsilon_i=\pm1)$, then, under any choice of signs of ϵ_i, $g\in G_g$. We can choose signs of ϵ_i so that $g*h$ is an odd subchain map of $(c_1,c_2,c_3,\ldots,c_{2g+1})$. □

From here to the end of this subsection, odd subchain maps mean only those of $(c_1, c_2, c_3, \ldots, c_{2g+1})$. The following lemma shows that any odd subchain map, whose length is at least 5 and which begins from $1, 2, 3, 4, 5$, is a product of shorter odd subchain maps and elements of G_g.

Lemma 6.7 *For any $6 \leq n_6 < n_7 < \cdots < n_{2k} \leq 2g+2$,*

$$(C_4^2) * [1,2,3,5][1,2,4,n_6,n_7,\ldots,n_{2k}](C_4 B_4 \overline{C_4}\ ) * [3,4,5,n_6,n_7,\ldots,n_{2k}] = \\ = [4,n_6,n_7,\ldots,n_{2k}][1,2,3,4,5,n_6,n_7,\ldots,n_{2k}]$$

Proof By (a) of Lemma 6.4, $\overline{C_4} * [3,4,5,\ldots,2k] = [3,4,5,\ldots,2k]$, and by (d) of Lemma 6.4,

$$[1,2,3,4][1,2,5,6,\ldots,2k] \cdot (B_4 \overline{C_4}\ ) * [3,4,5,\ldots,2k] = \\ = [5,6,\ldots,2k][1,2,3,4,\ldots,2k].$$

By applying C_4 to the above equation and remarking that $C_4 * [1,2,3,4] = (C_4^2) * (\overline{C_4} * [1,2,3,4]) = (C_4^2) * [1,2,3,5]$, we get,

$$(C_4^2) * [1,2,3,5] \cdot [1,2,4,6,\cdots,2k] \cdot (C_4 B_4 \overline{C_4}\ ) * [3,4,5,6,\cdots,2k] = \\ = [4,6,7,\cdots,2k][1,2,3,4,5,6,\cdots,2k].$$

After proper applications of $\overline{C_6}$, $\overline{C_7}$, ..., $\overline{C_{2g+1}}$, we get the equation we need. □

Lemma 6.8 *(1) When $i-k \geq 3$, $(\overline{C_{i-1}}\ C_{i-2} C_{i-1}) * [\ldots,k,i,j,\ldots] = [\ldots,k,i-2,j,\ldots]$.*
(2) When $i-k \geq 2$, $(C_i C_{i-1} \overline{C_i}\ ) * [\ldots,k,i,i+1,\ldots] = [\ldots,k,i-1,i,\ldots]$.

Proof Lemma 6.4 shows (1) and (2). □

For any odd subchain map $[i_1, i_2, \ldots, i_r]$, we introduce a notation $[[\tau_1, \tau_2, \ldots, \tau_{2g+2}]]$: $\tau_k = 1$ if k is a member of $\{i_1, i_2, \ldots, i_r\}$, and $\tau_k = 0$ if k is not a member of $\{i_1, i_2, \ldots, i_r\}$. For $[[\tau_1, \tau_2, \ldots, \tau_{2g+2}]]$, τ_i $(1 \leq i \leq 2g+2)$ is called the i-th *tack* of $[[\tau_1, \tau_2, \ldots, \tau_{2g+2}]]$, and if $\tau_i = 0$ (resp. 1) then τ_i is called a 0-tack (resp. a 1-tack). The number of 1-tacks in $[[\tau_1, \tau_2, \ldots, \tau_{2g+2}]]$ is called the *length* of $[[\tau_1, \tau_2, \ldots, \tau_{2g+2}]]$. Lemma 6.8 (1) means that, when $k \geq 3$, if there is a sequence of 0-tacks which begins from the $k+1$-st tack and whose length is at least 2, then the 1-tack subsequent to this 0-tack sequence is moved to left by 2-steps under the action of G_g. Lemma 6.8 (2) means that, when $k \geq 3$, if there is a sequence of 0-tacks which begins from the $k+1$-st tack and whose length is at least 1, then the adjacent two 1-tacks subsequent to this 0-tack sequence is moved to left by 1-step under the action of G_g. Therefore, for any $[[\tau_1, \tau_2, \ldots, \tau_{2g+2}]]$, we see,

$$[[\tau_1, \tau_2, \ldots, \tau_{2g+2}]] \underset{G_g}{\sim} [[\tau_1, \tau_2, \tau_3, 1, \ldots, 1, 0, 1, \ldots, 0, 1, 0, \ldots, 0]],$$

where $1, \ldots, 1$ is a sequence of 1-tacks (b denotes the length of this sequence), $0, 1, \ldots, 0, 1$ is a sequence arranged 0-tacks and 1-tacks alternatively (t denotes the number of 1-tacks in this sequence), $0, \ldots, 0$ is a sequence of 0-tacks. Since $C_1, C_2, C_3 \in G_g$, if there is one 1-tack among τ_1, τ_2, τ_3, then $[[\tau_1, \tau_2, \tau_3, \ldots]] \underset{G_g}{\sim} [[1, 0, 0, \ldots]]$, if there are two 1-tacks among τ_1, τ_2, τ_3, then $[[\tau_1, \tau_2, \tau_3, \ldots]] \underset{G_g}{\sim} [[1, 1, 0, \ldots]]$. The number of 1-tacks in τ_1, τ_2, τ_3 is denoted by h.

Lemma 6.9 *Any odd subchain map is a product of elements of G_g and the odd subchain maps whose h and b are (1) $h = 3, b = 1$, (2) $h = 3, b = 0$, (3) $h = 2, b = 0$, (4) $h = 1, b = 0$, (5) $h = 0, b = 0$.*

Proof We treat the case where $h = 3$. If $b \geq 2$, by Lemma 6.7, this odd subchain map is a product of elements of G_g and shorter odd subchain maps.

We treat the case where $h = 2$. If $b \geq 3$,

$$[[1, 1, 0, 1, 1, 1, \ldots]] \underset{C_3}{\longrightarrow} [[1, 1, 1, 0, 1, 1, \ldots]] \underset{\text{Lemma 6.8(2)}}{\longrightarrow} [[1, 1, 1, 1, 1, 0, \ldots]],$$

by Lemma 6.7, the last odd subchain map is a product of elements of G_g and shorter odd subchain maps. If $b = 2$,

$$[[1, 1, 0, 1, 1, 0, \ldots]] \underset{C_3}{\longrightarrow} [[1, 1, 1, 0, 1, 0, \ldots]],$$

the last odd subchain map is in the case where $h = 3, b = 0$. If $b = 1$, t should be at least 1, and

$$[[1, 1, 0, 1, 0, 1, 0, \ldots]] \underset{C_3}{\longrightarrow} [[1, 1, 1, 0, 0, 1, 0, \ldots]] \underset{\text{Lemma 6.8(1)}}{\longrightarrow} [[1, 1, 1, 1, 0, 0, 0, \ldots]],$$

the last odd subchain map is in the case where $h = 3, b = 1$.

We treat the case where $h = 1$. If $b \geq 5$,

$$[[1, 0, 0, 1, 1, 1, 1, 1, \ldots]] \underset{C_2 C_3}{\longrightarrow} [[1, 1, 0, 0, 1, 1, 1, 1, \ldots]] \underset{\text{Lemma 6.8(2)}}{\longrightarrow} [[1, 1, 0, 1, 1, 0, 1, 1, \ldots]] \underset{\text{Lemma 6.8(2)}}{\longrightarrow} [[1, 1, 0, 1, 1, 1, 1, 0, \ldots]] \underset{C_3}{\longrightarrow} [[1, 1, 1, 0, 1, 1, 1, 0, \ldots]] \underset{\text{Lemma 6.8(2)}}{\longrightarrow} [[1, 1, 1, 1, 1, 0, 1, 0, \ldots]],$$

by Lemma 6.7, the last odd subchain map is a product of elements of G_g and the shorter odd subchain maps. If $b=4$,

$$\begin{aligned}&[[1,0,0,1,1,1,1,0,\ldots]] \underset{C_2C_3}{\longrightarrow} [[1,1,0,0,1,1,1,0,\ldots]]\\ &\underset{\text{Lemma 6.8(2)}}{\longrightarrow} [[1,1,0,1,1,0,1,0,\ldots]] \underset{C_3}{\longrightarrow} [[1,1,1,0,1,0,1,0,\ldots]],\end{aligned}$$

the last odd subchain map is in the case where $h=3, b=0$. If $b=3$ and $t=0$,

$$\begin{aligned}&[[1,0,0,1,1,1,0,0,\ldots]] \underset{C_2C_3}{\longrightarrow} [[1,1,0,0,1,1,0,0,\ldots]]\\ &\underset{\text{Lemma 6.8(2)}}{\longrightarrow} [[1,1,0,1,1,0,0,0,\ldots]] \underset{C_3}{\longrightarrow} [[1,1,1,0,1,0,0,0,\ldots]],\end{aligned}$$

the last odd subchain map is in the case where $h=3, b=0$. If $b=3$ and $t\geq 2$,

$$\begin{aligned}&[[1,0,0,1,1,1,0,1,\ldots]] \underset{C_2C_3}{\longrightarrow} [[1,1,0,0,1,1,0,1,\ldots]]\\ &\underset{\text{Lemma 6.8(2)}}{\longrightarrow} [[1,1,0,1,1,0,0,1,\ldots]] \underset{\text{Lemma 6.8(1)}}{\longrightarrow} [[1,1,0,1,1,1,0,0,\ldots]]\\ &\underset{C_3}{\longrightarrow} [[1,1,1,0,1,1,0,0,\ldots]] \underset{\text{Lemma 6.8(2)}}{\longrightarrow} [[1,1,1,1,1,0,0,0,\ldots]],\end{aligned}$$

by Lemma 6.7, the last odd subchain map is a product of elements of G_g and shorter odd subchain maps. If $b=2$,

$$[[1,0,0,1,1,0,\ldots]] \underset{C_2C_3}{\longrightarrow} [[1,1,0,0,1,0,\ldots]],$$

the last odd subchain map is in the case where $h=2, b=0$. If $b=1$, t should be at least 2,

$$\begin{aligned}&[[1,0,0,1,0,1,0,1,\ldots]] \underset{C_2C_3}{\longrightarrow} [[1,1,0,0,0,1,0,1,\ldots]]\\ &\underset{\text{Lemma 6.8(1)}}{\longrightarrow} [[1,1,0,1,0,1,0,0,\ldots]],\end{aligned}$$

the last odd subchain map is in the case where $h=2, b=1$, which we treat before.

We treat the case where $h=0$. If $b\geq 7$,

$$\begin{aligned}&[[0,0,0,1,1,1,1,1,1,1,\ldots]] \underset{C_1C_2C_3}{\longrightarrow} [[1,0,0,0,1,1,1,1,1,1,\ldots]]\\ &\underset{\text{Lemma 6.8(2)}}{\longrightarrow} [[1,0,0,1,1,1,1,1,1,0,\ldots]] \underset{C_2C_3}{\longrightarrow} [[1,1,0,0,1,1,1,1,1,0,\ldots]]\\ &\underset{\text{Lemma 6.8(2)}}{\longrightarrow} [[1,1,0,1,1,1,1,0,1,0,\ldots]] \underset{C_3}{\longrightarrow} [[1,1,1,0,1,1,1,0,1,0,\ldots]]\\ &\underset{\text{Lemma 6.8(2)}}{\longrightarrow} [[1,1,1,1,1,0,1,0,1,0,\ldots]],\end{aligned}$$

by Lemma 6.7, the last odd subchain map is a product of G_g and shorter odd subchain maps. If $b=6$,

$$[[0,0,0,1,1,1,1,1,1,0,\dots]] \underset{C_1C_2C_3}{\longrightarrow} [[1,0,0,0,1,1,1,1,1,0,\dots]]$$
$$\underset{\text{Lemma 6.8(2)}}{\longrightarrow} [[1,0,0,1,1,1,1,0,1,0,\dots]] \underset{C_2C_3}{\longrightarrow} [[1,1,0,0,1,1,1,0,1,0,\dots]]$$
$$\underset{\text{Lemma 6.8(2)}}{\longrightarrow} [[1,1,0,1,1,0,1,0,1,0,\dots]] \underset{C_3}{\longrightarrow} [[1,1,1,0,1,0,1,0,1,0,\dots]],$$

the last odd subchain map is in the case where $h=3, b=0$. If $b=5$, t should be at least 1 and,

$$[[0,0,0,1,1,1,1,1,0,1,\dots]] \underset{C_1C_2C_3}{\longrightarrow} [[1,0,0,0,1,1,1,1,0,1,\dots]]$$
$$\underset{\text{Lemma 6.8(2)}}{\longrightarrow} [[1,0,0,1,1,1,1,0,0,1,\dots]] \underset{\text{Lemma 6.8(1)}}{\longrightarrow} [[1,0,0,1,1,1,1,1,0,0,\dots]]$$
$$\underset{C_2C_3}{\longrightarrow} [[1,1,0,0,1,1,1,1,0,0,\dots]] \underset{\text{Lemma 6.8(2)}}{\longrightarrow} [[1,1,0,1,1,1,1,0,0,0,\dots]]$$
$$\underset{C_3}{\longrightarrow} [[1,1,1,0,1,1,1,0,0,0,\dots]] \underset{\text{Lemma 6.8(2)}}{\longrightarrow} [[1,1,1,1,1,0,1,0,0,0,\dots]],$$

by Lemma 6.7, the last odd subchain map is a product of elements of G_g and shorter odd subchain maps. If $b=4$,

$$[[0,0,0,1,1,1,1,0,\dots]] \underset{C_1C_2C_3}{\longrightarrow} [[1,0,0,0,1,1,1,0,\dots]]$$
$$\underset{\text{Lemma 6.8(2)}}{\longrightarrow} [[1,0,0,1,1,0,1,0,\dots]] \underset{C_2C_3}{\longrightarrow} [[1,1,0,0,1,0,1,0,\dots]],$$

the last odd subchain map is in the case where $h=2, b=0$. If $b=3$ and $t=1$,

$$[[0,0,0,1,1,1,0,1,0,\dots]] \underset{C_1C_2C_3}{\longrightarrow} [[1,0,0,0,1,1,0,1,0,\dots]]$$
$$\underset{\text{Lemma 6.8(2)}}{\longrightarrow} [[1,0,0,1,1,0,0,1,0,\dots]] \underset{\text{Lemma 6.8(1)}}{\longrightarrow} [[1,0,0,1,1,1,0,0,0,\dots]]$$
$$\underset{C_2C_3}{\longrightarrow} [[1,1,0,0,1,1,0,0,0,\dots]] \underset{\text{Lemma 6.8(2)}}{\longrightarrow} [[1,1,0,1,1,0,0,0,0,\dots]]$$
$$\underset{C_3}{\longrightarrow} [[1,1,1,0,1,0,0,0,0,\dots]],$$

the last odd subchain map is in the case where $h=3, b=0$. If $b=3$ and $t\neq 1$, then t should be at least 3 and,

$$[[0,0,0,1,1,1,0,1,0,1,0,1,\dots]]$$
$$\underset{C_1C_2C_3}{\longrightarrow} [[1,0,0,0,1,1,0,1,0,1,0,1,\dots]]$$
$$\underset{\text{Lemma 6.8(2)}}{\longrightarrow} [[1,0,0,1,1,0,0,1,0,1,0,1,\dots]]$$

$$\xrightarrow[\text{Lemma 6.8(1)}]{} [[1,0,0,1,1,1,0,1,0,1,0,0,\ldots]]$$
$$\xrightarrow[C_2C_3]{} [[1,1,0,0,1,1,0,1,0,1,0,0,\ldots]]$$
$$\xrightarrow[\text{Lemma 6.8(2)}]{} [[1,1,0,1,1,0,0,1,0,1,0,0,\ldots]]$$
$$\xrightarrow[\text{Lemma 6.8(1)}]{} [[1,1,0,1,1,1,0,1,0,0,0,0,\ldots]]$$
$$\xrightarrow[C_3]{} [[1,1,1,0,1,1,0,1,0,0,0,0,\ldots]]$$
$$\xrightarrow[\text{Lemma 6.8(2)}]{} [[1,1,1,1,1,0,0,1,0,0,0,0,\ldots]],$$

by Lemma 6.7, the last odd subchain map is a product of elements of G_g and shorter odd subchain maps. If $b=2$,

$$[[0,0,0,1,1,0,\ldots]] \xrightarrow[C_1C_2C_3]{} [[1,0,0,0,1,0,\ldots]],$$

the last odd subchain map is in the case where $h=1, b=0$. If $b=1$, then t should be at least 3 and,

$$[[0,0,0,1,0,1,0,1,0,1,\ldots]] \xrightarrow[C_1C_2C_3]{} [[1,0,0,0,0,1,0,1,0,1,\ldots]]$$
$$\xrightarrow[\text{Lemma 6.8(1)}]{} [[1,0,0,1,0,1,0,1,0,0,\ldots]] \xrightarrow[C_2C_3]{} [[1,1,0,0,0,1,0,1,0,0,\ldots]]$$
$$\xrightarrow[\text{Lemma 6.8(1)}]{} [[1,1,0,1,0,1,0,0,0,0,\ldots]] \xrightarrow[C_3]{} [[1,1,1,0,0,1,0,0,0,0,\ldots]]$$
$$\xrightarrow[\text{Lemma 6.8(1)}]{} [[1,1,1,1,0,0,0,0,0,0,\ldots]],$$

the last odd subchain map is in the case where $h=3, b=1$.

This Lemma follows from the above case by case arguments and the induction on the length $(=h+b+t)$ of odd subchain maps. □

Lemma 6.10 *Any odd subchain maps of the* 6 *cases listed in Lemma 6.9 are products of elements of* G_g *and odd subchain maps* $[[1,1,1,1,0,\ldots,0]]$, $[[1,1,1,0,1,0,\ldots,0]]$, $[[1,1,1,0,1,0,1,0,1,0,\ldots,0]]$, *and* $[[1,1,0,0,1,0,1,0,\ldots,0]]$, *where* $0,\ldots,0$ *are sequences of* 0*-tacks.*

Proof By checking figures of chain maps, for examples $[[1,1,1,1,0,1,0,1,\ldots]]$ indicated in Figure 12 and $[[0,0,0,0,1,0,1,0,\ldots]]$ indicated in Figure 13, we see that if a odd subchain map begins from $[[0,0,0,0,\ldots,$ or $[[1,1,1,1,\ldots,$ then this map commutes with B_4, hence B_4* does not effect on this map.

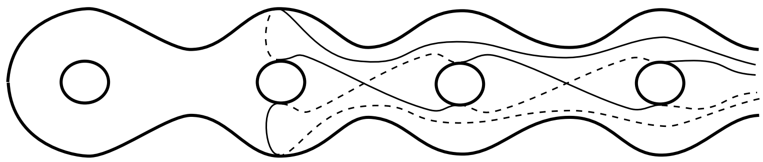


Figure 12

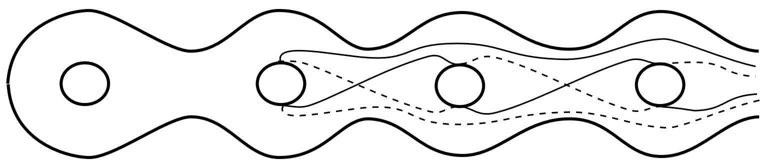


Figure 13

We treat the case where $h = 3$, $b = 1$. If $t = 0$, then this odd subchain map is $[[1, 1, 1, 1, 0, \ldots, 0]]$. If $t \neq 0$, then t should be at least 2 and,

$$[[1, 1, 1, 1, 0, 1, 0, 1, \ldots]] \underset{T_1}{\longrightarrow} [[1, 1, 1, 1, 1, 0, 1, 0, \ldots]],$$

by Lemma 6.7, the last odd subchain map is a product of elements of G_g and shorter odd subchain maps.

We treat the case where $h = 3$, $b = 0$. In this case, t should be an odd integer at least 1. If $t = 1$, then this map is $[[1, 1, 1, 0, 1, 0, \ldots, 0]]$. If $t = 3$, then this map is $[[1, 1, 1, 0, 1, 0, 1, 0, 1, 0, \ldots, 0]]$. If $t \geq 5$,

$$\begin{aligned}
&[[1,1,1,0,1,0,1,0,1,0,1,0,1,0,\ldots]] \\
&\underset{\overline{C_3}}{\longrightarrow} [[1,1,0,1,1,0,1,0,1,0,1,0,1,0,\ldots]] \\
&\underset{\text{Lemma 6.8(2)}}{\longrightarrow} [[1,1,0,0,1,1,1,0,1,0,1,0,1,0,\ldots]] \\
&\underset{\overline{C_3}\,\overline{C_2}}{\longrightarrow} [[1,0,0,1,1,1,1,0,1,0,1,0,1,0,\ldots]] \\
&\underset{\text{Lemma 6.8(2)}}{\longrightarrow} [[1,0,0,0,1,1,1,1,1,0,1,0,1,0,\ldots]] \\
&\underset{\overline{C_3}\,\overline{C_2}\,\overline{C_1}}{\longrightarrow} [[0,0,0,1,1,1,1,1,1,0,1,0,1,0,\ldots]] \\
&\underset{\text{Lemma 6.8(2)}}{\longrightarrow} [[0,0,0,0,1,1,1,1,1,1,1,0,1,0,\ldots]] \\
&\underset{\overline{T_1}}{\longrightarrow} [[0,0,0,0,1,1,1,1,1,1,0,1,0,1,\ldots]] \\
&\underset{\text{Lemma 6.8(2)}}{\longrightarrow} [[0,0,0,1,1,1,1,1,1,0,0,1,0,1,\ldots]]
\end{aligned}$$

$$
\begin{aligned}
&\underset{C_1C_2C_3}{\longrightarrow} [[1,0,0,0,1,1,1,1,1,0,0,1,0,1,\ldots]] \\
&\underset{\text{Lemma 6.8(1)}}{\longrightarrow} [[1,0,0,0,1,1,1,1,1,1,0,0,0,1,\ldots]] \\
&\underset{\text{Lemma 6.8(2)}}{\longrightarrow} [[1,0,0,1,1,1,1,1,1,0,0,0,0,1,\ldots]] \\
&\underset{C_2C_3}{\longrightarrow} [[1,1,0,0,1,1,1,1,1,0,0,0,0,1,\ldots]] \\
&\underset{\text{Lemma 6.8(2)}}{\longrightarrow} [[1,1,0,1,1,1,1,0,1,0,0,0,0,1,\ldots]] \\
&\underset{C_3}{\longrightarrow} [[1,1,1,0,1,1,1,0,1,0,0,0,0,1,\ldots]] \\
&\underset{\text{Lemma 6.8(2)}}{\longrightarrow} [[1,1,1,1,1,0,1,0,1,0,0,0,0,1,\ldots]],
\end{aligned}
$$

by Lemma 6.7, the last odd subchain map is a product of elements of G_g and shorter odd subchain maps.

We treat the case where $h=2$, $b=0$. In this case, t should be even integer at least 2. If $t=2$, this map is $[[1,1,0,0,1,0,1,0,\ldots,0]]$. If $t \geq 4$,

$$
\begin{aligned}
&[[1,1,0,0,1,0,1,0,1,0,1,0,\ldots]] \\
&\underset{\overline{C_3}\,\overline{C_2}}{\longrightarrow} [[1,0,0,1,1,0,1,0,1,0,1,0,\ldots]] \\
&\underset{\text{Lemma 6.8(2)}}{\longrightarrow} [[1,0,0,0,1,1,1,0,1,0,1,0,\ldots]] \\
&\underset{\overline{C_3}\,\overline{C_2}\,\overline{C_1}}{\longrightarrow} [[0,0,0,1,1,1,1,0,1,0,1,0,\ldots]] \\
&\underset{\text{Lemma 6.8(2)}}{\longrightarrow} [[0,0,0,0,1,1,1,1,1,0,1,0,\ldots]] \\
&\underset{\overline{T_1}}{\longrightarrow} [[0,0,0,0,1,1,1,1,0,1,0,1,\ldots]] \\
&\underset{\text{Lemma 6.8(2)}}{\longrightarrow} [[0,0,0,1,1,1,1,0,0,1,0,1,\ldots]] \\
&\underset{C_1C_2C_3}{\longrightarrow} [[1,0,0,0,1,1,1,0,0,1,0,1,\ldots]] \\
&\underset{\text{Lemma 6.8(1)}}{\longrightarrow} [[1,0,0,0,1,1,1,1,0,1,0,0,\ldots]] \\
&\underset{\text{Lemma 6.8(2)}}{\longrightarrow} [[1,0,0,1,1,1,1,0,0,1,0,0,\ldots]] \\
&\underset{\text{Lemma 6.8(1)}}{\longrightarrow} [[1,0,0,1,1,1,1,1,0,0,0,0,\ldots]] \\
&\underset{C_2C_3}{\longrightarrow} [[1,1,0,0,1,1,1,1,0,0,0,0,\ldots]] \\
&\underset{\text{Lemma 6.8(2)}}{\longrightarrow} [[1,1,0,1,1,1,1,0,0,0,0,0,\ldots]]
\end{aligned}
$$

$$\xrightarrow[C_3]{} [[1,1,1,0,1,1,1,0,0,0,0,0,\ldots]]$$
$$\xrightarrow[\text{Lemma 6.8(2)}]{} [[1,1,1,1,1,0,1,0,0,0,0,0,\ldots]],$$

by Lemma 6.7, the last odd subchain map is a product of elements of G_g and shorter odd subchain maps.

We treat the case where $h=1$, $b=0$. In this case, t should be an odd integer at least 3. If $t=3$,

$$[[1,0,0,0,1,0,1,0,1,0,\ldots]] \xrightarrow[\overline{C_3}\,\overline{C_2}\,\overline{C_1}]{} [[0,0,0,1,1,0,1,0,1,0,\ldots]]$$
$$\xrightarrow[\text{Lemma 6.8(2)}]{} [[0,0,0,0,1,1,1,0,1,0,\ldots]] \xrightarrow[\overline{T_1}]{} [[0,0,0,0,1,1,0,1,0,1,\ldots]]$$
$$\xrightarrow[\text{Lemma 6.8(2)}]{} [[0,0,0,1,1,0,0,1,0,1,\ldots]] \xrightarrow[\text{Lemma 6.8(1)}]{} [[0,0,0,1,1,1,0,1,0,0,\ldots]]$$
$$\xrightarrow[C_1C_2C_3]{} [[1,0,0,0,1,1,0,1,0,0,\ldots]] \xrightarrow[\text{Lemma 6.8(2)}]{} [[1,0,0,1,1,0,0,1,0,0,\ldots]]$$
$$\xrightarrow[\text{Lemma 6.8(1)}]{} [[1,0,0,1,1,1,0,0,0,0,\ldots]] \xrightarrow[C_2C_3]{} [[1,1,0,0,1,1,0,0,0,0,\ldots]]$$
$$\xrightarrow[\text{Lemma 6.8(2)}]{} [[1,1,0,1,1,0,0,0,0,0,\ldots]] \xrightarrow[C_3]{} [[1,1,1,0,1,0,0,0,0,0,\ldots]].$$

If $t \geq 5$,

$$[[1,0,0,0,1,0,1,0,1,0,1,0,\ldots]]$$
$$\xrightarrow[\overline{C_3}\,\overline{C_2}\,\overline{C_1}]{} [[0,0,0,1,1,0,1,0,1,0,1,0,\ldots]]$$
$$\xrightarrow[\text{Lemma 6.8(2)}]{} [[0,0,0,0,1,1,1,0,1,0,1,0,\ldots]]$$
$$\xrightarrow[\overline{T_1}]{} [[0,0,0,0,1,1,0,1,0,1,0,1,\ldots]]$$
$$\xrightarrow[\text{Lemma 6.8(2)}]{} [[0,0,0,1,1,0,0,1,0,1,0,1,\ldots]]$$
$$\xrightarrow[\text{Lemma 6.8(1)}]{} [[0,0,0,1,1,1,0,1,0,1,0,0,\ldots]]$$
$$\xrightarrow[C_1C_2C_3]{} [[1,0,0,0,1,1,0,1,0,1,0,0,\ldots]]$$
$$\xrightarrow[\text{Lemma 6.8(2)}]{} [[1,0,0,1,1,0,0,1,0,1,0,0,\ldots]]$$
$$\xrightarrow[\text{Lemma 6.8(1)}]{} [[1,0,0,1,1,1,0,1,0,0,0,0,\ldots]]$$
$$\xrightarrow[C_2C_3]{} [[1,1,0,0,1,1,0,1,0,0,0,0,\ldots]]$$
$$\xrightarrow[\text{Lemma 6.8(2)}]{} [[1,1,0,1,1,0,0,1,0,0,0,0,\ldots]]$$

$$\begin{aligned}
&\xrightarrow[\text{Lemma 6.8(1)}]{} [[1,1,0,1,1,1,0,0,0,0,0,0,\ldots]]\\
&\xrightarrow[C_3]{} [[1,1,1,0,1,1,0,0,0,0,0,0,\ldots]]\\
&\xrightarrow[\text{Lemma 6.8(2)}]{} [[1,1,1,1,1,0,0,0,0,0,0,0,\ldots]],
\end{aligned}$$

by Lemma 6.7, the last odd subchain map is a product of elements of G_g and shorter odd subchain maps.

We treat the case where $h=0$, $b=0$. In this case, t should be an even integer at least 4. If $t=4$,

$$\begin{aligned}
&[[0,0,0,0,1,0,1,0,1,0,1,0,\ldots]]\\
&\xrightarrow[\overline{T_1}]{} [[0,0,0,0,0,1,0,1,0,1,0,1,\ldots]]\\
&\xrightarrow[\text{Lemma 6.8(1)}]{} [[0,0,0,1,0,1,0,1,0,1,0,0,\ldots]]\\
&\xrightarrow[C_1C_2C_3]{} [[1,0,0,0,0,1,0,1,0,1,0,0,\ldots]]\\
&\xrightarrow[\text{Lemma 6.8(1)}]{} [[1,0,0,1,0,1,0,1,0,0,0,0,\ldots]]\\
&\xrightarrow[C_2C_3]{} [[1,1,0,0,0,1,0,1,0,0,0,0,\ldots]]\\
&\xrightarrow[\text{Lemma 6.8(1)}]{} [[1,1,0,1,0,1,0,0,0,0,0,0,\ldots]]\\
&\xrightarrow[C_3]{} [[1,1,1,0,0,1,0,0,0,0,0,0,\ldots]]\\
&\xrightarrow[\text{Lemma 6.8(1)}]{} [[1,1,1,1,0,0,0,0,0,0,0,0,\ldots]].
\end{aligned}$$

If $t \geq 6$,

$$\begin{aligned}
&[[0,0,0,0,1,0,1,0,1,0,1,0,1,0,1,0,\ldots]]\\
&\xrightarrow[\overline{T_1}]{} [[0,0,0,0,0,1,0,1,0,1,0,1,0,1,0,1,\ldots]]\\
&\longrightarrow \text{(as in the previous case)} \longrightarrow [[1,1,1,1,0,0,0,0,0,0,0,0,1,0,1,\ldots]]\\
&\xrightarrow[\text{Lemma 6.8(1)}]{} [[1,1,1,1,0,1,0,1,\ldots]],
\end{aligned}$$

the last odd subchain map is in the case where $h=3, b=1$, which we treat before. □

Lemma 6.11 *The odd subchain maps* $[[1,1,1,1,0,\ldots,0]]$, $[[1,1,1,0,1,0,\ldots,0]]$ *and* $[[1,1,0,0,1,0,1,0,\ldots,0]]$ *are elements of* G_g.

Proof In a proof of this Lemma, we use "braid relation", which is explained as follows. Let a and b are simple closed curves on Σ_g intersecting transversely in one point, then $T_aT_bT_a^{-1} = T_b^{-1}T_aT_b$, in other word, $T_a * T_b = \overline{T_b} * T_a$.

Let b'_4 be the simple closed curve on Σ_g indicated in Figure 8 and let $B'_4 = T_{b'_4}$. The odd subchain map $[[1,1,1,1,0,\ldots,0]]$ is equal to $B_4\overline{B'_4}$. Since $b'_4 = C_4C_3C_2C_1C_1C_2C_3C_4(b_4)$,

$$\begin{aligned} B_4\overline{B'_4} &= B_4C_4C_3C_2C_1C_1C_2C_3C_4\overline{B_4}\,\overline{C_4}\,\overline{C_3}\,\overline{C_2}\,\overline{C_1}\,\overline{C_1}\,\overline{C_2}\,\overline{C_3}\,\overline{C_4} \\ &= (B_4C_4C_3C_2) * (C_1C_1) \cdot (B_4C_4C_3) * (C_2C_2) \cdot (B_4C_4) * (C_3C_3) \cdot \\ &\quad \cdot B_4 * (C_4C_4) \cdot (\overline{C_4}\,\overline{C_3}\,\overline{C_2}) * (\overline{C_1}\,\overline{C_1}) \cdot (\overline{C_4}\,\overline{C_3}) * (\overline{C_2}\,\overline{C_2}) \cdot \\ &\quad \cdot \overline{C_4} * (\overline{C_3}\,\overline{C_3}) \cdot \overline{C_4}\,\overline{C_4}. \end{aligned}$$

This equation means that $B_4\overline{B'_4}$ is a product of squares Dehn twists. By using braid relations of $\mathcal{M}_g$, we can see that these squares of Dehn twists are elements of G_g as follows,

$$\begin{aligned} (B_4C_4C_3C_2) * (C_1C_1) &= (\overline{C_1} \cdot \overline{C_2} \cdot \overline{C_3} \cdot B_4) * (C_4C_4) \\ &= (\overline{C_1} \cdot \overline{C_2} \cdot \overline{C_3}) * (B_4C_4\overline{B_4} \cdot B_4C_4\overline{B_4}), \\ (B_4C_4C_3) * (C_2C_2) &= (\overline{C_2} \cdot \overline{C_3} \cdot B_4) * (C_4C_4) \\ &= (\overline{C_2} \cdot \overline{C_3}) * (B_4C_4\overline{B_4} \cdot B_4C_4\overline{B_4}), \\ (B_4C_4) * (C_3C_3) &= (\overline{C_3} \cdot B_4) * (C_4C_4) = \overline{C_3} * (B_4C_4\overline{B_4} \cdot B_4C_4\overline{B_4}), \\ B_4 * (C_4C_4) &= B_4C_4\overline{B_4} \cdot B_4C_4\overline{B_4}, \\ (\overline{C_4}\,\overline{C_3}\,\overline{C_2}) * (C_1C_1) &= (C_1 \cdot C_2 \cdot C_3) * (C_4C_4), \\ (\overline{C_4}\,\overline{C_3}) * (C_2C_2) &= (C_2 \cdot C_3) * (C_4C_4), \\ \overline{C_4} * (C_3C_3) &= C_3 * (C_4C_4). \end{aligned}$$

Since $\overline{C_4} * [[1,1,1,1,0,\ldots,0]] = [[1,1,1,0,1,0,\ldots,0]]$,

$$\begin{aligned} &[[1,1,1,0,1,0,\ldots,0]] = \overline{C_4} * (B_4\overline{B'_4}) \\ &= (\overline{C_4}\,B_4C_4C_3C_2) * (C_1C_1) \cdot (\overline{C_4}\,B_4C_4C_3) * (C_2C_2) \cdot (\overline{C_4}\,B_4C_4) * (C_3C_3) \cdot \\ &\cdot (\overline{C_4}\,B_4) * (C_4C_4) \cdot (\overline{C_4}\,\overline{C_4}\,\overline{C_3}\,\overline{C_2}) * (\overline{C_1}\,\overline{C_1}) \cdot (\overline{C_4}\,\overline{C_4}\,\overline{C_3}) * (\overline{C_2}\,\overline{C_2}) \cdot \\ &\cdot (\overline{C_4}\,\overline{C_4}) * (\overline{C_3}\,\overline{C_3}) \cdot \overline{C_4} * (\overline{C_4}\,\overline{C_4}) \\ &= (\overline{C_4}\,B_4C_4 \cdot C_3 \cdot C_2) * (C_1C_1) \cdot (\overline{C_4}\,B_4C_4 \cdot C_3) * (C_2C_2) \cdot \\ &\cdot (\overline{C_4}\,B_4C_4) * (C_3C_3) \cdot (\overline{C_4}\,B_4C_4) * (C_4C_4) \cdot (\overline{C_4}\,\overline{C_4} \cdot \overline{C_3} \cdot \overline{C_2}) * (\overline{C_1}\,\overline{C_1}) \cdot \\ &\cdot (\overline{C_4}\,\overline{C_4} \cdot \overline{C_3}) * (\overline{C_2}\,\overline{C_2}) \cdot (\overline{C_4}\,\overline{C_4}) * (\overline{C_3}\,\overline{C_3}) \cdot (\overline{C_4}\,\overline{C_4}) \end{aligned}$$

This equation shows that $[[1,1,1,0,1,0,\ldots,0]] \in G_g$.

Since $\overline{C_4}\,\overline{C_3}\,\overline{C_6}\,\overline{C_5}\,\overline{C_4} * [[1,1,1,1,0,\dots,0]] = [[1,1,0,0,1,0,1,0,\dots,0]]$,

$$\begin{aligned}
&[[1,1,0,0,1,0,1,0,\dots,0]] = \overline{C_4}\,\overline{C_3}\,\overline{C_6}\,\overline{C_5}\,\overline{C_4} * (B_4\overline{B_4'}) \\
&= (\overline{C_4}\,\overline{C_3}\,\overline{C_6}\,\overline{C_5}\,\overline{C_4}\,B_4C_4C_3C_2) * (C_1C_1)\cdot(\overline{C_4}\,\overline{C_3}\,\overline{C_6}\,\overline{C_5}\,\overline{C_4}\,B_4C_4C_3) * (C_2C_2)\cdot \\
&\cdot(\overline{C_4}\,\overline{C_3}\,\overline{C_6}\,\overline{C_5}\,\overline{C_4}\,B_4C_4) * (C_3C_3)\cdot(\overline{C_4}\,\overline{C_3}\,\overline{C_6}\,\overline{C_5}\,\overline{C_4}\,B_4) * (C_4C_4)\cdot \\
&\cdot(\overline{C_4}\,\overline{C_3}\,\overline{C_6}\,\overline{C_5}\,\overline{C_4}\,\overline{C_4}\,\overline{C_3}\,\overline{C_2}) * (\overline{C_1}\,\overline{C_1})\cdot(\overline{C_4}\,\overline{C_3}\,\overline{C_6}\,\overline{C_5}\,\overline{C_4}\,\overline{C_4}\,\overline{C_3}) * (\overline{C_2}\,\overline{C_2})\cdot \\
&\cdot(\overline{C_4}\,\overline{C_3}\,\overline{C_6}\,\overline{C_5}\,\overline{C_4}\,\overline{C_4}) * (\overline{C_3}\,\overline{C_3})\cdot(\overline{C_4}\,\overline{C_3}\,\overline{C_6}\,\overline{C_5}\,\overline{C_4}) * (\overline{C_4}\,\overline{C_4}).
\end{aligned}$$

This equation describes $[[1,1,0,0,1,0,1,0,\dots,0]]$ as a product of squares of Dehn twists. By using braid relations of $\mathcal{M}_g$, we show that these squares of Dehn twists are elements of G_g as follows,

$$\begin{aligned}
&(\overline{C_4}\,\overline{C_3}\,\overline{C_6}\,\overline{C_5}\,\overline{C_4}\,B_4C_4C_3C_2) * (C_1C_1) \\
&\qquad = (\overline{C_1}\cdot\overline{C_4}\,B_4C_4\cdot\overline{C_2}\cdot C_3\cdot\overline{C_6}\,C_5C_6) * (C_4C_4), \\
&(\overline{C_4}\,\overline{C_3}\,\overline{C_6}\,\overline{C_5}\,\overline{C_4}\,B_4C_4C_3) * (C_2C_2) \\
&\qquad = (\overline{C_4}\,B_4C_4\cdot\overline{C_2}\cdot C_3\cdot\overline{C_6}\,C_5C_6) * (C_4C_4), \\
&(\overline{C_4}\,\overline{C_3}\,\overline{C_6}\,\overline{C_5}\,\overline{C_4}\,B_4C_4) * (C_3C_3) = (\overline{C_4}\,B_4C_4\cdot C_3\cdot\overline{C_4}\,\overline{C_4}\cdot\overline{C_6}) * (C_5C_5) \\
&\qquad = (\overline{C_4}\,B_4C_4\cdot C_3\cdot\overline{C_4}\,\overline{C_4}) * (\overline{C_6}\,C_5C_6\cdot\overline{C_6}\,C_5C_6), \\
&(\overline{C_4}\,\overline{C_3}\,\overline{C_6}\,\overline{C_5}\,\overline{C_4}\,B_4) * (C_4C_4) \\
&\qquad = (C_3\cdot\overline{C_4}\,B_4C_4\cdot\overline{C_6}\,C_5C_6\cdot\overline{C_4}\,\overline{C_4}) * (C_3C_3), \\
&(\overline{C_4}\,\overline{C_3}\,\overline{C_6}\,\overline{C_5}\,\overline{C_4}\,\overline{C_4}\,\overline{C_3}\,\overline{C_2}) * (\overline{C_1}\,\overline{C_1}) \\
&\qquad = (C_1\cdot C_3\cdot C_2\cdot\overline{C_4}\,\overline{C_4}\cdot\overline{C_6}\,\overline{C_5}\,C_6\cdot\overline{C_4}\,\overline{C_4}) * (C_3C_3), \\
&(\overline{C_4}\,\overline{C_3}\,\overline{C_6}\,\overline{C_5}\,\overline{C_4}\,\overline{C_4}\,\overline{C_3}) * (\overline{C_2}\,\overline{C_2}) \\
&\qquad = (C_3\cdot C_2\cdot\overline{C_4}\,\overline{C_4}\cdot\overline{C_6}\,\overline{C_5}\,C_6\cdot\overline{C_4}\,\overline{C_4}) * (C_3C_3), \\
&(\overline{C_4}\,\overline{C_3}\,\overline{C_6}\,\overline{C_5}\,\overline{C_4}\,\overline{C_4}) * (\overline{C_3}\,\overline{C_3}) = (C_3\cdot\overline{C_6}\,C_5C_6\cdot\overline{C_6}\,C_5C_6) * (C_4C_4), \\
&(\overline{C_4}\,\overline{C_3}\,\overline{C_6}\,\overline{C_5}\,\overline{C_4}) * (\overline{C_4}\,\overline{C_4}) = (C_3\cdot\overline{C_4}\,C_5C_4) * (C_6C_6).
\end{aligned}$$

□

Lemma 6.12 *The odd subchain map $[[1,1,1,0,1,0,1,0,1,0,\dots,0]]$ is an element of G_g.*

Proof We can show that this odd subchain map is G_g-equivalent to $[[0,0,0,0,1,1,1,1,1,1,0,\dots,0]]$ as follows,

$$[[1,1,1,0,1,0,1,0,1,0,\dots,0]] \underset{\overline{C_3}}{\longrightarrow} [[1,1,0,1,1,0,1,0,1,0,\dots,0]]$$

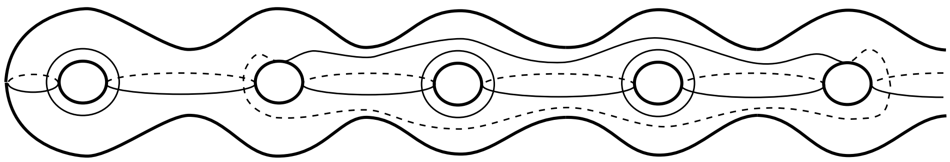


Figure 14

$$\xrightarrow[\text{Lemma 6.8(2)}]{} [[1,1,0,0,1,1,1,0,1,0,\ldots,0]]$$
$$\xrightarrow[\overline{C_3}\,\overline{C_2}]{} [[1,0,0,1,1,1,1,0,1,0,\ldots,0]]$$
$$\xrightarrow[\text{Lemma 6.8(2)}]{} [[1,0,0,0,1,1,1,1,1,0,\ldots,0]]$$
$$\xrightarrow[\overline{C_3}\,\overline{C_2}\,\overline{C_1}]{} [[0,0,0,1,1,1,1,1,1,0,\ldots,0]]$$
$$\xrightarrow[\text{Lemma 6.8(2)}]{} [[0,0,0,0,1,1,1,1,1,1,0,\ldots,0]].$$

If $g = 4$, $[[0,0,0,0,1,1,1,1,1,1]] = B_4\overline{B'_4} = [[1,1,1,1,0,0,0,0,0,0]]$, which we have already treated in Lemma 6.11. If $g \geq 5$, as we see in Figure 14,

$$[[0,0,0,0,1,1,1,1,1,1,0,\ldots,0]] = [[1,1,1,1,0,0,0,0,0,0,1,\ldots,1]],$$

in the notation of the last odd subchain map, … is a sequence of 1-tacks. By Lemma 6.8 (2),

$$[[1,1,1,1,0,0,0,0,0,0,1,\ldots,1]] \underset{G_g}{\sim} [[1,1,1,1,1,\ldots,1,0,0,0,0,0,0]],$$

which is a product of elements of G_g and shorter odd subchain maps. □

Therefore, Lemma 6.2 is proved.

6.2 Step 2 for the case where $g \geq 3$

Let Φ_2 be the natural homomorphism from $\mathcal{M}_g$ to $\mathrm{Sp}(2g,\mathbb{Z}_2)$ defined by the action of $\mathcal{M}_g$ on the $\mathbb{Z}_2$-coefficient first homology group $H_1(\Sigma_g;\mathbb{Z}_2)$. In this section, we will show the following lemma.

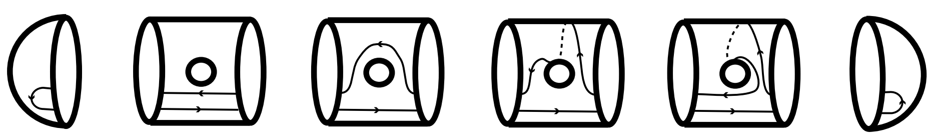
Lemma 6.13 $\ker\Phi_2$ *is a subgroup of* G_g*.*

We denote the kernel of the natural homomorphism from $\mathrm{Sp}(2g,\mathbb{Z})$ to $\mathrm{Sp}(2g,\mathbb{Z}_2)$ by $\mathrm{Sp}^{(2)}(2g)$. We set a basis of $H_1(\Sigma_g;\mathbb{Z})$ as in Figure 9, and define the intersection form $(,)$ on $H_1(\Sigma_g;\mathbb{Z})$ to satisfy $(x_i,y_j)=\delta_{i,j}$, $(x_i,x_j)=(y_i,y_j)=0$ $(1\le i,j,\le g)$. An element a of $H_1(\Sigma_g;\mathbb{Z})$ is called *primitive* if there is no element $n(\ne 0,\pm 1)$ of $\mathbb{Z}$, and no element b of $H_1(\Sigma_g;\mathbb{Z})$ such that $a=nb$. For a primitive element a of $H_1(\Sigma_g;\mathbb{Z})$, we define an isomorphism $T_a : H_1(\Sigma_g;\mathbb{Z})\to H_1(\Sigma_g;\mathbb{Z})$ by $T_a(v)=v+(a,v)a$. This isomorphism is the action of Dehn twist about a simple closed curve representing a on $H_1(\Sigma_g;\mathbb{Z})$. We call T_a^2 *the square transvection about* a. Johnson [15] showed the following result.

Lemma 6.14 $\mathrm{Sp}^{(2)}(2g)$ *is generated by square transvections.*

In [11], we showed,

Lemma 6.15 $\mathrm{Sp}^{(2)}(2g)$ *is generated by the square transvections about the primitive elements* $\sum_{i=1}^{g}(\epsilon_i x_i+\delta_i y_i)$, *where* $\epsilon_i=0,1$ *and* $\delta_i=0,1$.



(-) (0) (1) (2) (3) (+)

Figure 15

For each element $[(\epsilon_1,\delta_1),\cdots,(\epsilon_g,\delta_g)]=\sum_{i=1}^{g}(\epsilon_i x_i+\delta_i y_i)$ (where $\epsilon_i=0,1$, $\delta_i=0,1$) of $H_1(\Sigma_g;\mathbb{Z})$, we construct an oriented simple closed curve on Σ_g which represent this homology class. For each i-th block, if $(\epsilon_i,\delta_i)=(0,0)$, we prepare (0) of Figure 15, if $(\epsilon_i,\delta_i)=(0,1)$, we prepare (1) of Figure 15, if $(\epsilon_i,\delta_i)=(1,1)$, we prepare (2) of Figure 15, if $(\epsilon_i,\delta_i)=(1,0)$, we prepare (3) of Figure 15. After that, we glue them along the boundaries and cap the left boundary component by (-) of Figure 15 and the right boundary component by (+) of Figure 15. We denote this oriented simple closed curve on Σ_g by $\{(\epsilon_1,\delta_1),\cdots,(\epsilon_g,\delta_g)\}$. Here, we remark that the action of $T_{\{(\epsilon_1,\delta_1),\cdots,(\epsilon_g,\delta_g)\}}$ on $H_1(\Sigma_g;\mathbb{Z})$ equals $T_{[(\epsilon_1,\delta_1),\cdots,(\epsilon_g,\delta_g)]}$, and, for any ϕ of $\mathcal{M}_g$, $\phi\circ T_{\{(\epsilon_1,\delta_1),\cdots,(\epsilon_g,\delta_g)\}}\circ\phi^{-1}=T_{\phi(\{(\epsilon_1,\delta_1),\cdots,(\epsilon_g,\delta_g)\})}$.

Lemma 6.16 *For any* $\{(\epsilon_1,\delta_1),\cdots,(\epsilon_g,\delta_g)\}$, *there is an element* ϕ *of* G_g *such* *that*

$$\begin{aligned}\phi(\{(\epsilon_1,\delta_1),\cdots,(\epsilon_g,\delta_g)\}) &= \{(0,0),(0,1),(0,0),(0,0),\cdots,(0,0)\}\\ or\ &= \{(0,0),(1,1),(0,0),(0,0),\cdots,(0,0)\}\\ or\ &= \{(0,0),(0,0),(1,1),(0,0),\cdots,(0,0)\}\\ or\ &= \{(0,1),(0,0),(0,0),\cdots,(0,0)\}\\ or\ &= \{(1,1),(0,0),(0,0),\cdots,(0,0)\}\\ or\ &= \{(0,0),(0,0),(0,0),\cdots,(0,0)\}.\end{aligned}$$

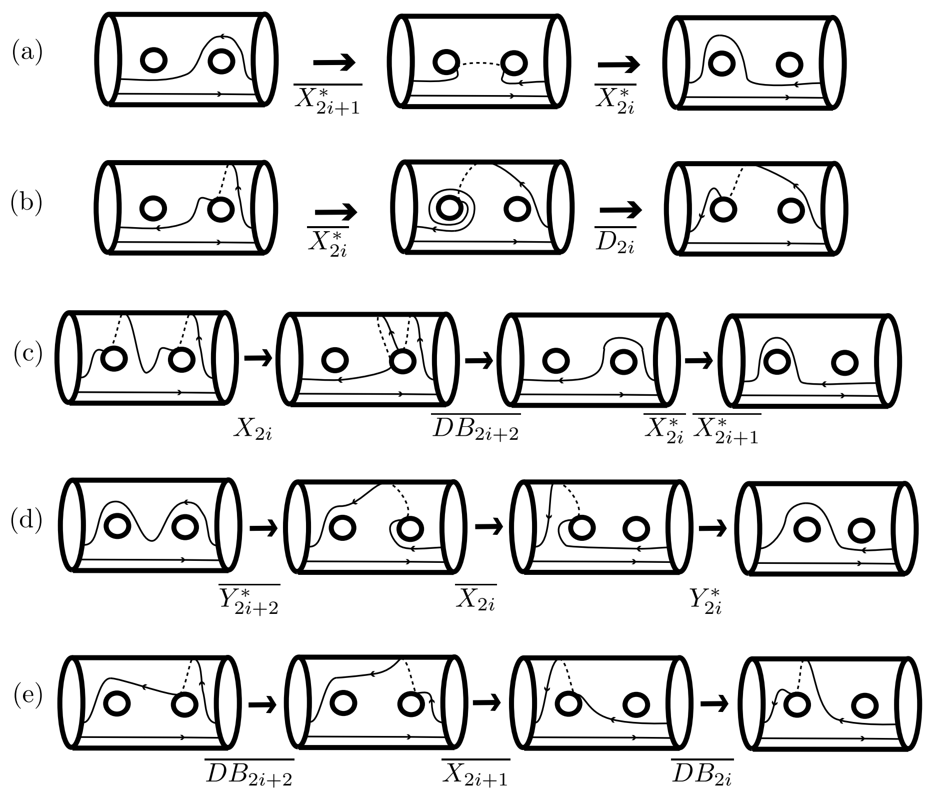


Figure 16

Proof If the i-th block is (3), by the action of $\overline{Y_{2i}}$ if $2 \leq i \leq g-1$, $C_2\overline{C_1}\,\overline{C_2}$ if $i=1$, and $C_{2g}\overline{C_{2g+1}}\,\overline{C_{2g}}$ if $i=g$, this block is changed to (1). Therefore, it suffices to show this lemma in the case where each block is not (3). First we investigate actions of elements of G_g on adjacent blocks, say the i-th block and the $i+1$-st block, where $i \geq 2$. Each picture of Figure 16 shows the action of G_g on this adjacent blocks.

$$(a) \text{ shows } \{\bullet\bullet\bullet, (0,0), (0,1), \bullet\bullet\bullet\} \underset{G_g}{\sim} \{\bullet\bullet\bullet, (0,1), (0,0), \bullet\bullet\bullet\},$$

$$(b) \text{ shows } \{\bullet\bullet\bullet, (0,0), (1,1), \bullet\bullet\bullet\} \underset{G_g}{\sim} \{\bullet\bullet\bullet, (1,1), (0,1), \bullet\bullet\bullet\},$$

$$(c) \text{ shows } \{\bullet\bullet\bullet, (1,1), (1,1), \bullet\bullet\bullet\} \underset{G_g}{\sim} \{\bullet\bullet\bullet, (0,1), (0,0), \bullet\bullet\bullet\},$$

$$(d) \text{ shows } \{\bullet\bullet\bullet, (0,1), (0,1), \bullet\bullet\bullet\} \underset{G_g}{\sim} \{\bullet\bullet\bullet, (0,1), (0,0), \bullet\bullet\bullet\},$$

$$(e) \text{ shows } \{\bullet\bullet\bullet, (0,1), (1,1), \bullet\bullet\bullet\} \underset{G_g}{\sim} \{\bullet\bullet\bullet, (1,1), (0,0), \bullet\bullet\bullet\},$$

where $\bullet\bullet\bullet$ indicates the part which is not changed by the action of G_g. Let $x = \{(\epsilon_1, \delta_1), \cdots, (\epsilon_g, \delta_g)\}$, each of whose block is $(0,0)$ or $(0,1)$ or $(1,1)$. If there are the j-th blocks $(1,1)$ $(j \geq 2)$, by (b) and (e), they are gathered to a sequence of $(1,1)$ blocks which begins from the second block. If there are the j-th blocks $(0,1)$ $(j \geq 2)$, by (a), they are gathered to a sequence of $(0,1)$ blocks subsequent to the previous sequence of $(1,1)$ blocks. Hence, we showed,

$$x \underset{G_g}{\sim} \{(\epsilon_1, \delta_1), (1,1), \cdots, (1,1), (0,1), \cdots, (0,1), (0,0), \cdots, (0,0)\}.$$

By (a) and (d), the sequence of $(0,1)$ blocks is altered to $(0,1), (0,0), \cdots, (0,0)$ or $(0,0), \cdots, (0,0)$. By (c), the sequence of $(1,1)$ blocks is altered to $(1,1)$, $(0,1), (0,0), \cdots, (0,1), (0,0)$ (when the length of the sequence is odd) or to $(0,1), (0,0), \cdots, (0,1), (0,0)$ (when the length of the sequence is even). By (a) and (d), $(1,1), (0,1), (0,0), \cdots, (0,1), (0,0)$ is altered to $(1,1), (0,1), (0,0), \cdots,$ $(0,0), (0,0)$ or $(1,1), (0,0), (0,0), \cdots, (0,0), (0,0)$, and $(0,1), (0,0), \cdots, (0,1),$ $(0,0)$ to $(0,1), (0,0), \cdots, (0,0), (0,0)$ or $(0,0), (0,0), \cdots, (0,0), (0,0)$. Therefore, we showed,

$$\begin{aligned}
x &\underset{G_g}{\sim} \{(\epsilon_1, \delta_1), (1,1), (0,0), (0,0), \cdots, (0,0), (0,0), (0,0), \cdots, (0,0)\},\\
\text{or} &\underset{G_g}{\sim} \{(\epsilon_1, \delta_1), (1,1), (0,0), (0,0), \cdots, (0,0), (0,1), (0,0), \cdots, (0,0)\},\\
\text{or} &\underset{G_g}{\sim} \{(\epsilon_1, \delta_1), (1,1), (0,1), (0,0), \cdots, (0,0), (0,0), (0,0), \cdots, (0,0)\},\\
\text{or} &\underset{G_g}{\sim} \{(\epsilon_1, \delta_1), (1,1), (0,1), (0,0), \cdots, (0,0), (0,1), (0,0), \cdots, (0,0)\},\\
\text{or} &\underset{G_g}{\sim} \{(\epsilon_1, \delta_1), (0,1), (0,0), (0,0), \cdots, (0,0), (0,0), (0,0), \cdots, (0,0)\},\\
\text{or} &\underset{G_g}{\sim} \{(\epsilon_1, \delta_1), (0,1), (0,0), (0,0), \cdots, (0,0), (0,1), (0,0), \cdots, (0,0)\},\\
\text{or} &\underset{G_g}{\sim} \{(\epsilon_1, \delta_1), (0,0), (0,0), (0,0), \cdots, (0,0), (0,0), (0,0), \cdots, (0,0)\},
\end{aligned}$$

$$\text{or} \underset{G_g}{\sim} \{(\epsilon_1,\delta_1),(0,0),(0,0),(0,0),\cdots,(0,0),(0,1),(0,0),\cdots,(0,0)\}.$$

In the second case,

$$\begin{aligned}&\{(\epsilon_1,\delta_1),(1,1),(0,0),\cdots,(0,0),(0,1),(0,0),\cdots,(0,0)\}\\ \underset{G_g}{\sim}\ &\{(\epsilon_1,\delta_1),(1,1),(0,1),(0,0),\cdots,(0,0)\}(\text{ by } (a)).\end{aligned}$$

In the 4-th case,

$$\begin{aligned}&\{(\epsilon_1,\delta_1),(1,1),(0,1),(0,0),\cdots,(0,0),(0,1),(0,0),\cdots,(0,0)\}\\ \underset{G_g}{\sim}\ &\{(\epsilon_1,\delta_1),(1,1),(0,1),(0,1),(0,0),\cdots,(0,0)\}(\text{ by } (a))\\ \underset{G_g}{\sim}\ &\{(\epsilon_1,\delta_1),(1,1),(0,1),(0,0),(0,0),\cdots,(0,0)\}(\text{ by } (d)).\end{aligned}$$

In the 6-th case,

$$\begin{aligned}&\{(\epsilon_1,\delta_1),(0,1),(0,0),\cdots,(0,0),(0,1),(0,0),\cdots,(0,0)\}\\ \underset{G_g}{\sim}\ &\{(\epsilon_1,\delta_1),(0,1),(0,1),(0,0),\cdots,(0,0)\}(\text{ by } (a))\\ \underset{G_g}{\sim}\ &\{(\epsilon_1,\delta_1),(0,1),(0,0),(0,0),\cdots,(0,0)\}(\text{ by } (d)).\end{aligned}$$

In the 8-th case,

$$\begin{aligned}&\{(\epsilon_1,\delta_1),(0,0),\cdots,(0,0),(0,1),(0,0),\cdots,(0,0)\}\\ \underset{G_g}{\sim}\ &\{(\epsilon_1,\delta_1),(0,1),(0,0),\cdots,(0,0)\}(\text{ by } (a)).\end{aligned}$$

Therefore,

$$\begin{aligned}x \underset{G_g}{\sim}\ &\{(\epsilon_1,\delta_1),(1,1),(0,0),(0,0),\cdots,(0,0)\},\\ \text{or } \underset{G_g}{\sim}\ &\{(\epsilon_1,\delta_1),(1,1),(0,1),(0,0),\cdots,(0,0)\},\\ \text{or } \underset{G_g}{\sim}\ &\{(\epsilon_1,\delta_1),(0,1),(0,0),(0,0),\cdots,(0,0)\},\\ \text{or } \underset{G_g}{\sim}\ &\{(\epsilon_1,\delta_1),(0,0),(0,0),(0,0),\cdots,(0,0)\}.\end{aligned}$$

There are 7 cases remained to consider,

$$\begin{aligned}&\{(0,0),(1,1),(0,1),(0,0),\cdots,(0,0)\}, \quad \{(0,1),(1,1),(0,0),(0,0),\cdots,(0,0)\},\\ &\{(0,1),(1,1),(0,1),(0,0),\cdots,(0,0)\}, \quad \{(0,1),(0,1),(0,0),(0,0),\cdots,(0,0)\},\\ &\{(1,1),(1,1),(0,0),(0,0),\cdots,(0,0)\}, \quad \{(1,1),(1,1),(0,1),(0,0),\cdots,(0,0)\},\\ &\{(1,1),(0,1),(0,0),(0,0),\cdots,(0,0)\}.\end{aligned}$$

By (b), the first one is G_g-equivalent to $\{(0,0),(0,0),(1,1),(0,0),\cdots,(0,0)\}$. Here, we observe actions of G_g on the first and the second blocks,

$$\{(0,1),(1,1),\cdots\} \underset{C_1}{\longrightarrow} \{(1,1),(1,1),\cdots\} \underset{\overline{DB_4}\cdot C_3\cdot C_2}{\longrightarrow} \{(0,0),(0,1),\cdots\},$$
$$\{(0,1),(0,1),\cdots\} \underset{C_1}{\longrightarrow} \{(1,1),(0,1),\cdots\} \underset{C_3C_2}{\longrightarrow} \{(0,0),(1,1),\cdots\}.$$

By the above observation, we see,

$$\begin{aligned}
\{(0,1),(1,1),(0,0),\cdots,(0,0)\} &\underset{G_g}{\sim} \{(1,1),(1,1),(0,0),\cdots,(0,0)\} \\
&\underset{G_g}{\sim} \{(0,0),(0,1),(0,0),\cdots,(0,0)\}, \\
\{(0,1),(1,1),(0,1),\cdots,(0,0)\} &\underset{G_g}{\sim} \{(1,1),(1,1),(0,1),\cdots,(0,0)\} \\
&\underset{G_g}{\sim} \{(0,0),(0,1),(0,1),\cdots,(0,0)\} \\
&\underset{G_g}{\sim} \{(0,0),(0,1),(0,0),\cdots,(0,0)\}, \\
\{(0,1),(0,1),(0,0),\cdots,(0,0)\} &\underset{G_g}{\sim} \{(1,1),(0,1),(0,0),\cdots,(0,0)\} \\
&\underset{G_g}{\sim} \{(0,0),(1,1),(0,0),\cdots,(0,0)\}.
\end{aligned}$$

Hence, we showed that any x is G_g-equivalent to the elements listed in the statement of this Lemma. □

Since

$$\begin{aligned}
&T^2_{\{(0,1),(0,0),\cdots,(0,0)\}} = D_2, \quad T^2_{\{(1,1),(0,0),\cdots,(0,0)\}} = (C_1C_2C_1^{-1})^2, \\
&T^2_{\{(0,0),(1,1),(0,0),\cdots,(0,0)\}} = (Y_2^*)^2, \quad T^2_{\{(0,0),(0,1),(0,0),\cdots,(0,0)\}} = D_4, \\
&T^2_{\{(0,0),(0,0),(1,1),(0,0),\cdots,(0,0)\}} = (Y_4^*)^2, \quad T^2_{\{(0,0),\cdots,(0,0)\}} = id,
\end{aligned}$$

these are elements of G_g. By this fact and Lemma 6.2, Lemma 6.13 is proved.

6.3 Step 3 for the case where $g \geq 3$

As in the previous subsection, let $\Phi_2\colon \mathcal{M}_g \to \mathrm{Sp}(2g,\mathbb{Z}_2)$ be the natural homomorphism. Let $q_1\colon H_1(\Sigma_g;\mathbb{Z}_2) \to \mathbb{Z}_2$ be the quadratic form associated with the intersection form $(\,,)_2$ of $H_1(\Sigma_g;\mathbb{Z}_2)$ which satisfies, for the basis x_i, y_i of $H_1(\Sigma_g;\mathbb{Z}_2)$ indicated on Figure 9, $q_1(x_1) = q_1(y_1) = 1$, and $q_1(x_i) = q_1(y_i) = 0$ when $i \neq 1$. We define $\mathrm{O}_{q_1}(2g,\mathbb{Z}_2) = \{\phi \in \mathrm{Aut}(H_1(\Sigma_g;\mathbb{Z}_2)) | q_1(\phi(x)) = q_1(x)$ for any $x \in H_1(\Sigma_g;\mathbb{Z}_2)\}$, then $\mathcal{SP}_g[q_1] = \Phi_2^{-1}(\mathrm{O}_{q_1}(2g,\mathbb{Z}_2))$. Because of Lemma 6.13, if we show $\Phi_2(G_g) = \mathrm{O}_{q_1}(2g,\mathbb{Z}_2)$, then $G_g = \mathcal{SP}_g[q_1]$ follows.

For any $z \in H_1(\Sigma_g;\mathbb{Z}_2)$ such that $q_1(z)=1$, we define $\mathbb{T}_z(x) = x + (z,x)_2\, z$. Then $\mathbb{T}_z$ is an element of $\mathrm{O}_{q_1}(2g,\mathbb{Z}_2)$, and we call this a $\mathbb{Z}_2$*-transvection about* z. Dieudonné [4] showed the following Theorem (see also [7, Chap.14]).

Theorem 6.17 [4, Proposition 14 on p.42] *When* $g \geq 3$, $\mathrm{O}_{q_1}(2g,\mathbb{Z}_2)$ *is generated by* $\mathbb{Z}_2$*-transvections.*

Let Λ_g be the set of z of $H_1(\Sigma_g;\mathbb{Z}_2)$ such that $q(z)=1$. For any elements z_1 and z_2 of Λ_g, we define $z_1 \Box z_2 = z_1 + (z_2,z_1)_2\, z_2$. Here, we remark that $\mathbb{T}_{z_1}^2 = \mathrm{id}$, $\mathbb{T}_{z_2}\mathbb{T}_{z_1}\mathbb{T}_{z_2}^{-1} = \mathbb{T}_{z_1\Box z_2}$ and $(z_1\Box z_2)\Box z_2 = z_1$. An element $\epsilon_1 x_1 + \delta_1 y_1 + \cdots + \epsilon_g x_g + \delta_g y_g$ of $H_1(\Sigma_g;\mathbb{Z}_2)$ is denoted by $[(\epsilon_1,\delta_1),\cdots,(\epsilon_g,\delta_g)]$, and each (ϵ_i,δ_i) is called the i-th block. We remark that $q([(\epsilon_1,\delta_1),\cdots,(\epsilon_g,\delta_g)]) = (\epsilon_1+\delta_1+\epsilon_1\delta_1) + \epsilon_2\delta_2 + \cdots \epsilon_g\delta_g$.

Lemma 6.18 *Under the operation* $\Box$, Λ_g *is generated by* x_1, y_1, x_1+x_2, x_i+y_i $(2 \leq i \leq g)$, $x_i+y_i+x_{i+1}$ $(2 \leq i \leq g-1)$, *and* $x_i + x_{i+1} + y_{i+1}$ $(2 \leq i \leq g-1)$.

Proof For an element $[(\epsilon_1,\delta_1),\cdots,(\epsilon_g,\delta_g)]$ of $H_1(\Sigma_g;\mathbb{Z}_2)$, let the j-th block be the right most block which is $(1,1)$. When $j \geq 3$, there exist 4 cases of the combination of the $(j-1)$-st block and the j-th block: $[\cdots,(1,1),(1,1),\cdots]$, $[\cdots,(0,0),(1,1),\cdots]$, $[\cdots,(0,1),(1,1),\cdots]$, $[\cdots,(1,0),(1,1),\cdots]$. In each case, we can reduce j at least 1. In fact,

$$\begin{aligned}
&[\cdots,(1,1),(1,1),\cdots]\Box(x_{j-1}+x_j+y_j) = [\cdots,(0,1),(0,0),\cdots],\\
&[\cdots,(0,0),(1,1),\cdots]\Box(x_{j-1}+y_{j-1}+x_j) = [\cdots,(1,1),(0,1),\cdots],\\
&[\cdots,(0,1),(1,1),\cdots]\Box(x_{j-1}+x_j+y_j) = [\cdots,(1,1),(0,0),\cdots],\\
&([\cdots,(1,0),(1,1),\cdots]\Box(x_{j-1}+y_{j-1}))\Box(x_{j-1}+x_j+y_j)\\
&\qquad\qquad = [\cdots,(1,1),(0,0),\cdots].
\end{aligned}$$

When $j=2$, since $q([(\epsilon_1,\delta_1),\cdots,(\epsilon_g,\delta_g)])=1$, $[(\epsilon_1,\delta_1),\cdots,(\epsilon_g,\delta_g)]$ must be $[(0,0),(1,1),\cdots]$. Because of an equation

$$([(0,0),(1,1),\cdots]\Box(x_1+x_2))\Box y_1 = [(1,1),(0,1),\cdots],$$

we can reduce j to 1. When $j=1$, if every i-th $(i \geq 2)$ block is $(0,0)$, then it is x_1+y_1, which is equal to $x_1\Box y_1$. If there exist at least one of the i-th

$(i \geq 2)$ blocks which are $(1,0)$ or $(0,1)$, then,

$$[\cdots,(0,0),\overset{i}{(1,0)},\cdots]\square(x_{i-1}+x_i+y_i)=[\cdots,(1,0),(0,1),\cdots],$$
$$[\cdots,(1,0),\overset{i}{(0,0)},\cdots]\square(x_{i-1}+y_{i-1}+x_i)=[\cdots,(0,1),(1,0),\cdots],$$
$$[\cdots,(0,0),\overset{i}{(0,1)},\cdots]\square(x_{i-1}+x_i+y_i)=[\cdots,(1,0),(1,0),\cdots],$$
$$[\cdots,(0,1),\overset{i}{(0,0)},\cdots]\square(x_{i-1}+y_{i-1}+x_i)=[\cdots,(1,0),(1,0),\cdots].$$

Therefore, we can alter this to an element, each i-th $(i \geq 2)$ block of which is $(1,0)$ or $(0,1)$. If the i-th block of this is $(0,1)$, then

$$[\cdots,(0,1),\cdots]\square(x_i+y_i)=[\cdots,(1,0),\cdots].$$

Therefore, it suffices to consider the case where the first block is $(1,1)$ and other blocks are $(1,0)$. In this case,

$$([\cdots,(1,0),(1,0)]\square(x_{g-1}+y_{g-1}+x_g))\square(x_{g-1}+y_{g-1})=[\cdots,(1,0),(0,0)].$$

By applying the same operation repeatedly, we get $[(1,1),(1,0),(0,0),\cdots,(0,0)]$, which is equal to $y_1\square(x_1+x_2)$. □

This lemma and Theorem 6.17 shows that

Corollary 6.19 $\mathrm{O}_{q_1}(2g,\mathbb{Z}_2)$ *is generated by* $\mathbb{T}_{x_1}$, $\mathbb{T}_{y_1}$, $\mathbb{T}_{x_1+x_2}$, $\mathbb{T}_{x_i+y_i}$ $(2 \leq i \leq g)$, $\mathbb{T}_{x_i+y_i+x_{i+1}}$ $(2 \leq i \leq g-1)$, *and* $\mathbb{T}_{x_i+x_{i+1}+y_{i+1}}$ $(2 \leq i \leq g-1)$. □

Since G_g is a subgroup of $\mathcal{SP}_g[q_1]$, $\Phi_2(G_g) \subset \mathrm{O}_{q_1}(2g,\mathbb{Z}_2)$. On the other hand, the fact that $\Phi_2(C_1)=\mathbb{T}_{x_1}$, $\Phi_2(C_2)=\mathbb{T}_{y_1}$, $\Phi_2(C_3)=\mathbb{T}_{x_1+x_2}$, $\Phi_2(X_{2i})=\mathbb{T}_{x_i+y_i+x_{i+1}}$ $(2 \leq i \leq g-1)$, $\Phi_2(X_{2i+1})=\mathbb{T}_{x_i+x_{i+1}+y_{i+1}}$ $(2 \leq i \leq g-1)$, $\Phi_2(Y_{2j})=\mathbb{T}_{x_j+y_j}$ $(2 \leq j \leq g-1)$, $\Phi_2(X_{2g})=\mathbb{T}_{x_g+y_g}$, and Corollary 6.19, show $\Phi_2(G_g) \supset \mathrm{O}_{q_1}(2g,\mathbb{Z}_2)$. Therefore we proved that $\mathcal{SP}_g[q_1]=G_g$ when $g \geq 3$.

6.4 Genus 2 case: Reidemeister-Schreier method

Birman and Hilden showed the following Theorem.

Theorem 6.20 [2] *$\mathcal{M}_2$ is generated by C_1,C_2,C_3,C_4,C_5 and its defining relations are:*

(1) $C_iC_j=C_jC_i$, *if* $|i-j| \geq 2$, $i,j=1,2,3,4,5$,

(2) $C_iC_{i+1}C_i=C_{i+1}C_iC_{i+1}$, $i=1,2,3,4$,

(3) $(C_1C_2C_3C_4C_5)^6 = 1$,

(4) $(C_1C_2C_3C_4C_5C_5C_4C_3C_2C_1)^2 = 1$,

(5) $C_1C_2C_3C_4C_5C_5C_4C_3C_2C_1 \rightleftarrows C_i$, $i = 1, 2, 3, 4, 5$,

where $\rightleftarrows$ *means "commute with".*

We call (1) (2) of the above relations *braid relations*. We will use the well-known method, called *the Reidemeister–Schreier method* [18, §2.3], to show $\mathcal{SP}_2[q_1] \subset G_2$. We review (a part of) this method.

Let G be a group generated by finite elements $g_1, \ldots, g_m$ and H be a finite index subgroup of G. For two elements a, b of G, we write $a \equiv b \bmod H$ if there is an element h of H such that $a = hb$. A finite subset S of G is called *a coset representative system* for G mod H, if, for each elements g of G, there is only one element $\overline{\overline{g}} \in S$ such that $g \equiv \overline{\overline{g}} \bmod H$. The set $\{sg_i\overline{\overline{sg_i}}^{-1} \mid i = 1, \ldots, m, \ s \in S\}$ generates H.

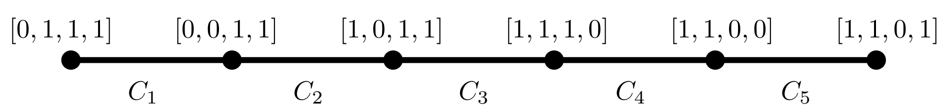


Figure 17

For the sake of giving a coset representative system for $\mathcal{M}_2$ modulo $\mathcal{SP}_2[q_1]$, we will draw a graph Γ which represents the action of $\mathcal{M}_2$ on the quadratic forms of $H_1(\Sigma_2;\mathbb{Z}_2)$ with Arf invariants 1. Let $[\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4]$ denote the quadratic form q' of $H_1(\Sigma_2;\mathbb{Z}_2)$ such that $q'(x_1) = \epsilon_1$, $q'(y_1) = \epsilon_2$, $q'(x_2) = \epsilon_3$, $q'(y_2) = \epsilon_4$. Each vertex of Γ corresponds to a quadratic form. For each generator C_i of $\mathcal{M}_2$, we denote its action on $H_1(\Sigma_2;\mathbb{Z}_2)$ by $(C_i)_*$. For the quadratic form q' indicated by the symbol $[\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4]$, let $\delta_1 = q'((C_i)_*x_1)$, $\delta_2 = q'((C_i)_*y_1)$, $\delta_3 = q'((C_i)_*x_2)$, and $\delta_4 = q'((C_i)_*y_2)$. Then, we connect two vertices, corresponding to $[\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4]$, $[\delta_1, \delta_2, \delta_3, \delta_4]$ respectively, by the edge with the letter C_i. We remark that this action is a right action. For simplicity, we omit the edge whose ends are the same vertex. As a result, we get a graph Γ as in Figure 17. The words $S = \{1,\ C_5,\ C_4,\ C_4C_3,\ C_4C_3C_2,\ C_4C_3C_2C_1\}$, which correspond to the edge paths beginning from $[1, 1, 0, 0]$ on Γ, define a coset representative system for $\mathcal{M}_2$ modulo $\mathcal{SP}_2[q_1]$. For each element g of $\mathcal{M}_2$, we can give a $\overline{\overline{g}} \in S$ with using this graph. For example, say $g = C_2C_4C_5C_3$, we follow an edge path assigned to this word which begins from $[1, 1, 0, 0]$, (note that we read words from left to right) then we arrive at the vertex

Table 1: Generators of $\mathcal{SP}_2[q_1]$

	C_1	C_2	C_3
1	1	1	1
C_5	C_1	C_2	C_3
C_4	C_1	C_2	1
C_4C_3	C_1	1	$C_3^{-1}D_4C_3$
$C_4C_3C_2$	1	$C_2^{-1}C_3^{-1}D_4C_3C_2$	C_2
$C_4C_3C_2C_1$	$C_1^{-1}C_2^{-1}C_3^{-1}D_4C_3C_2C_1$	C_1	C_2

	C_4	C_5
1	1	1
C_5	1	D_5
C_4	D_4	$D_5^{-1}X_4D_5$
C_4C_3	C_3	$D_5^{-1}X_4D_5$
$C_4C_3C_2$	C_3	$D_5^{-1}X_4D_5$
$C_4C_3C_2C_1$	C_3	$D_5^{-1}X_4D_5$

$[1,0,1,1]$. The element in T which begins from $[1,1,0,0]$ and ends at $[1,0,1,1]$ is C_4C_3. Hence, $\overline{\overline{C_2C_4C_5C_3}} = C_4C_3$. We list in Table 1 the set of generators $\{sC_i\overline{\overline{sC_i}}^{-1} \mid i = 1,\dots,5,\ s \in S\}$ of $\mathcal{SP}_g[q_1]$. In Table 1, vertical direction is a coset representative system S, horizontal direction is a set of generators $\{C_1,\ C_2,\ C_3,\ C_4,\ C_5\}$. We can check this table by Figure 17 and braid relations. For example,

$$\begin{aligned}
&C_4C_3C_2C_1\cdot C_2\overline{\overline{C_4C_3C_2C_1\cdot C_2}}^{-1} = C_4C_3C_2C_1C_2(C_4C_3C_2C_1)^{-1}\\
&= C_4C_3C_2C_1C_2C_1^{-1}C_2^{-1}C_3^{-1}C_4^{-1} = C_4C_3C_2C_2^{-1}C_1C_2C_2^{-1}C_3^{-1}C_4^{-1}\\
&= C_4C_3C_1C_3^{-1}C_4^{-1} = C_1.
\end{aligned}$$

This table shows that $\mathcal{SP}_2[q_1] \subset G_2$.

7 Proof of Theorem 5.1

We embed H_{g-1} standardly in $S^3 = \partial D_4$ such that there is a 2-sphere separating $F_{3,3}$ and H_{g-1}, and make a connected sum $F_{3,3}\#\partial H_{g-1}$ as indicated in Figure 18. Then, we can see $(\mathbb{CP}^2, K_3\#\Sigma_{g-1}) = (\mathbb{CP}^2, (F_{3,3}\#\partial H_{g-1})\cup D_3)$, where K_3 is the non-singular plane curve of degree 3 and D_3 is parallel three disks which is used to construct K_3 in §4. We identify $K_3\#\Sigma_{g-1}$ with Σ_g so that

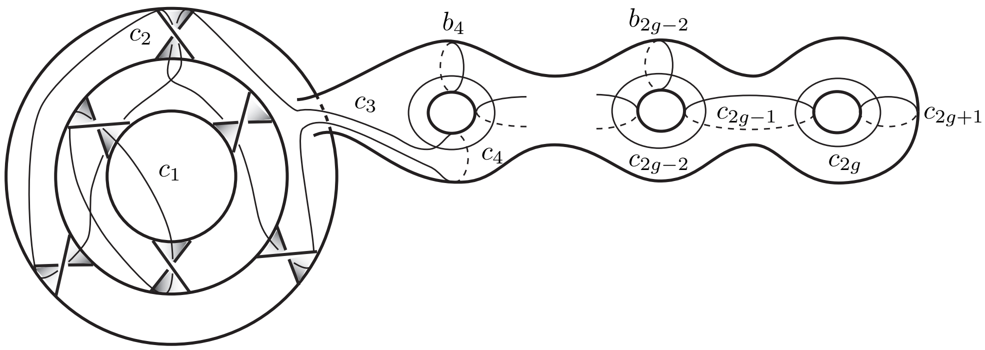


Figure 18

simple closed curves with the same symbol are identified. Then $q_{K_3\#\Sigma_{g-1}} = q_1$. We will show that each elements of $\mathcal{SP}_g[q_{K_3\#\Sigma_{g-1}}] = \mathcal{SP}_g[q_1]$ is extendable.

Each regular neighborhood of c_1, c_2, c_3, $C_{i+1}(c_i)$ $(4 \leq i \leq 2g)$, and $C_{2j}(b_{2j})$ $(2 \leq j \leq g-1)$ is Hopf band. Therefore, by Proposition 2.1, C_1, C_2, C_3, $C_{i+1}C_i\overline{C_{i+1}}$ $(4 \leq i \leq 2g)$, and $C_{2j}B_{2j}\overline{C_{2j}}$ $(2 \leq j \leq g-1)$ are elements of $\mathcal{E}(\mathbb{CP}^2, K_3\#\Sigma_{g-1})$. Each regular neighborhood of c_i $(4 \leq i \leq 2g+1)$, b_{2j} $(2 \leq i \leq g-1)$ is an annulus standardly embedded in $S^3 = \partial D^4$. We can deform this annulus as indicated in Figure 1. Therefore, C_i^2 $(4 \leq i \leq 2g+1)$, B_{2j}^2 $(2 \leq j \leq g-1)$ are elements of $\mathcal{E}(\mathbb{CP}^2, K_3\#\Sigma_{g-1})$. Finally, the extendability of $B_4C_5C_7\dots C_{2g+1}$ follows from the proof of Lemma 2.2 in [11]. Therefore, we showed $\mathcal{SP}_g[q_{K_3\#\Sigma_{g-1}}] \subset \mathcal{E}(\mathbb{CP}^2, K_3\#\Sigma_{g-1})$. On the other hand, by the definition of the Rokhlin quadratic form $q_{K_3\#\Sigma_{g-1}}$, we see $\mathcal{E}(\mathbb{CP}^2, K_3\#\Sigma_{g-1})$ $\subset \mathcal{SP}_g[q_{K_3\#\Sigma_{g-1}}]$. Theorem 5.1 follows.

Acknowledgments

The author would like to express his gratitude to Professors Masaharu Ishikawa, Masahico Saito, and Akira Yasuhara for fruitful discussions and comments. The author would also like to thank the referee, whose comments and corrections improved the paper. This research was partially supported by Grant-in-Aid for Encouragement of Young Scientists (No. 16740038), Ministry of Education, Culture, Sports, Science and Technology, Japan.

References

[1] **S. Akbulut and R. Kirby**, *Branched covers of surfaces in 4-manifolds*, Math. Ann. 252 (1980), 111–131. MathReview

[2] **J.S. Birman and H. Hilden**, *On the mapping class group of closed surface as covering spaces*, In "Advances in the theory of Riemann surfaces" Ann. of Math. Studies 66 (1971), 81–115. MathReview

[3] **J. Boyle**, *Classifying 1-handles attached to knotted surfaces*, Trans. Amer. Math. Soc. 306 (1988), 475–487. MathReview

[4] **J. Dieudonné**, *Sur les groupes classiques*, Actualites Sci. Ind., no. 1040 = Publ. Inst. Math. Univ. Strasbourg (N.S.) no. 1 (1945). Hermann et Cie., Paris, 1948. MathReview

[5] **M. Freedman and R. Kirby**, *A geometric proof of Rochlin's theorem*, Proc. Symp. Pure Math. 32 (1978), 85–97. MathReview

[6] **R. Gompf and A. Stipsicz**, *4-manifolds and Kirby calculus*, Grad. Stud. in Math. 20, American Mathematical Society, 1999. MathReview

[7] **L.C. Grove**, *Classical groups and geometric algebra*, Grad. Stud. in Math. 39, American Mathematical Society, 2002. MathReview

[8] **J.L. Harer**, *Stability of the homology of the moduli spaces of Riemann surfaces with spin structure*, Math. Ann. 287 (1990), 323–334. MathReview

[9] **J.L. Harer**, *The rational Picard group of the moduli space of Riemann surfaces with spin structure*, Contemp. Math. 150 (1993), 107–136. MathReview

[10] **S. Hirose**, *On diffeomorphisms over T^2-knot*, Proc. of A.M.S. 119 (1993), 1009–1018. MathReview

[11] **S. Hirose**, *On diffeomorphisms over trivially embedded in the 4-sphere*, Algebr. Geom. Topol. 2 (2002) 791–824. MathReview

[12] **L.K. Hua and I. Reiner**, *On the generators of the symplectic modular group,* Trans. Amer. Math. Soc. 65 (1949), 415–426. MathReview

[13] **Z. Iwase**, *Dehn surgery along a torus T^2-knot. II*, Japan. J. Math. 16 (1990), 171–196. MathReview

[14] **D. Johnson**, *The structure of the Torelli Group I: A finite set of generators for $\mathcal{I}$*, Ann. of Math. 118 (1983), 423–442. MathReview

[15] **D. Johnson**, *The structure of the Torelli Group III: The abelianization of $\mathcal{I}$*, Topology 24 (1985), 127–144. MathReview

[16] **W.B.R. Lickorish**, *A finite set of generators for the homeotopy group of a 2-manifold*, Proc. Cambridge Philos. Soc. 60 (1964), 769–778, Corrigendum: Proc. Cambridge Philos. Soc. 62 (1966), 679–681. MathReview

[17] **Y. Matsumoto**, *An elementary proof of Rochlin's signature theorem and its extension by Guillou and Marin*, A la Recherche de la Topologie Perdue, Progress in Math., 62 (1986), 119–139. MathReview

[18] **W. Magnus, A. Karras and D. Solitar**, *Combinatorial Group Theory*, reprint of the 1976 second edition. Dover Publications, Inc. Mineola, NY (2004) MathReview

[19] **J.M. Montesinos**, *On twins in the four-sphere I*, Quart. J. Math. Oxford (2), 34 (1983), 171–199. MathReview

[20] **V.R. Rohlin**, *Proof of a conjecture of Gudkov*, Funkcional. Anal. i Priložen. 6 (1972) 62–64 (Russian); translation: Functional Analysis and its Applications, 6 (1972), 136–138. MathReview

[21] **C.P. Rourke and D.P. Sullivan**, *On the Kervaire obstruction*, Ann. of Math. (2) 94 (1971), 397–413. MathReview

Department of Mathematics, Faculty of Science and Engineering
Saga University, Saga, 840-8502 Japan.

Email: `hirose@ms.saga-u.ac.jp`

Received: 13 February 2005 Revised: 28 April 2005