ISSN 1364-0380 (on line) 1465-3060 (printed)

Geometry & Topology
Volume 6 (2002) 409–424
Published: 15 September 2002

On the Cut Number of a 3–manifold

Shelly L Harvey

Department of Mathematics
University of California at San Diego
La Jolla, CA 92093-0112, USA

Email: `sharvey@math.ucsd.edu`

URL: `http://math.ucsd.edu/~sharvey`

Abstract

The question was raised as to whether the cut number of a 3–manifold X is bounded from below by $\frac{1}{3}\beta_1(X)$. We show that the answer to this question is “no.” For each $m \geq 1$, we construct explicit examples of closed 3–manifolds X with $\beta_1(X) = m$ and cut number 1. That is, $\pi_1(X)$ cannot map onto any non-abelian free group. Moreover, we show that these examples can be assumed to be hyperbolic.

AMS Classification numbers Primary: 57M27, 57N10

Secondary: 57M05, 57M50, 20F34, 20F67

Keywords: 3–manifold, fundamental group, corank, Alexander module, virtual betti number, free group

Proposed: Cameron Gordon | Received: 27 February 2002
Seconded: Joan Birman, Walter Neumann | Accepted: 22 August 2002

1 Introduction

Let X be a closed, orientable n–manifold. The *cut number of* X, $c(X)$, is defined to be the maximal number of components of a closed, 2–sided, orientable hypersurface $F \subset X$ such that $X - F$ is connected. Hence, for any $n \le c(X)$, we can construct a map $f\colon X \to \bigvee_{i=1}^{n} S^1$ such that the induced map on π_1 is surjective. That is, there exists a surjective map $f_*\colon \pi_1(X) \twoheadrightarrow F(c)$, where $F(c)$ is the free group with $c = c(X)$ generators. Conversely, if we have any epimorphism $\phi\colon \pi_1(X) \twoheadrightarrow F(n)$, then we can find a map $f\colon X \to \bigvee_{i=1}^{n} S^1$ such that $f_* = \phi$. After making the f transverse to a non-wedge point x_i on each S^1, $f^{-1}(X)$ will give n disjoint surfaces $F = \cup F_i$ with $X - F$ connected. Hence one has the following elementary group-theoretic characterization of $c(X)$.

Proposition 1.1 *$c(X)$ is the maximal n such that there is an epimorphism $\phi\colon \pi_1(X) \twoheadrightarrow F(n)$ onto the free group with n generators.*

Example 1.2 Let $X = S^1 \times S^1 \times S^1$ be the 3–torus. Since $\pi_1(X) = \mathbb{Z}^3$ is abelian, $c(X) = 1$.

Using Proposition 1.1, we show that the cut number is additive under connected sum.

Proposition 1.3 *If $X = X_1 \# X_2$ is the connected sum of X_1 and X_2 then*

$$c(X) = c(X_1) + c(X_2).$$

Proof Let $G_i = \pi_1(X_i)$ for $i = 1, 2$ and $G = \pi_1(X) = G_1 * G_2$. It is clear that G maps surjectively onto $F(c(X_1)) * F(c(X_2)) = F(c(X_1 + X_2))$. Therefore $c(X) \ge c(X_1) + c(X_2)$.

Now suppose that there exists a map $\phi\colon G \twoheadrightarrow F(n)$. Let $\phi_i\colon G_i \to F(n)$ be the composition $G_i \to G_1 * G_2 \xrightarrow{\cong} G \twoheadrightarrow F(n)$. Since ϕ is surjective and $G = G_1 * G_2$, $\mathrm{Im}(\phi_1)$ and $\mathrm{Im}(\phi_2)$ generate $F(n)$. Morever, $\mathrm{Im}(\phi_i)$ is a subgroup of a free group, hence is free of rank less than or equal to $c(X_i)$. It follows that $n \le c(X_1) + c(X_2)$. In particular, when n is maximal we have $c(X) = n \le c(X_1) + c(X_2)$. □

In this paper, we will only consider 3–manifolds with $\beta_1(X) \ge 1$. Consider the surjective map $\pi_1(X) \twoheadrightarrow H_1(X)/\{\mathbb{Z}\text{–torsion}\} = \mathbb{Z}^{\beta_1(X)}$. Since $\beta_1(X) \ge 1$, we can find a surjective map from $\mathbb{Z}^{\beta_1(X)}$ onto $\mathbb{Z}$. It follows from Proposition 1.1 that $c(X) \geq 1$. Moreover, every map $\phi: \pi_1(X) \twoheadrightarrow F(n)$ gives rise to an epimorphism $\bar{\phi}: H_1(X) \twoheadrightarrow H_1\left(\bigvee_{i=1}^n S^1\right) \cong \mathbb{Z}^n$ It follows that $\beta_1(X) \geq n$ which gives us the well known result:

$$1 \leq c(X) \leq \beta_1(X). \tag{1}$$

It has recently been asked whether a (non-trivial) lower bound exists for the cut number. We make the following observations.

Remark 1.4 If S is a closed, orientable surface then $c(S) = \frac{1}{2}\beta_1(S)$.

Remark 1.5 If X has solvable fundamental group then $c(X) = 1$ and $\beta_1(X) \leq 3$.

Remark 1.6 Both c and β_1 are additive under connected sum (Proposition 1.3).

Therefore it is natural to ask the following question first asked by A Sikora and T Kerler. This question was motivated by certain results and conjectures on the divisibility of quantum 3–manifold invariants by P Gilmer–T Kerler [2] and T Cochran–P Melvin [1].

Question 1.7 Is $c(X) \geq \frac{1}{3}\beta_1(X)$ for all closed, orientable 3–manifolds X?

We show that the answer to this question is "as far from yes as possible." In fact, we show that for each $m \geq 1$ there exists a closed, *hyperbolic* 3–manifold with $\beta_1(X) = m$ and $c(X) = 1$. We actually prove a stronger statement.

Theorem 3.1 *For each $m \geq 1$ there exist closed 3–manifolds X with $\beta_1(X) = m$ such that for any infinite cyclic cover $X_\infty \to X$*, $\mathrm{rank}_{\mathbb{Z}[t^{\pm 1}]} H_1(X_\infty) = 0$.

We note the condition stated in the Theorem 3.1 is especially interesting because of the following theorem of J Howie [3]. Recall that a group G is *large* if some subgroup of finite index has a non-abelian free homomorphic image. Howie shows that if G has an infinite cyclic cover whose rank is at least 1 then G is large.

Theorem 1.8 (Howie [3]) *Suppose that $\tilde{K}$ is a connected regular covering complex of a finite 2–complex K, with nontrivial free abelian covering transformation group A. Suppose also that $H_1\left(\tilde{K};\mathbb{Q}\right)$ has a free $\mathbb{Q}[A]$–submodule of rank at least* 1. *Then $G = \pi_1(K)$ is large.*

Using the proof of Theorem 3.1 we show that the fundamental group of the aforementioned 3–manifolds cannot map onto F/F_4 where F is the free group with 2 generators and F_4 is the 4^{th} term of the lower central series of F.

Proposition 3.3 *Let X be as in Theorem 3.1, $G = \pi_1(X)$ and F be the free group on 2 generators. There is no epimorphism from G onto F/F_4.*

Independently, A Sikora has recently shown that the cut number of a "generic" 3–manifold is at most 2 [8]. Also, C Leininger and A Reid have constructed specific examples of genus 2 surface bundles X satisfying (i) $\beta_1(X) = 5$ and $c(X) = 1$ and (ii) $\beta_1(X) = 7$ and $c(X) = 2$ [6].

Acknowledgements I became interested in the question as to whether the cut number of a 3–manifold was bounded below by one-third the first betti number after hearing it asked by A Sikora at a problem session of the 2001 Georgia Topology Conference. The question was also posed in a talk by T Kerler at the 2001 Lehigh Geometry and Topology Conference. The author was supported by NSF DMS-0104275 as well as by the Bob E and Lore Merten Watt Fellowship.

2 Relative Cut Number

Let ψ be a primitive class in $H^1(X;\mathbb{Z})$. Since $H^1(X;\mathbb{Z}) = \mathrm{Hom}(\pi_1(X);\mathbb{Z})$, we can assume ψ is a surjective homomorphism, $\psi: \pi_1(X) \twoheadrightarrow \mathbb{Z}$. Since X is an orientable 3–manifold, every element in $H_2(X;\mathbb{Z})$ can be represented by an embedded, oriented, 2–sided surface [10, Lemma 1]. Therefore, if $\psi \in H^1(X;\mathbb{Z}) = H_2(X;\mathbb{Z})$ there exists a surface (not unique) dual to ψ. The *cut number of X relative to ψ*, $c(X;\psi)$, is defined as the maximal number of components of a closed, 2–sided, oriented surface $F \subset X$ such that $X - F$ is connected and one of the components of F is dual to ψ. In the above definition, we could have required that "any number" of components of F be dual to ψ as opposed to just "one." We remark that since $X - F$ is connected, these two conditions are equivalent. Similar to $c(X)$, we can describe $c(X;\psi)$ group theoretically.

Proposition 2.1 *$c(X;\psi)$ is the maximal n such that there is an epimorphism $\phi: \pi_1(X) \twoheadrightarrow F(n)$ onto the free group with n generators that factors through ψ (see diagram on next page).*

$$\begin{array}{ccc} {}_1(X) & \longrightarrow & \mathbb{Z} \\ \Big\downarrow & \nearrow & \\ F(n) & & \end{array}$$

It follows immediately from the de nitions that $c(X;\ )\quad c(X)$ for all primitive . Now let F be any surface with $c(X)$ components and let be dual to one of the components, then $c(X;\ ) = c(X)$. Hence

$$c(X) = \max \quad c(X;\ )\, j \quad \text{is a primitive element of } H^1(X;\mathbb{Z}) \quad . \qquad (2)$$

In particular, if $c(X;\ ) = 1$ for all then $c(X) = 1$.

We wish to nd su cient conditions for $c(X;\ ) = 1$. In [5, page 44], T Kerler develops a skein theoretic algorithm to compute the one-variable Alexander polynomial $_{X;}$ from a surgery presentation of X. As a result, he shows that if $c(X;\ )\quad 2$ then the Frohman{Nicas TQFT evaluated on the cut cobordism is zero, implying that $_{X;} = 0$. Using the fact that $\mathbb{Q}\ t^{\ 1}$ is a principal ideal domain one can prove that $_{X;} = 0$ is equivalent to $\mathrm{rank}_{\mathbb{Z}[t\ ^1]}H_1(X\ )\quad 1$. We give an elementary proof of the equivalent statement of Kerler's.

Proposition 2.2 *If* $c(X;\ )\quad 2$ *then* $\mathrm{rank}_{\mathbb{Z}[t\ ^1]}H_1(X\ )\quad 1$.

Proof Suppose $c(X;\ )\quad 2$ then there is a surjective map $\ :\ {}_1(X) \twoheadrightarrow F(n)$ that factors through with $n\quad 2$. Let $\bar{\ }: F(n) \twoheadrightarrow \mathbb{Z}$ be the homomorphism such that $= \bar{\ }\quad .\quad$ surjective implies that $_{j\ker}\ :\ \ker\ \twoheadrightarrow \ker\bar{\ }$ is surjective. Writing $\mathbb{Z}$ as the multiplicative group generated by t, we can consider $\frac{\ker}{[\ker\ ;\ker\ ]}$ and $\frac{\ker\bar{\ }}{[\ker\bar{\ };\ker\bar{\ }]}$ as modules over $\mathbb{Z}\ t^{\ 1}$. Here, the t acts by conjugating by an element that maps to t by or $\bar{\ }$. Moreover, $_{j\ker}\ :\ \frac{\ker}{[\ker\ ;\ker\ ]} \twoheadrightarrow \frac{\ker\bar{\ }}{[\ker\bar{\ };\ker\bar{\ }]}$ is surjective hence

$$\mathrm{rank}_{\mathbb{Z}[t\ ^1]}\ \frac{\ker}{[\ker\ ;\ker\ ]}\qquad \mathrm{rank}_{\mathbb{Z}[t\ ^1]}\ \frac{\ker\bar{\ }}{\ker\bar{\ };\ker\bar{\ }}\ ! = n-1.$$

Since $n\quad 2$, $\mathrm{rank}_{\mathbb{Z}[t\ ^1]}H_1(X\ ) = \mathrm{rank}_{\mathbb{Z}[t\ ^1]}\ \frac{\ker}{[\ker\ ;\ker\ ]}\quad 1$. □

Corollary 2.3 *If* ${}_1(X) \twoheadrightarrow F{=}F^{00}$ *where* F *is a free group of rank 2 then there exists a* $\ :\ {}_1(X) \twoheadrightarrow \mathbb{Z}$ *such that* $\mathrm{rank}_{\mathbb{Z}[t\ ^1]}H_1(X\ )\quad 1$.

Proof This follows immediately from the proof of Proposition 2.2 after noticing that $F'' \subseteq [\ker \bar{\phi}, \ker \bar{\phi}]$ and $\mathrm{Hom}(F/F''; \mathbb{Z}) = \mathrm{Hom}(F; \mathbb{Z})$. □

3 The Examples

We construct closed 3–manifolds all of whose infinite cyclic covers have first homology that is $\mathbb{Z}[t^{\pm 1}]$–torsion. The 3–manifolds we consider are 0–surgery on an m–component link that is obtained from the trivial link by tying a Whitehead link interaction between each two components.

Theorem 3.1 *For each $m \geq 1$ there exist closed 3–manifolds X with $\beta_1(X) = m$ such that for any infinite cyclic cover $X_\psi \to X$, $\mathrm{rank}_{\mathbb{Z}[t^{\pm 1}]} H_1(X_\psi) = 0$.*

It follows from Propostion 2.2 that the cut number of the manifolds in Theorem 3.1 is 1. In fact, Corollary 2.3 implies that $\pi_1(X)$ does not map onto F/F'' where F is a free group of rank 2. Moreover, the proof of this theorem shows that $\pi_1(X)$ does not even map onto F/F_4 where F_n is the n^{th} term of the lower central series of F (see Proposition 3.3).

By a theorem of Ruberman [7], we can assume that the manifolds with cut number 1 are hyperbolic.

Corollary 3.2 *For each $m \geq 1$ there exist closed, orientable, hyperbolic 3–manifolds Y with $\beta_1(Y) = m$ such that for any infinite cyclic cover $Y_\psi \to Y$, $\mathrm{rank}_{\mathbb{Z}[t^{\pm 1}]} H_1(Y_\psi) = 0$.*

Proof Let X be one of the 3–manifolds in Theorem 3.1. By [7, Theorem 2.6], there exists a degree one map $f\colon Y \to X$ where Y is hyperbolic and f is an isomorphism on H_*. Denote by $G = \pi_1(X)$ and $P = \pi_1(Y)$. It is then well-known that f is surjective on π_1. It follows from Stalling's theorem [9, page 170] that the kernel of f_* is $P_\omega \equiv \cap P_n$. Now, suppose $\psi\colon P \overset{f_*}{\twoheadrightarrow} G \overset{\bar{\psi}}{\twoheadrightarrow} \mathbb{Z}$ defines an infinite cyclic cover of Y. Then $H_1(Y_\psi) \twoheadrightarrow H_1\left(X_{\bar{\psi}}\right)$ has kernel $P_\omega/[\ker\psi, \ker\psi]$. To show that $\mathrm{rank}_{\mathbb{Z}[t^{\pm 1}]} H_1(Y_\psi) = 0$ it suffices to show that P_ω vanishes under the map $H_1(Y_\psi) \to H_1(Y_\psi) \otimes_{\mathbb{Z}[t^{\pm 1}]} \mathbb{Q}[t^{\pm 1}] \to H_1(Y_\psi) \otimes_{\mathbb{Z}[t^{\pm 1}]} \mathbb{Q}(t)$ since then $\mathrm{rank}_{\mathbb{Z}[t^{\pm 1}]} H_1(Y_\psi) = \mathrm{rank}_{\mathbb{Z}[t^{\pm 1}]} H_1\left(X_{\bar{\psi}}\right) = 0$.

Note that $H_1(Y_\psi) \otimes_{\mathbb{Z}[t^{\pm 1}]} \mathbb{Q}[t^{\pm 1}] \cong \bigoplus_{i=1}^{n} \mathbb{Q}[t^{\pm 1}] \oplus T$ where T is a $\mathbb{Q}[t^{\pm 1}]$ torsion module. Moreover, P_n is generated by elements of the form $\gamma =$

$[p_1, [p_2, [p_3, \ldots, [p_{n-2}, \;]]]]$ where $\in P_2 \subset \ker$. Therefore

$$[y] = (\ (p_i) - 1) \cdots (\ (p_{n-2}) - 1)[\ ]$$

in $H_1(Y)$ which implies that $P_n \subset J^{n-2}(H_1(Y))$ for $n \geq 2$ where J is the augmentation ideal of $\mathbb{Z}[t^{\pm 1}]$. It follows that any element of P_ω considered as an element of $H_1(Y) \otimes_{\mathbb{Z}[t^{\pm 1}]} \mathbb{Q}[t^{\pm 1}]$ is infinitely divisible by $t-1$ and hence is torsion. ☐

Proof of Theorem 3.1 Let $L = \sqcup L_i$ be the oriented trivial link with m components in S^3 and $\sqcup D_i$ be oriented disjoint disks with $\partial D_i = L_i$. The fundamental group of $S^3 - L$ is freely generated by $\{x_i\}$ where x_i is a meridian curve of L_i which intersects D_i exactly once and $D_i \cdot x_i = 1$. For all i, j with $1 \leq i < j \leq m$ let $_{ij} : I \to S^3$ be oriented disjointly embedded arcs such that $_{ij}(0) \in L_i$ and $_{ij}(1) \in L_j$ and $_{ij}(I)$ does not intersect $\sqcup D_i$. For each arc $_{ij}$, let y_{ij} be the curve embedded in a small neighborhood of $_{ij}$ representing the class $[x_i, x_j]$ as in Figure 1. Let X be the 3-manifold obtained performing

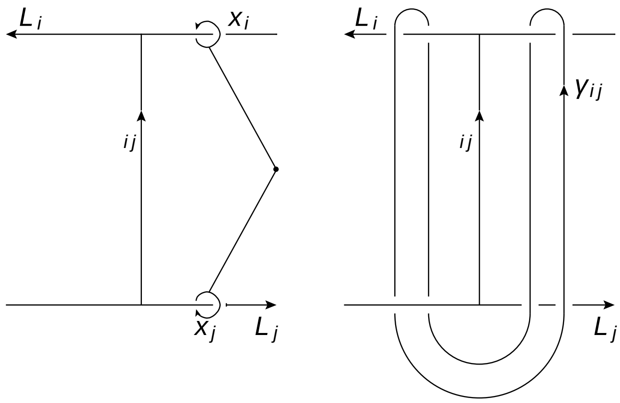

Figure 1

0-framed Dehn surgery on L and -1-framed Dehn surgery on each $y = \sqcup y_{ij}$. See Figure 2 for an example of X when $m = 5$.

Denote by X_0, the manifold obtained by performing 0-framed Dehn surgery on L. Let W be the 4-manifold obtained by adding a 2-handle to $X_0 \times I$ along each curve $y_{ij} \times \{1\}$ with framing coefficient -1. The boundary of W is $\partial W = X_0 \sqcup -X$. We note that

$$\pi_1(W) = \langle x_1, \ldots, x_m \mid [x_i, x_j] = 1 \text{ for all } 1 \leq i < j \leq m \rangle = \mathbb{Z}^m.$$

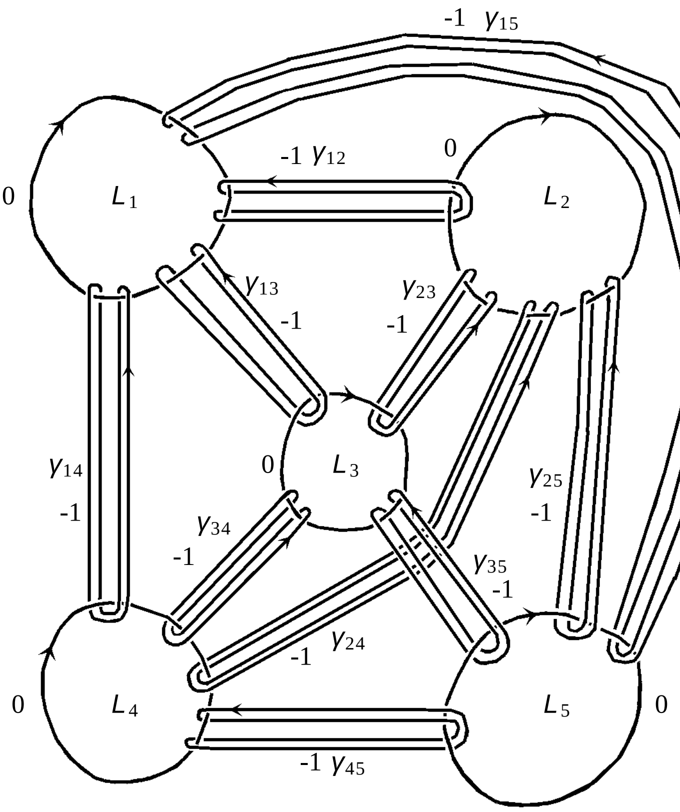

Figure 2: The surgered manifold X when $m = 5$

Let $\{x_{ik}; \ _{ijl}\}$ be the generators of $_1\left(S^3 - (L \sqcup y)\right)$ that are obtained from a Wirtinger presentation where x_{ik} are meridians of the i^{th} component of L and $_{ijl}$ are meridians of the $(i; j)^{th}$ component of y. Note that $\{x_{ik}; \ _{ijl}\}$ generate $G \quad _1(X)$. For each $1 \quad i \quad m$ let $\overline{x}_i = x_{i1}$ and $\overline{}_{ij}$ be the speci c $_{ijl}$ that is denoted in Figure 3. We will use the convention that

$$[a; b] = aba^{-1}b^{-1}$$

and

$$a^b = bab^{-1}.$$

We can choose a projection of the trivial link so that the arcs $_{ij}$ do not pass under a component of L. Since $\overline{}_{ij}$ is equal to a longitude of the curve y_{ij} in X, we have $\overline{}_{ij} = x_{in_{ij}}; \ x_{jn_{ji}} \ ^{-1}$ for some n_{ij} and n_{ji} and where is a product of conjugates of meridian curves $\overline{}_{lk}$ and $\overline{}_{lk}^{-1}$. Moreover, we can nd

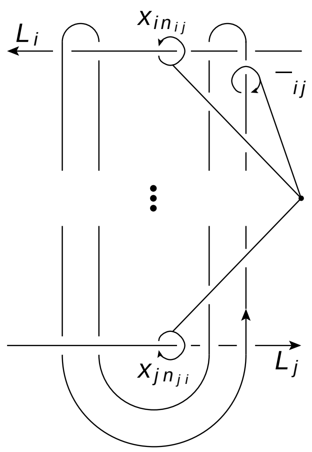

Figure 3

a projection of $L \sqcup \gamma$ so that the individual components of L do not pass under or over one another. Hence $x_{ij} = \omega \overline{x}_i \omega^{-1}$ where ω is a product of conjugates of the meridian curves μ_{lk} and μ_{lk}^{-1}. As a result, we have

$$\begin{aligned} \mu_{ij} &= x_{in_{ij}}, \; x_{jn_{ji}} {}^{-1} \quad (3)\\ &= \omega_1 \overline{x}_i \omega_1^{-1}, \; \omega_2 \overline{x}_j \omega_2^{-1} {}^{-1} \\ &= \overline{x}_i, \omega_1^{-1} \omega_2 \overline{x}_j \omega_2^{-1} {}^{-1} \omega_1^{-1} {}^{\omega_1} \end{aligned}$$

for some , ω_1, and ω_2.

We note that $\mu_{ij} = x_{in_{ij}}, x_{jn_{ji}} {}^{-1}$ hence $\mu_{ij} \in G'$ for all $i < j$. Setting $v = \omega_1^{-1} \omega_2$ and using the equality

$$[a, bc] = [a, b]\,[a, c]^b \quad (4)$$

we see that

$$\begin{aligned} \mu_{ij} &= \overline{x}_i, v\overline{x}_j v^{-1} {}^{\omega_1} \quad (5)\\ &= \overline{x}_i, v\overline{x}_j v^{-1} \mod G'' \\ &= [\overline{x}_i, [v, \overline{x}_j]\,\overline{x}_j] \\ &= [\overline{x}_i, [v, \overline{x}_j]]\,[\overline{x}_i, \overline{x}_j]^{[v, \overline{x}_j]} \\ &= [\overline{x}_i, [v, \overline{x}_j]]\,[\overline{x}_i, \overline{x}_j] \mod G'' \end{aligned}$$

since $\omega_1, v \in G'$.

Consider the dual relative handlebody decomposition (W, X). W can be obtained from X by adding a 0-framed 2-handle to $X \quad I$ along each of the meridian curves $\gamma_{ij}\times\{1\}$. (3) implies that γ_{ij} is trivial in $H_1(X)$ hence the inclusion map $j\colon X \to W$ induces an isomorphism $j_*\colon H_1(X) \xrightarrow{\cong} H_1(W)$. Therefore if $\phi\colon G \twoheadrightarrow \Gamma$ where Γ is abelian then there exists a $\psi\colon \pi_1(W) \twoheadrightarrow \Gamma$ such that $\psi \circ j_* = \phi$.

Suppose $\phi\colon G \twoheadrightarrow \langle t\rangle \cong \mathbb{Z}$ and $\psi\colon \pi_1(W) \twoheadrightarrow \langle t\rangle$ is an extension of ϕ to $\pi_1(W)$. Let X_∞ and W_∞ be the infinite cyclic covers of W and X corresponding to ψ and ϕ respectively. Consider the long exact sequence of pairs,

$$\to H_2(W_\infty, X_\infty) \xrightarrow{\partial} H_1(X_\infty) \to H_1(W_\infty) \to \tag{6}$$

Since $\pi_1(W) \cong \mathbb{Z}^m$, $H_1(W_\infty) \cong \mathbb{Z}^{m-1}$ where t acts trivially so that $H_1(W_\infty)$ has rank 0 as a $\mathbb{Z}[t^{\pm 1}]$–module. $H_2(W_\infty, X_\infty) \cong \left(\mathbb{Z}[t^{\pm 1}]\right)^{\binom{m}{2}}$ generated by the core of each 2–handle (extended by $\gamma_{ij}\times I$) attached to X. Therefore, Im∂ is generated by a lift of γ_{ij} in $H_1(X_\infty)$ for all $1 \le i < j \le m$. To show that $H_1(X_\infty)$ has rank 0 it suffices to show that each of the γ_{ij} are $\mathbb{Z}[t^{\pm 1}]$–torsion in $H_1(X_\infty)$.

Let $F = \langle \bar{x}_1, \dots, \bar{x}_m\rangle$ be the free group of rank m and $f\colon F \to G$ be defined by $f(\bar{x}_i) = \bar{x}_i$. We have the following $\binom{m}{3}$ Jacobi relations in F/F'' [4, Proposition 7.3.6]. For all $1 \le i < j < k \le m$,

$$[\bar{x}_i,[\bar{x}_j,\bar{x}_k]][\bar{x}_j,[\bar{x}_k,\bar{x}_i]][\bar{x}_k,[\bar{x}_i,\bar{x}_j]] = 1 \mod F''.$$

Using f, we see that these relations hold in G/G'' as well. From (5), we can write

$$[\bar{x}_i,\bar{x}_j] = [[v_{ij},\bar{x}_j],\bar{x}_i]\,\gamma_{ij} \mod G''.$$

Hence for each $1 \le i < j < k \le m$ we have the Jacobi relation $J(i,j,k)$ in G/G'',

$$\begin{aligned}
1 &= [\bar{x}_i,[\bar{x}_j,\bar{x}_k]]\left[\bar{x}_j,[\bar{x}_i,\bar{x}_k]^{-1}\right][\bar{x}_k,[\bar{x}_i,\bar{x}_j]] \mod G'' \\
&= \left[\bar{x}_i,[[v_{jk},\bar{x}_k],\bar{x}_j]\gamma_{jk}\right]\left[\bar{x}_j,\gamma_{ik}^{-1}[\bar{x}_i,[v_{ik},\bar{x}_k]]\right] \\
&\quad \left[\bar{x}_k,[[v_{ij},\bar{x}_j],\bar{x}_i]\gamma_{ij}\right] \mod G'' \\
&= [\bar{x}_i,[[v_{jk},\bar{x}_k],\bar{x}_j]]\,[\bar{x}_i,\gamma_{jk}]\,[\bar{x}_j,\gamma_{ik}^{-1}]\,[\bar{x}_j,[\bar{x}_i,[v_{ik},\bar{x}_k]]] \\
&\quad [\bar{x}_k,[[v_{ij},\bar{x}_j],\bar{x}_i]]\,[\bar{x}_k,\gamma_{ij}] \mod G'' \\
&= [\bar{x}_i,\gamma_{jk}]\,[\bar{x}_j,\gamma_{ik}^{-1}]\,[\bar{x}_k,\gamma_{ij}]\,[\bar{x}_i,[[v_{jk},\bar{x}_k],\bar{x}_j]][\bar{x}_j,[\bar{x}_i,[v_{ik},\bar{x}_k]]] \\
&\quad [\bar{x}_k,[[v_{ij},\bar{x}_j],\bar{x}_i]] \mod G''.
\end{aligned} \tag{7}$$

Moreover, for each component of the trivial link L_i the longitude, l_i, of L_i is trivial in G and is a product of commutators of γ_{ij} with a conjugate of $\bar{x}_j$. We can write each of the longitudes (see Figure 4) as

$$
\begin{aligned}
l_i &= \prod_{j<i} \;{}_j\; {}_j^{-1}\bar{\ }_{ji}^{-1}\; {}_j \prod_{k>i} \bar{\ }_{ik}\; {}_k \bmod G'' \\
&= \prod_{j<i} {}_j^{-1} x_{jn_{ji}}^{-1} \bar{\ }_{ji} x_{jn_{ji}}\; {}_j \quad {}_j^{-1}\bar{\ }_{ji}^{-1}\; {}_j \\
&\quad \prod_{k>i} \bar{\ }_{ik} \quad {}_k x_{kn_{ki}}^{-1}\; {}_k^{-1}\bar{\ }_{ik}^{-1}\; {}_k x_{kn_{ki}}\; {}_k^{-1} \\
&= \prod_{j<i} \Big[x_{jn_{ji}}^{-1}; \bar{\ }_{ji} \Big]^{{}_j^{-1}} \prod_{k>i} \Big[\bar{\ }_{ik};\; {}_k x_{kn_{ki}}^{-1}\; {}_k^{-1} \Big] \\
&= \prod_{j<i} \Big[\bar{x}_j^{-1}; \bar{\ }_{ji} \Big] \prod_{k>i} \bar{\ }_{ik}; \bar{x}_k^{-1} \quad \bmod G''.
\end{aligned} \tag{8}
$$

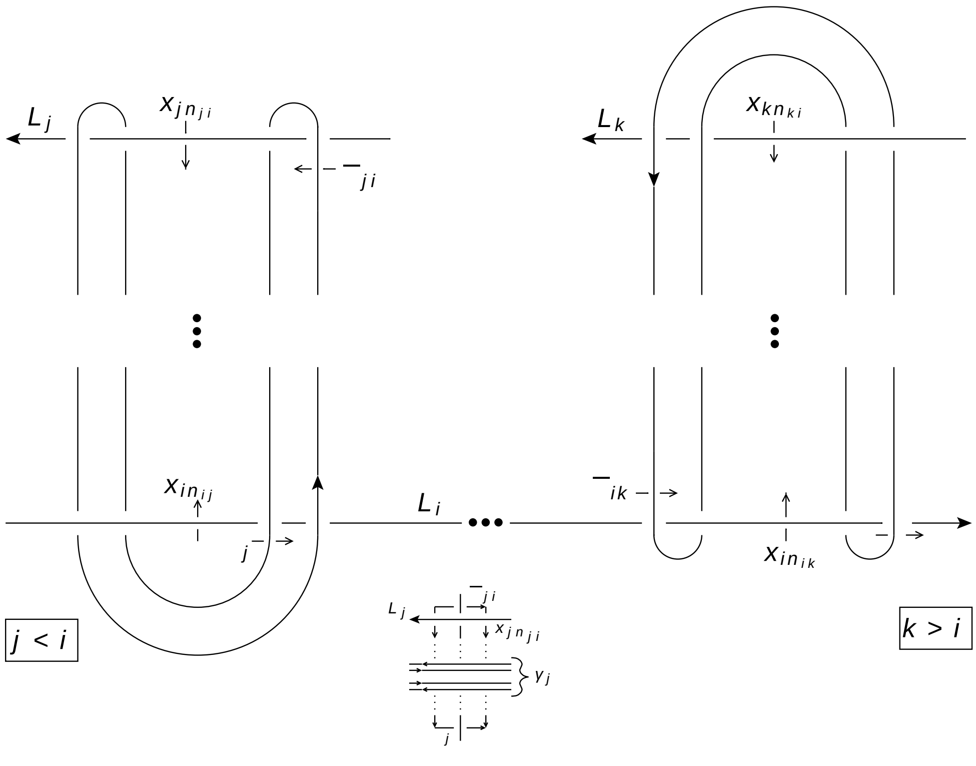

Figure 4

It follows that
$$\prod_{j<i}\left[\overline{x}_j^{-1},\gamma_{ji}\right]\prod_{k>i}\left[\gamma_{ik},\overline{x}_k^{-1}\right] = 1 \mod G''.$$

Since $G'' \subset [\ker\psi,\ker\psi]$, the relations in (7) and (8) hold in $H_1(X_\psi)$ $(= \ker\psi/[\ker\psi,\ker\psi])$ as well. Suppose $\psi\colon G \twoheadrightarrow \mathbb{Z}$ is defined by sending $\overline{x}_i \mapsto t^{n_i}$. Since ψ is surjective, $n_N \neq 0$ for some N. We consider a subset of $\binom{m}{2}$ relations in $H_1(X_\psi)$ that we index by (i,j) for $1 \le i < j \le m$. When $i = N$ or $j = N$ we consider the $m-1$ relations

$$\text{(i)}\quad R_{iN} = l_i \quad\text{and}\quad \text{(ii)}\quad R_{Nj} = l_j^{-1}.$$

Rewriting l_i as an element of the $\mathbb{Z}\left[t^{\pm 1}\right]$-module $H_1(X_\psi)$ generated by $\{\gamma_{ij} \mid 1 \le i < j \le m\}$ from (8) we have

$$\begin{aligned}
R_{iN} &= \sum_{j<i}\left(t^{-n_j}-1\right)\gamma_{ji} + \sum_{k>i}\left(1-t^{-n_k}\right)\gamma_{ik}\\
&= \sum_{j<i} t^{-n_j}\left(1-t^{n_j}\right)\gamma_{ji} + \sum_{k>i} t^{-n_k}\left(t^{n_k}-1\right)\gamma_{ik}\\
&= \sum_{j<i}\left[(1-t^{n_j}) + \left(t^{-n_j}-1\right)(1-t^{n_j})\right]\gamma_{ji} +\\
&\quad \sum_{k>i}\left[(t^{n_k}-1) + \left(t^{-n_k}-1\right)(t^{n_k}-1)\right]\gamma_{ik}. \qquad (9)
\end{aligned}$$

Similarily, we have

$$\begin{aligned}
R_{Nj} &= \sum_{i<j}\left[(t^{n_i}-1) + \left(t^{-n_i}-1\right)(t^{n_i}-1)\right]\gamma_{ij} +\\
&\quad \sum_{k>j}\left[(1-t^{n_k}) + \left(t^{-n_k}-1\right)(1-t^{n_k})\right]\gamma_{jk}. \qquad (10)
\end{aligned}$$

For the other $\binom{m-1}{3}$ relations, we use the Jacobi relations from (7). Define R_{ij} to be

$$R_{ij} = \begin{cases} J(N,i,j) & \text{for } N<i<j\\ J(i,N,j)^{-1} & \text{for } i<N<j\\ J(i,j,N) & \text{for } i<j<N \end{cases}.$$

We can write these relations as

$$R_{ij} = \begin{cases}
(t^{n_j}-1)\gamma_{Ni} + (1-t^{n_i})\gamma_{Nj} + (t^{n_N}-1)\gamma_{ij} + \\
(t^{n_N}-1)(t^{n_i}-1)(t^{n_j}-1)(\bar\gamma_{ij}+\bar\gamma_{Nj}-\bar\gamma_{Nj}) & \text{for } N<i<j\\
(1-t^{n_j})\gamma_{iN} + (t^{n_N}-1)\gamma_{ij} + (1-t^{n_i})\gamma_{Nj} + \\
(t^{n_N}-1)(t^{n_i}-1)(t^{n_j}-1)(-\bar\gamma_{iN}-\bar\gamma_{Nj}+\bar\gamma_{ij}) & \text{for } i<N<j\\
(t^{n_N}-1)\gamma_{ij} + (1-t^{n_j})\gamma_{iN} + (t^{n_i}-1)\gamma_{jN} + \\
(t^{n_N}-1)(t^{n_i}-1)(t^{n_j}-1)(\bar\gamma_{ij}+\bar\gamma_{jN}-\bar\gamma_{iN}) & \text{for } i<j<N
\end{cases} \qquad (11)$$

where $\tilde{v}_{ij}$ is a lift of v_{ij}.

For $1 \leq i < j \leq m$ order the pairs ij by the dictionary ordering. That is, $ij < lk$ provided either $i < l$ or $j < k$ when $i = l$. The relations above give us an $\binom{m}{2} \times \binom{m}{2}$ matrix M with coefficients in $\mathbb{Z}[t^{\pm 1}]$. The $(ij;kl)^{th}$ component of M is the coefficient of $\bar{\gamma}_{kl}$ in R_{ij}. We claim for now that

$$M = (t^{n_N} - 1)I + (t-1)S + (t-1)^2 E \tag{12}$$

for some "error" matrix E where I is the identity matrix and S is a skew-symmetric matrix. For an example, when $m = 4$ and $N = 1$, M is the matrix

$$\begin{bmatrix} t^{n_1}-1 & 0 & 0 & 1-t^{n_3} & 1-t^{n_4} & 0 \\ 0 & t^{n_1}-1 & 0 & t^{n_2}-1 & 0 & 1-t^{n_4} \\ 0 & 0 & t^{n_1}-1 & 0 & t^{n_2}-1 & t^{n_3}-1 \\ t^{n_3}-1 & 1-t^{n_2} & 0 & t^{n_1}-1 & 0 & 0 \\ t^{n_4}-1 & 0 & 1-t^{n_2} & 0 & t^{n_1}-1 & 0 \\ 0 & t^{n_4}-1 & 1-t^{n_3} & 0 & 0 & t^{n_1}-1 \end{bmatrix} + (t-1)^2 E.$$

The proof of (12) is left until the end.

We will show that M is non-singular as a matrix over the quotient field $\mathbb{Q}(t)$. Consider the matrix $A = \frac{1}{t-1}M$. We note that A is a matrix with entries in $\mathbb{Z}[t^{\pm 1}]$ and $A(1)$ evaluated at $t = 1$ is

$$A(1) = NI + S(1).$$

To show that M is non-singular, it suffices to show that $A(1)$ is non-singular. Consider the quadratic form $q\colon \mathbb{Q}^{\binom{m}{2}} \to \mathbb{Q}^{\binom{m}{2}}$ defined by $q(z) \equiv z^T A(1) z$ where z^T is the transpose of z. Since $A(1) = NI + S(1)$ where $S(1)$ is skew-symmetric we have,

$$q(z) = N \sum z_i^2.$$

Moreover, $N \neq 0$ so $q(z) = 0$ if and only if $z = 0$. Let z be a vector satisfying $A(1)z = 0$. We have $q(z) = z^T A(1) z = z^T 0 = 0$ which implies that $z = 0$. Therefore M is a non-singular matrix. This implies that each element $\bar{\gamma}_{ij}$ is $\mathbb{Z}[t^{\pm 1}]$–torsion which will complete the the proof once we have established the above claim.

We ignore entries in M that lie in J^2 where J is the augmentation ideal of $\mathbb{Z}[t^{\pm 1}]$ since they only contribute to the error matrix E. Using (9), (10), and (11) above we can explicitely write the entries in M (mod J^2). Let $m_{ij;lk}$ denote the $(ij;lk)$ entry of M (mod J^2).

Case 1 ($j = N$): From (9) we have

$$m_{iN;li} = 1 - t^{n_l},\ m_{iN;ik} = t^{n_k} - 1,$$

and $m_{iN;lk} = 0$ when neither l nor k is equal to N.

Case 2 ($i = N$): From (10) we have

$$m_{Nj;lj} = t^{n_l} - 1,\ m_{Nj;jk} = 1 - t^{n_k},$$

and $m_{Nj;lk} = 0$ when neither l nor k is equal to N.

Case 3 ($N < i < j$): From (11) we have

$$m_{ij;Ni} = t^{n_j} - 1,\ m_{ij;Nj} = 1 - t^{n_i},\ m_{ij;ij} = t^{n_N} - 1,$$

and $m_{ij;lk} = 0$ otherwise.

Case 4 ($i < N < j$): From (11) we have

$$m_{ij;iN} = 1 - t^{n_j},\ m_{ij;ij} = t^{n_N} - 1,\ m_{ij;Nj} = 1 - t^{n_i},$$

and $m_{ij;lk} = 0$ otherwise.

Case 5 ($i < j < N$): From (11) we have

$$m_{ij;ij} = t^{n_N} - 1,\ m_{ij;iN} = 1 - t^{n_j},\ m_{ij;jN} = t^{n_i} - 1,$$

and $m_{ij;lk} = 0$ otherwise.

We first note that in each of the cases, the diagonal entries $m_{ij;ij}$ are all $t^{n_N} - 1$. Next, we will show that the off diagonal entries have the property that $m_{ij;lk} = -m_{lk;ij}$ for $ij < lk$. This will complete the proof of the claim since we see that each entry is divisible by $t - 1$.

We verify the skew symmetry in Cases 1 and 3. The other cases are similar and we leave the verifications to the reader.

Case 1 ($j = N$):

$$m_{iN;li} = 1 - t^{n_l} = -m_{li;iN} \text{ (case 5)}$$

and

$$m_{iN;ik} = t^{n_k} - 1 = -m_{ik;iN} \text{ (case 4).}$$

Case 3 ($N < i < j$):

$$m_{ij;Ni} = t^{n_j} - 1 = -m_{Ni;ij} \text{ (case 2)}$$

and

$$m_{ij;Nj} = 1 - t^{n_i} = -m_{Nj;ij} \text{ (case 2).}$$ □

Proposition 3.3 *Let X be as in Theorem 3.1, $G=\pi_1(X)$ and F be the free group on 2 generators. There is no epimorphism from G onto F/F_4.*

Proof Let $F=\langle x,y\rangle$ be the free group and $\phi\colon F/F_4 \twoheadrightarrow \langle t\rangle$ be defined by $x\mapsto t$ and $y\mapsto 1$. Suppose that there exists a surjective map $\psi\colon G\twoheadrightarrow F/F_4$. Let $N=\ker\phi$ and $H=\ker(\phi\circ\psi)$. Since ψ is surjective we get an epimorphism of $\mathbb{Z}[t^{\pm1}]$–modules $\tilde\psi\colon H/H' \twoheadrightarrow N/N'$. From (6) we get the short exact sequence

$$0\to \mathrm{Im}\,\partial \xrightarrow{i} H_1(X_\infty)\to H_1(W_\infty)\to 0.$$

Let J be the augmentation ideal of $\mathbb{Z}[t^{\pm1}]$. We compute $N/N'=\mathbb{Z}[t^{\pm1}]/J^3$ so that $\tilde\psi\colon H_1(X_\infty)\twoheadrightarrow \mathbb{Z}[t^{\pm1}]/J^3$. Let $\alpha\in H_1(X_\infty)$ such that $\tilde\psi(\alpha)=1$. Since every element in $H_1(W_\infty)=\bigoplus_{i=1}^{m-1}\frac{\mathbb{Z}[t^{\pm1}]}{J}$ is $(t-1)$–torsion, $(t-1)\alpha\in \mathrm{Im}\,\partial$ hence $t-1\in\mathrm{Im}(\tilde\psi\circ i)$. Recall that in the proof of the Theorem 3.1, we showed that there exists a surjective $\mathbb{Z}[t^{\pm1}]$–module homomorphism $\rho\colon P\twoheadrightarrow \mathrm{Im}\,\partial$ where P is finitely presented as

$$0\to \mathbb{Z}[t^{\pm1}]^{\binom{m}{2}} \xrightarrow{(t-1)A} \mathbb{Z}[t^{\pm1}]^{\binom{m}{2}} \to P\to 0.$$

Let $g\colon P\to \mathbb{Z}[t^{\pm1}]/J^3$ defined by $g=\tilde\psi\circ i\circ\rho$. Since ρ is surjective, $t-1\in\mathrm{Im}\,g$. After tensoring with $\mathbb{Q}[t^{\pm1}]$, we get a map $g\colon P\otimes_{\mathbb{Z}[t^{\pm1}]}\mathbb{Q}[t^{\pm1}]\to \mathbb{Q}[t^{\pm1}]/J^3$. It is easy to see that either g is surjective or the image of g is the submodule generated by $t-1$. Note that the submodule generated by $t-1$ is isomorphic $\mathbb{Q}[t^{\pm1}]/J^2$. Hence, in either case, we get a surjective map $h\colon P\otimes_{\mathbb{Z}[t^{\pm1}]}\mathbb{Q}[t^{\pm1}]\to\mathbb{Q}[t^{\pm1}]/J^2$.

Consider the $\mathbb{Q}[t^{\pm1}]$–module P' presented by A. Let $h'\colon \mathbb{Q}[t^{\pm1}]^{\binom{m}{2}}\to \mathbb{Q}[t^{\pm1}]/J^2$ be defined by $h'=(t-1)h\circ\pi$. Since

$$h'(A(\cdot))=(t-1)h(\pi(A(\cdot)))=h(\pi((t-1)A(\cdot)))=h(0)=0;$$

this defines a map $h'\colon P'\to\mathbb{Q}[t^{\pm1}]/J^2$ whose image is the submodule generated by $t-1$. It follows that P' maps onto $\mathbb{Q}[t^{\pm1}]/J$. Setting $t=1$, the vector space over $\mathbb{Q}$ presented by $A(1)$ maps onto $\mathbb{Q}$. Therefore $\det(A(1))=0$. However, it was previously shown that $A(1)$ was non-singular which is a contradiction. □

Corollary 3.4 *For any closed, orientable 3–manifold Y with $P/P_4=G/G_4$ where $P=\pi_1(Y)$ and $G=\pi_1(X)$ is the fundamental group of the examples in Theorem 3.1, $c(Y)=1$.*

Using Proposition 3.3, it is much easier to show that there exist *hyperbolic* 3–manifolds with cut number 1.

Corollary 3.5 *For each $m \geq 1$ there exist closed, orientable, hyperbolic 3–manifolds Y with $\beta_1(Y) = m$ such that $\pi_1(Y)$ cannot map onto F/F_4 where F is the free group on 2 generators.*

Proof Let X be one of the 3–manifolds in Theorem 3.1. By [7, Theorem 2.6], there exists a degree one map $f\colon Y \to X$ where Y is hyperbolic and f is an isomorphism on H_*. Denote by $G = \pi_1(X)$ and $P = \pi_1(Y)$. It follows from Stalling's theorem [9] that f induces an isomorphism $f_*\colon P/P_n \to G/G_n$. In particular this is true for $n = 4$ which completes the proof. □

References

[1] T Cochran, P Melvin, *Quantum cylotomic orders of 3–manifolds*, Topology 40 (2001) 95–125

[2] P Gilmer, T Kerler, *Cut numbers of 3–manifolds via quantum orders in $\mathbb{Z}[\zeta_5]$*, preprint

[3] J Howie, *Free subgroups in groups of small deficiency*, J. Group Theory 1 (1998) 95–112

[4] A Kawauchi, *A survey of knot theory*, Birkhauser Verlag, Basel (1996)

[5] T Kerler, *Homology TQFTs and the Alexander–Reidemeister invariant of 3–manifolds via Hopf algebras and skein theory*, preprint

[6] C Leininger, A Reid, *The co-rank conjecture for 3–manifold groups*, Algebr. Geom. Topol. 2 (2002) 37–50

[7] D Ruberman, *Seifert surfaces of knots in S^4*, Pacific J. Math. 145 (1990)

[8] A S Sikora, *Cut numbers of 3–manifolds*, preprint

[9] J Stallings, *Homology and central series of groups*, J. Algebra 2 (1965) 170–181

[10] W P Thurston, *A norm for the homology of 3–manifolds*, Mem. Amer. Math. Soc. 59 (1986) i–vi, 99–130