ISSN 1472-2739 (on-line) 1472-2747 (printed)

Algebraic & Geometric Topology
Volume 5 (2005) 741–750
Published: 23 July 2005

Boundary slopes (nearly) bound cyclic slopes

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Abstract Let r_m and r_M be the least and greatest finite boundary slopes of a hyperbolic knot K in S^3. We show that any cyclic surgery slopes of K must lie in the interval $(r_m - 1/2, r_M + 1/2)$.

AMS Classification 57M25; 57M27

Keywords Dehn surgery, character variety, exceptional surgery, boundary slope

1 Introduction

In [M1, M2] we observed that the Seifert surgeries of $(-3, 3, n)$ pretzel knots follow an interesting pattern, summarised in the table below. For each positive integer n, the Seifert surgeries of the $(-3, 3, n)$ pretzel lie between the boundary slopes 0 and $8/(n+1)$. Indeed, all integral slopes in the interval $(0, 8/(n+1))$ are Seifert.

n	1	2	3	4	5	6	≥ 7
$8/(n+1)$	4	8/3	2	8/5	4/3	8/7	≤ 1
Seifert Surgeries	$1, 2, 3$	$1, 2$	1	1	1	1	none

Some other famous knots also share this pattern; all integral slopes between two boundary slopes (shown in bold typeface) are Seifert:

Knot	Non-trivial exceptional surgeries
Figure 8	**−4**, −3, −2, −1, **0**, 1, 2, 3, **4**
(-2,3,7)	**16**, 17, 18, **37/2**, 19, **20**
Twist Knots	**0**, 1, 2, 3, **4**

Based on this and other evidence, Kimihiko Motegi posed the question: "Are Seifert surgeries bounded by boundary slopes?" In other words, if slope t is a Seifert surgery slope, are there necessarily boundary slopes r_m and r_M with

$r_m \leq t \leq r_M$? In [IMS], we construct a parameter c, such that $r_m - c \leq t \leq r_M + c$. More precisely, we can reformulate [IMS, Corollary 3] as follows. (In part 3 of Theorem 1, the constant s is the minimal Culler-Shalen norm defined in Section 2 below while A counts the characters of non-abelian representations of the knot exterior that factor through the surgery. We refer the reader to [IMS] for details.)

Theorem 1 (Corollary 3 of [IMS]) *Let r_m and r_M be the least and greatest finite boundary slopes of a hyperbolic knot K and t a non-trivial exceptional surgery slope. Then $r_m - c \leq t \leq r_M + c$ where c depends on the type of slope t.*

(1) *If t is cyclic, $c = 1$.*

(2) *If $t = a/b$ is finite, $c = 3/b$.*

(3) *If $t = a/b$ is a Seifert fibred slope, $c = (1 + 2A/s)/b$.*

In this formulation, Motegi's conjecture corresponds to showing $c = 0$ for a Seifert fibred surgery. That $c = 1$ for a cyclic surgery was first shown by Dunfield [Du]. In the current article, we show that for a cyclic surgery, we can take $c = 1/2$.

Theorem 2 *If t is a non-trivial cyclic surgery on a hyperbolic knot K in S^3 and r_m and r_M are the least and greatest finite boundary slopes of K, then $r_m - \frac{1}{2} < t < r_M + \frac{1}{2}$.*

Moreover, the theorem applies more generally to a hyperbolic knot in a manifold with cyclic fundamental group whose exterior satisfies $H^1(M, \mathbb{Z}_2) = \mathbb{Z}_2$.

In light of Theorem 1, it is natural to extend Motegi's question about Seifert surgeries to exceptional surgeries in general:

Question 1 *Do boundary slopes bound exceptional slopes?*

In other words, for a hyperbolic knot K in S^3 do all non-trivial exceptional surgery slopes lie in the interval $[r_m, r_M]$ between the least and greatest finite boundary slopes? Note that, by the Cabling Conjecture, K should have no reducible surgeries and that toroidal surgeries are themselves boundary slopes and will, therefore, necessarily lie in $[r_m, r_M]$. The real question is whether other types of exceptional surgeries (i.e., cyclic, finite, Seifert fibred) must also lie in this interval. It is the Seifert case that gives the largest values for c

in Theorem 1 and, in any case, cyclic and finite surgeries are thought to be examples of Seifert surgeries. Thus, an affirmative answer to Motegi's question is likely to imply the same for all types of exceptional surgeries.

In Section 2 we provide definitions and discuss the geometry of the Culler-Shalen norm. In Section 3, we prove Theorem 2.

2 Definition, geometry of the Culler-Shalen norm

Let K be a hyperbolic knot in S^3 and let $M = S^3 \setminus N(K)$ denote the knot exterior. Fixing the usual meridian, longitude basis $\{\mu, \lambda\}$, the element $\gamma = a\mu + b\lambda$ of $H_1(\partial M;\mathbb{Z})$ will be represented as (a,b). This class can be identified with the *"slope"* $r_\gamma = a/b$ in $\mathbb{Q} \cup \{\frac{1}{0}\}$. We will occasionally wish to change our framing which amounts to replacing λ by $k\mu + \lambda$ and to changing coordinates by $(a,b) \mapsto (a - bk, b)$.

Let $M(r)$ denote the manifold obtained by Dehn surgery along slope r (i.e., $M(r)$ is constructed by attaching a solid torus to M such that the boundaries of meridional disks are curves of slope r in ∂M). We will call r a *cyclic* (respectively *finite*) *slope* if $\pi_1(M(r))$ is cyclic (resp. finite). If $M(r)$ admits the structure of a Seifert fibred space, we call r a *Seifert fibred slope.* Since $M(\frac{1}{0}) = S^3$, we refer to meridional surgery along slope $r_\mu = \frac{1}{0}$ as *trivial surgery.*

If there is an essential surface Σ in M that meets ∂M in a non-empty set of parallel curves of slope r, we call r a *boundary slope.* If there is such a Σ that is not a fibre in a fibration of M over S^1, r is a *strict boundary slope.* For example, by applying the loop theorem to a Seifert surface of K, we observe that 0 is a boundary slope. We will say r is a *finite boundary slope* if it is a boundary slope and $r \neq \frac{1}{0}$.

The proof of Theorem 2 depends on the geometry of the Culler-Shalen norm of K. We introduce some of the main properties of this norm and refer the reader to [CGLS, Chapter 1] for a more complete account.

Let $R = \mathrm{Hom}(\pi, \mathrm{SL}_2(\mathbb{C}))$ denote the set of $\mathrm{SL}_2(\mathbb{C})$-representations of the fundamental group π of M. Then R is an affine algebraic set, as is X, the set of characters of representations in R.

For $\gamma \in \pi$, define the regular function $I_\gamma : X \to \mathbb{C}$ by $I_\gamma(\chi_\rho) = \chi_\rho(\gamma) = \mathrm{trace}(\rho(\gamma))$. By the Hurewicz isomorphism, a class $\gamma \in L = H_1(\partial M, \mathbb{Z})$ determines an element of $\pi_1(\partial M)$, and therefore an element of π well-defined up to conjugacy. A *norm curve* is a one-dimensional irreducible component of X on which no I_γ ($\gamma \in L \setminus \{0\}$) is constant. For example, the irreducible component, X_0, that contains the character of the holonomy representation is a norm curve.

The terminology reflects the fact that we may associate to X_0 a norm $\|\cdot\|$ on $H_1(\partial M, \mathbb{R})$ called a *Culler-Shalen norm* in the following manner. Let $\widetilde{X}_0$ be the smooth projective model of X_0 which is birationally equivalent to X_0. The birational map is regular at all but a finite number of points of $\widetilde{X}_0$ which are called *ideal points* of $\widetilde{X}_0$. The function $f_\gamma = I_\gamma^2 - 4$ is again regular and so can be pulled back to $\widetilde{X}_0$. For $\gamma \in L$, $\|\gamma\|$ is the degree of $f_\gamma : \widetilde{X}_0 \to \mathbb{CP}^1$. The norm is extended to $H_1(\partial M, \mathbb{R})$ by linearity.

Let $s = \min_{0 \neq \gamma \in H_1(\partial M, \mathbb{Z})} \|\gamma\|$ denote the *minimal norm*. The norm disc of radius s is a convex, finite-sided polygon P that is symmetric about the origin. We will call P the *fundamental polygon*. The ideal points of $\widetilde{X}_0$ can be associated with a set $\mathcal{B}$ of strict boundary slopes of the knot and the vertices of P occur at rational multiples of the classes of slopes in $\mathcal{B}$. It follows that $\mathcal{B}$ must contain at least two slopes. One of the main results of [CGLS] is that if r_γ is a cyclic slope that is not a strict boundary slope then $\|\gamma\| = s$. Moreover, r_γ is either integral or trivial (i.e., $r_\gamma = \frac{1}{0}$).

3 Proof of Theorem 2

We will prove two propositions before coming to the proof of the theorem. The main external inputs for our argument are the Cyclic Surgery Theorem [CGLS] and Theorems 4.1 and 4.2 of [Du]. Thus, although we formulate our results in terms of a knot in S^3, they carry over to the case of a hyperbolic knot in a manifold with cyclic fundamental group whose exterior satisfies $H^1(M, \mathbb{Z}_2) = \mathbb{Z}_2$.

Proposition 1 *Let K be a hyperbolic knot in S^3. Let r_M be the greatest finite boundary slope of K. Suppose γ is a non-trivial cyclic class with $r_\gamma = n > 0$. Then $n \leq r_M + \frac{1}{2}$.*

Remark In fact, we will show that $n \leq r + \frac{1}{2}$ where r is the greatest finite boundary slope associated to the norm curve X_0. In particular, r is a strict boundary slope.

Proof Let X_0 be the norm curve that contains the character of the holonomy representation and let $\mathcal{B}$ be the associated set of boundary slopes. If $r_\gamma \in \mathcal{B}$, then $n = r_\gamma \leq r_M$. So we may assume $r_\gamma \notin \mathcal{B}$.

Suppose, for a contradiction, that $n > r_M + \frac{1}{2}$. Without loss of generality, we may assume $r_M \in \mathcal{B}$ (otherwise replace r_M by the greatest finite boundary slope in $\mathcal{B}$). Our goal is to argue that $(n-1,1)$ is in the interior of P (see Figure 1) where P is the fundamental polygon of the Culler-Shalen norm $\|\cdot\|$

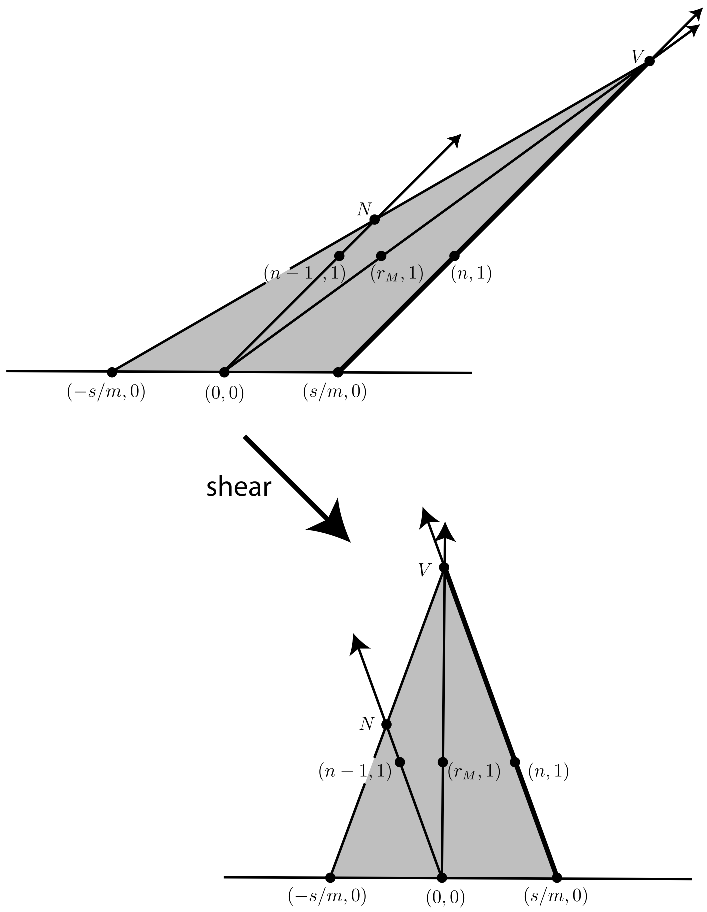

Figure 1: The geometry of P assuming $n > r_M + \frac{1}{2}$

associated to X_0. Let us outline the argument. First, it has already been shown that r_M must be non-integral [Du]. Construct the line from $(s/m, 0)$ through $(n,1)$, where $m = \|\mu\|$ and s is the minimal norm. This line will form part of the boundary of P and continues to the vertex V determined by the boundary slope r_M. By convexity, the line joining $(-s/m, 0)$ to V is also in P. It follows that $(n-1,1)$ is in the interior of P so that $\|(n-1,1)\| < s$. (This is "obvious" if one applies a shear as in Figure 1.) This is a contradiction as s is defined to be the minimal norm. Thus, we conclude $n \leq r_M + \frac{1}{2}$.

Now let's fill in the details. Since $n > r_M + \frac{1}{2}$, if r_M is an integer, then $r_M \leq n - 1$. However, by [Du, Theorem 4.2] there is a strict boundary slope $r_\delta \in \mathcal{B}$ with $|n - r_\delta| < 1$. Then $r_M < r_\delta$ in contradiction to the choice of r_M. Therefore, r_M is not an integer. Moreover, since there is a strict boundary slope δ with $|n - r_\delta| < 1$ we have $n - 1 < r_M < n$.

We next construct the vertex V of P corresponding to the boundary slope r_M. Let $\|\mu\| = m$. (If $r_\mu = \frac{1}{0} \notin \mathcal{B}$, then $m = s$.) The point $(s/m, 0)$ is then in ∂P. Since r_M is maximal among slopes in $\mathcal{B}$, and $r_M < n < \infty$, the segment joining $(s/m, 0)$ and $(n, 1)$ is part of the boundary of P. This segment has equation $y = (x - s/m)/(n - s/m)$. It continues to the line $y = x/r_M$. (Since there are no strict boundary slopes between n and r_M, the segment has no vertex before it reaches the line $y = x/r_M$ corresponding to the boundary slope r_M.) These lines meet at the point

$$V = \frac{s/m}{r_M + s/m - n}(r_M, 1)$$

which is therefore a vertex of P.

Since P is convex, the segment joining V and $(-s/m, 0)$ (both in P) is contained in P. We argue that the point N where this segment crosses $y = x/(n-1)$ is above the line $y = 1$. Indeed, the segment has the equation

$$y = \frac{x + s/m}{2r_M + s/m - n}.$$

It meets the line $y = x/(n-1)$ at the point

$$N = \frac{s/m}{2(r_M - n) + 1 + s/m}(n - 1, 1)$$

which is therefore in P. Let y_N denote the y coordinate of N.

$$\begin{aligned} n > r_M + \frac{1}{2} \quad &\Rightarrow \quad 0 > 2(r_M - n) + 1 \\ &\Rightarrow \quad s/m + 2(r_M - n) + 1 < s/m \\ &\Rightarrow \quad y_N = \frac{s/m}{2(r_M - n) + 1 + s/m} > 1 \end{aligned}$$

Since $y_N > 1$, the point $(n-1, 1)$ is in the interior of P and, therefore, $\|(n-1, 1)\| < s$. This is a contradiction as s is defined to be the minimal norm. We conclude that $n \leq r_M + \frac{1}{2}$. □

We will now show that Proposition 1 can be strengthened to a strict inequality if the meridian is not a strict boundary class for the norm curve X_0. The argument makes use of the idea of the diameter D of the set of boundary slopes. Culler and Shalen [CS] showed that $D \geq 2$. This inequality is sharp by an example of Dunfield of a knot in a manifold with cyclic fundamental group (see [CS]). For hyperbolic knots in S^3, the smallest known diameter is $D = 8$ for the Figure 8 knot.

Proposition 2 *Let X_0 be the norm curve containing the character of the holonomy representation for the hyperbolic knot $K \in S^3$. Let $\mathcal{B}$ be the associated boundary slopes and suppose that $r_\mu = \frac{1}{0} \notin \mathcal{B}$. Let γ be a non-trivial cyclic class with $r_\gamma = n > 0$. Let r_m and r_M denote the least and greatest boundary slopes in $\mathcal{B}$. Let $D = \mathit{Diam}(\mathcal{B}) = r_M - r_m$. Then $n \leq r_M + 1 - \frac{1}{2}(D - \sqrt{D(D-2)}) < r_M + \frac{1}{2}$.*

Remark The difference between n and r_M goes to zero as D approaches 2. For $D = 8$ (Figure 8 knot) we have $1 - \frac{1}{2}(D - \sqrt{D(D-2)}) = -3 + 2\sqrt{3} \approx 0.46$.

Proof Now $\|\mu\| = s$ and $\pm(1,0) \in \partial P$. If $n \leq r_M$, the proposition holds. So, using Proposition 1, we'll assume $r_M < n \leq r_M + \frac{1}{2}$. Let's change the framing so that n becomes 1. We will use $\sim$ to refer to measurements in the new framing. Thus, $\tilde{n} = 1$, $\tilde{r}_M = r_M - n + 1$ and $\tilde{r}_m = r_m - n + 1$. The line through $(\tilde{n}, 1)$ and $(1,0)$ is then vertical and and meets $y = x/\tilde{r}_M$ at $V = (1, 1/\tilde{r}_M)$ (see Figure 2).

The segment in ∂P which passes through V, $(\tilde{n}, 1)$, and $(1,0)$ continues to the line $y = x/\tilde{r}_m$ as there are no boundary slopes between $\tilde{r}_m$ and $\tilde{r}_M$ to provide a vertex. The intersection point $W' = (1, 1/\tilde{r}_m)$ is therefore a vertex of P as is its reflection $W = -W'$.

Since P is convex, the segment joining V and W is contained in P. It meets the line $y = 0$ at the point

$$M = (0, \frac{\tilde{r}_m - \tilde{r}_M}{2\tilde{r}_M \tilde{r}_m}).$$

Since $\|(0,1)\| \geq s$, the y coordinate of M cannot exceed 1 (note that $D = \tilde{r}_M - \tilde{r}_m$):

$$1 \geq \frac{D}{2\tilde{r}_M(D - \tilde{r}_M)}.$$

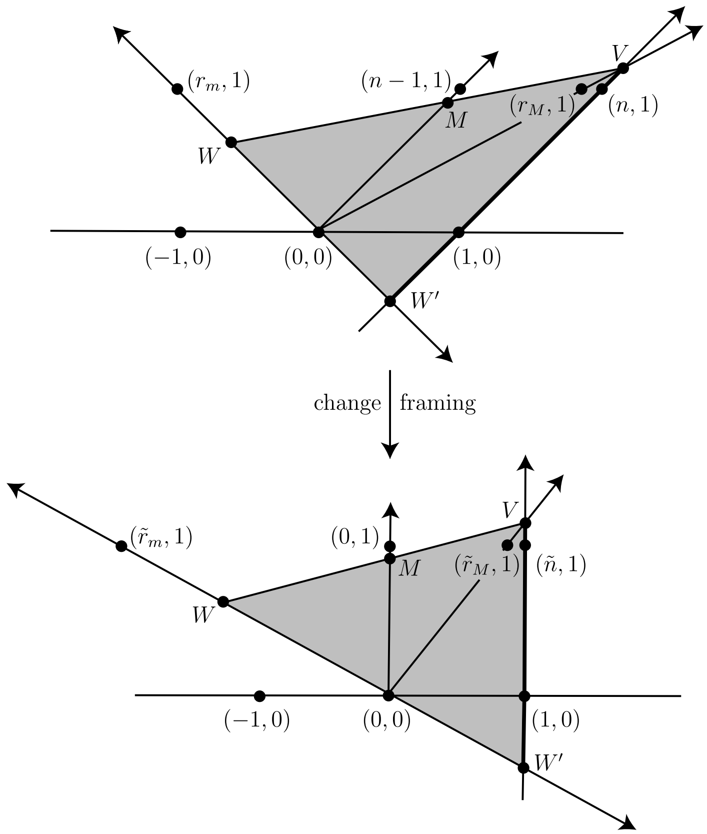

Figure 2: The geometry of P assuming $\|\mu\| = s$

By the previous proposition, $\tilde{r}_M \geq \frac{1}{2}$. Since $n \geq r_M$, we have $\tilde{r}_M \leq 1$. Recall [CS] that $D \geq 2$.

$$\begin{aligned} 1 \geq \frac{D}{2\tilde{r}_M(D - \tilde{r}_M)} \quad &\Rightarrow \quad 2\tilde{r}_M(D - \tilde{r}_M) \geq D \\ &\Rightarrow \quad 2\tilde{r}_M^2 - 2D\tilde{r}_M + D \leq 0 \\ &\Rightarrow \quad \tilde{r}_M \geq \frac{1}{2}(D - \sqrt{D(D-2)}) \end{aligned}$$

Since $\tilde{r}_M = r_M - n + 1$, we have, $n \leq r_M + 1 - \frac{1}{2}(D - \sqrt{D(D-2)})$, as required. □

We are now in a position to prove Theorem 2.

Let K be a hyperbolic knot in S^3 and let $r_\gamma = n$ be a non-trivial cyclic surgery slope. Then, by the Cyclic Surgery Theorem [CGLS], n is an integer and, without loss of generality, we may assume $n \geq 0$. Since 0 is a boundary slope, $r_m \leq 0$ so that $r_m - \frac{1}{2} < n$. Similarly, if $n = 0$, $r_M + \frac{1}{2} > n$. Thus, in order to prove Theorem 2, it is enough to show $n < r_M + \frac{1}{2}$ when $0 < n = r_\gamma$ is a cyclic surgery slope.

By Proposition 1, $n \leq r_M + \frac{1}{2}$, so it remains only to show that equality is not possible. Suppose then (for a contradiction) that $n = r_M + \frac{1}{2}$ and change the framing so that n goes to $\tilde{n} = 1$. Then $\tilde{r}_M = \frac{1}{2}$. Following the argument in the proof of Proposition 1, the line from the point $E = (s/m, 0)$ through V is part of the boundary of P. Since $m = \|\mu\|$, m is at least as big as the minimal norm s. Thus, E lies on the half open segment $((0,0),(1,0)]$. Similarly, $\|(1,2)\| \geq s$ and, therefore, V, which lies on the boundary of P, must be in the half open segment $((0,0),(1,2)]$. However, the line through E and V also passes through $(\tilde{n}, 1) = (1,1)$. The only consistent way to account for all these facts is to have $V = (1,2)$ and $E = (1,0)$. In other words, $\|\mu\| = s$. Reviewing the argument of Proposition 2, we see that $\|\mu\| = s$ is exactly the extra input needed to deduce that $n < r_M + \frac{1}{2}$. Thus, we conclude that $n < r_M + \frac{1}{2}$. This is absurd since we began by assuming $n = r_M + \frac{1}{2}$. The contradiction shows that, in fact, equality is not possible in Proposition 1. This completes the proof of Theorem 2.

Acknowledgments I am grateful to Kimihiko Motegi for suggesting the question that motivated this study and to Masaharu Ishikawa and Koya Shimokawa for helpful conversations. I thank the referee for many suggestions that significantly improved this paper.

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Received: 16 November 2004