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A chain rule in the calculus of homotopy functors

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Abstract

We formulate and prove a chain rule for the *derivative*, in the sense of Goodwillie, of compositions of weak homotopy functors from simplicial sets to simplicial sets. The derivative spectrum $\partial F(X)$ of such a functor F at a simplicial set X can be equipped with a right action by the loop group of its domain X, and a free left action by the loop group of its codomain $Y = F(X)$. The derivative spectrum $\partial(E \circ F)(X)$ of a composite of such functors is then stably equivalent to the balanced smash product of the derivatives $\partial E(Y)$ and $\partial F(X)$, with respect to the two actions of the loop group of Y. As an application we provide a non-manifold computation of the derivative of the functor $F(X) = Q(\mathrm{Map}(K, X)_+)$.

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1 Introduction

The calculus of functors was introduced by Goodwillie in [6] as a language to keep track of stable range calculations of certain geometrically defined homotopy functors, such as stable pseudo-isotopy theory. The input for the theory is a homotopy functor

$$f\colon \mathcal{U} \longrightarrow \mathcal{T}$$

from spaces to based spaces. At an object $X \in \mathcal{U}$ it is then possible to associate the "best excisive approximation" to f near X. This so-called *linearization* of f at X is a functor

$$P_X f\colon \mathcal{U}/X \longrightarrow \mathcal{T}$$

from spaces over X to based spaces, which maps homotopy pushout squares to homotopy pullback squares. The associated reduced functor is called the *differential* of f at X, and is denoted by $D_X f$. Choosing a base point $x \in X$, the composite functor

$$L\colon \mathcal{T} \xrightarrow{i} \mathcal{U}/X \xrightarrow{D_X f} \mathcal{T}$$

that takes a based space T to $D_X f(X \vee_x T) = \mathrm{hofib}(P_X f(X \vee_x T) \to P_X f(X))$ is a *linear* functor, whose homotopy groups $L_*(T) = \pi_*(L(T))$ define a generalized homology theory. Each such homology theory is represented by a spectrum, and the spectrum associated to this particular homology theory L_* is called the *derivative* $\partial f(X)$ of f at (X, x).

The goal of this paper is to establish a chain rule for the derivative of a composite functor. This is a reasonable goal, since many naturally occurring functors are composites. For example, the topological Hochschild homology $\mathrm{THH}(X)$ of a space X has the homotopy type of $Q(\Lambda X_+)$, where $\Lambda X = \mathrm{Map}(S^1, X)$ is the free loop space (see [2, 3.7]). We can view this as the composite of the two functors $f(X) = \Lambda X$ and $e(Y) = Q(Y_+) = \mathrm{colim}_n\, \Omega^n \Sigma^n(Y_+)$.

In order to even state a chain rule, some modification has to be made to the above set-up. In particular, we will relax the condition that the functor f takes values in based spaces, considering instead weak homotopy functors

$$f\colon \mathcal{U} \longrightarrow \mathcal{U}$$

from spaces to spaces. (All our spaces will be compactly generated.) Then for any space X we let $Y = f(X)$, and choose base points $x \in X$ and $y \in Y$. We then study the derivative $\partial^x_y f(X)$ of f at X, with respect to the base points x and y. Of course, if $f(X)$ naturally comes equipped with a base point, then we may take that point as y.

Thus consider functors $e, f\colon \mathcal{U} \to \mathcal{U}$, with composite $e \circ f\colon \mathcal{U} \to \mathcal{U}$. Let X be a space, and set $Y = f(X)$, $Z = e(Y)$. Choose base points $x \in X$, $y \in Y$ and $z \in Z$. Suppose that f and e are bounded below, stably excisive functors (section 3), that e satisfies the colimit axiom (section 3), and that Y is path connected. Let $\Omega_y(Y)$ denote the geometric realization of the Kan loop group (section 8) of the total singular simplicial set of Y. This is a topological group, weakly homotopy equivalent to the usual loop space of (Y, y). (More precisely, $\Omega_y(Y)$ is a group object in the category of compactly generated topological spaces.) Then it turns out that, by choosing the models right (section 9, see also remark 12.4), the derivative $\partial^x_y f(X)$ admits a left $\Omega_y(Y)$–action and the derivative $\partial^y_z e(Y)$ admits a right $\Omega_y(Y)$–action. It thus makes sense to form the homotopy orbit spectrum for the diagonal $\Omega_y(Y)$–action on the smash product of spectra $\partial^y_z e(Y) \wedge \partial^x_y f(X)$.

Theorem 1.1 (Chain Rule) *Let $e, f\colon \mathcal{U} \to \mathcal{U}$ be bounded below, stably excisive functors, with $Y = f(X)$ and $Z = e(Y)$, and suppose that e satisfies the colimit axiom. Suppose that Y is path connected, and choose base points $x \in X$, $y \in Y$ and $z \in Z$. Then the composite $e \circ f$ is bounded below and stably excisive, and its derivative spectrum at X with respect to x and z is described by a stable equivalence*

$$\partial^x_z(e \circ f)(X) \simeq \partial^y_z e(Y) \wedge_{h\Omega_y(Y)} \partial^x_y f(X).$$

The subscript $h\Omega_y(Y)$ denotes homotopy orbits with respect to the diagonal action of the topological group $\Omega_y(Y)$.

This is theorem 12.3 specialized to the case when Y is path connected.

If X_x is the path component of x in X, and Z_y is the path component of z in Z, then the topological group $\Omega_x(X_x)$ acts on $\partial^x_z(e \circ f)(X)$ and $\partial^x_y f(X)$ from the right, the topological group $\Omega_z(Z_y)$ acts on $\partial^x_z(e \circ f)(X)$ and $\partial^y_z e(Y)$ from the left, and the chain rule gives a stable equivalence of spectra with left $\Omega_z(Z_y)$–action and right $\Omega_x(X_x)$–action.

It is technically easier to discuss these group actions on spectra that are formed in the category $\mathcal{S}$ of based simplicial sets than for spectra formed in $\mathcal{T}$. The reason is that the definition of the right action by $\Omega_y(Y)$ on $\partial^y_z e(Y)$ basically requires e to be a continuous functor. It is awkward to achieve continuity from a weak homotopy functor in the topological context. However, for functors between simplicial sets it is easy to promote a weak homotopy functor to a simplicial functor, which suffices to define the right action in the simplicial context. See definition 9.6. We therefore choose to develop the whole theory for weak homotopy functors $F\colon \mathcal{S} \to \mathcal{S}$ from the category $\mathcal{S}$ of simplicial sets to itself, rather than for functors $f\colon \mathcal{U} \to \mathcal{U}$. In this case, the chain rule appears as theorem 11.4.

It is also possible to start with functors $\Phi\colon \mathcal{S}/X \to \mathcal{S}$ and $\Psi\colon \mathcal{S}/Y \to \mathcal{S}$, with X a simplicial set and $Y = \Phi(X)$, that may or may not factor through the forgetful functors $u\colon \mathcal{S}/X \to \mathcal{S}$ and $u\colon \mathcal{S}/Y \to \mathcal{S}$, respectively. The latter is the most convenient general framework, and the body of the paper is written in this context. Thus theorem 11.3 is really our main theorem, from which the other forms of the chain rule are easily deduced.

The contents of the paper are as follows. In section 2 we define the categories of simplicial sets and spectra that we shall work with, and fix terminology like "bounded below" and "stably excisive" in section 3. Then in section 4 we start with a stably excisive weak homotopy functor $\Phi\colon \mathcal{S}/X \to \mathcal{S}$ and construct its "best excisive approximation" $P^0\Phi$, adapting [6, section 1]. Some modification is needed, since we want $P^0\Phi$ to take values in $\mathcal{S}/Y$ in order to be able to compose with Ψ. In section 5 we recall the Goodwillie derivative $\partial\Phi(X)$. In section 6 we show that in order to prove a chain rule expressing $\partial(\Psi\circ\Phi)(X)$ in terms of $\partial\Psi(Y)$ and $\partial\Phi(X)$ we may replace Φ and Ψ by their respective best excisive approximations (see proposition 6.2). This leads us to study linear functors $\Phi\colon \mathcal{R}(X) \to \mathcal{R}(Y)$ and $\Psi\colon \mathcal{R}(Y) \to \mathcal{R}(Z)$, where $\mathcal{R}(X)$ is the category of retractive simplicial sets over X, and $Z = \Psi(Y)$. In section 7 we reduce further to the case where X, Y and Z are all connected.

When (Y, y) is based and connected, there is a natural equivalence $\mathcal{R}(Y) \simeq \mathcal{R}(*; \Omega_y(Y))$ (see proposition 8.1), where $\mathcal{R}(*; \Omega_y(Y))$ is the category of based, free $\Omega_y(Y)$–simplicial sets, which we study in section 8. We are thus led to study linear functors $\phi\colon \mathcal{R}(*; \Omega_x(X)) \to \mathcal{R}(*; \Omega_y(Y))$ and $\psi\colon \mathcal{R}(*; \Omega_y(Y)) \to \mathcal{R}(*; \Omega_z(Z))$, and their composite $\psi\circ\phi$. The Goodwillie derivative $\partial\phi$ of such a functor ϕ has a natural free left $\Omega_y(Y)$–action. Using a canonical enrichment of ϕ to a simplicial functor ϕ, we show in section 9 that $\partial\phi$ also has a natural right $\Omega_x(X)$–action (see proposition 9.6). Then, in section 10 we establish a version of the *Brown–Whitehead representability theorem* (see [6, 1.3]) that represents a linear functor like ϕ above in terms of its Goodwillie derivative $\partial\phi$, equipped with these left and right actions, under the assumption that ϕ satisfies a "colimit axiom". See propositions 10.1 and 10.4. In section 11 we bring these structures and representations together, to prove the chain rule for bounded below, excisive functors Φ and Ψ in proposition 11.1, and for bounded below, stably excisive functors Φ and Ψ in theorem 11.3. The translation to functors to and from topological spaces goes via the usual equivalence $\mathcal{S} \simeq \mathcal{U}$, and is found in section 12.

We give a list of examples in section 13, including a purely homotopy-theoretic derivation in example 13.4 of the "stable homotopy of mapping spaces" functor

$X \mapsto Q(\mathrm{Map}(K, X)_+)$, which was previously investigated in [6, section 2], [9], and [1] using manifold or configuration space techniques. Our answer apparently takes a different form from that given in the cited papers, but in [10] the first author shows that the two answers are indeed equivalent.

The paper is written using fairly strong explicit hypotheses on the functors, such as being bounded below and stably excisive, in line with the style of [6] and [7]. Yet, many of the functors one typically considers satisfy these hypotheses. Our main technical reason for doing so occurs at the end of the proof of proposition 11.1, where we wish to ensure that one functor respects certain stable equivalences of spectra arising from another functor. A side effect is that all proofs become explicit, appealing directly to homotopy excision rather than to closed model category theory. Conceivably some of these conditions could be relaxed by reference to the framework of simplicial functors, as in [11], but the work leading to the present paper precedes that preprint. Likewise, the present work can be incorporated into the more general language of pointed simplicial algebraic theories, as in [13]. The second author's Master student H. Fausk [5] proved a version of the chain rule in the special case when $Y = f(X)$ is contractible.

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2 Categories of simplicial sets

Let $\mathcal{S}$ be the category of simplicial sets, and let X be a (fixed) simplicial set. The category $\mathcal{S}/X$ of *simplicial sets over* X has objects the simplicial sets X' equipped with a map $X' \to X$, and morphisms the maps $X' \to X''$ commuting with the structure map to X. The category $\mathcal{S}/X$ has the identity map $X \to X$ as a terminal object.

The category $\mathcal{R}(X)$ of *retractive simplicial sets over* X has objects the simplicial sets X' with maps $r\colon X' \to X$ and $s\colon X \to X'$ such that $rs\colon X \to X$ is the identity map, and morphisms the maps $X' \to X''$ that commute with both structure maps r and s. The category $\mathcal{R}(X)$ has the identity map $X \to X$ as an initial and terminal object. We briefly denote this base point object by X. In the case $X = *$ (a one-point simplicial set), $\mathcal{R}(*)$ is isomorphic to the category $\mathcal{S}_*$ of based simplicial sets.

Let G be a simplicial group. A G-simplicial set W is a simplicial set with an action $G \times W \to W$. For any G-simplicial set W let $\mathcal{R}(W; G)$ be the category of relatively free, retractive G–simplicial sets over W. It has objects (W', r, s), where W' is a G–simplicial set, $r\colon W' \to W$ and $s\colon W \to W'$ are maps of G–simplicial sets, $rs\colon W \to W$ is the identity map, and W' may be obtained from W by *attaching free G–cells*, ie, by repeated pushouts along the inclusions $G \times \partial\Delta^n \subset G \times \Delta^n$. When $G = 1$ is the trivial group, $R(W;1) = R(W)$ as before. When $W = *$, the objects of $R(*;G)$ are precisely the based, free G–simplicial sets. (Cf [14, page 378].)

Let $u\colon S/X \to S$, $v\colon R(X) \to S/X$ and $w\colon R(W;G) \to R(W)$ be the obvious forgetful functors.

Consider any functor $\Phi\colon S/X \to S$. Let $Y = \Phi(X)$ be its value at the terminal object X (equipped with the identity map $X \to X$). Then there is a canonical lift of $\Phi\colon S/X \to S$ over $u\colon S/Y \to S$ to a functor $S/X \to S/Y$, which we also denote by Φ. Furthermore, there is a canonical lift of $\Phi \circ v\colon R(X) \to S/Y$ over $v\colon R(Y) \to S/Y$ to a functor $R(X) \to R(Y)$, which we again denote by Φ. The latter functor takes the chosen initial and terminal object X of $R(X)$ to the chosen initial and terminal object Y of $R(Y)$. Such functors are called *pointed*.

Functors $\Phi\colon S/X \to S$ sometimes arise from functors $F\colon S \to S$ as composites $\Phi = F \circ u$, but will in general depend on the structure map to X. We have a commutative diagram:

(2.1)
$$\begin{array}{ccccc} R(X) & \xrightarrow{v} & S/X & \xrightarrow{u} & S \\ \downarrow{\scriptstyle\Phi} & & \downarrow{\scriptstyle\Phi} & \searrow^{\Phi} & \downarrow{\scriptstyle F} \\ R(Y) & \xrightarrow{v} & S/Y & \xrightarrow{u} & S \end{array}$$

In this paper, a *spectrum* $\mathbf{L}$ is a sequence $\{n \mapsto L_n\}$ of based simplicial sets L_n, and based structure maps $\sigma\colon \Sigma L_n = L_n \wedge S^1 \to L_{n+1}$ for $n \geq 0$, as in [3, 2.1]. Here it will be convenient to interpret S^1 as $\Delta^1 \cup_{\partial\Delta^1} \Delta^1$, rather than as $\Delta^1/\partial\Delta^1$. To be definite, we take the 0-th vertex of $\partial\Delta^1$ as the base point of S^1. Let $S^n = S^1 \wedge \cdots \wedge S^1$ (with n copies of S^1), and let $CS^n = S^n \wedge \Delta^1$ be the cone on S^n. We write Sp for the category of spectra.

A map of spectra $f\colon \mathbf{L} \to \mathbf{M}$ is a *strict equivalence* if each map $f_n\colon L_n \to M_n$ is a weak equivalence. It will be called a *meta-stable equivalence* if there exist integers c and κ such that $f_n\colon L_n \to M_n$ is $(2n - c)$–connected for all $n \geq \kappa$ (cf section 3). And f is a *stable equivalence* if it induces an isomorphism $\pi_*(f)\colon \pi_*(\mathbf{L}) \to \pi_*(\mathbf{M})$ on all homotopy groups. Clearly strict equivalences are meta-stable, and meta-stable equivalences are stable.

Let G be a simplicial group, as above. A *spectrum with G–action* $\mathbf{L}$ is a sequence $\{n \mapsto L_n\}$ of G–simplicial sets with a G–fixed base point, and based

G–maps $\Sigma L_n = L_n \wedge S^1 \to L_{n+1}$ for $n \geq 0$, where G acts trivially on S^1. Let Sp^G be the category of spectra with G–action.

A *free G–spectrum* $\mathbf{L}$ is a sequence $\{n \mapsto L_n\}$ of based, free G–simplicial sets, and based G–maps $\Sigma L_n = L_n \wedge S^1 \to L_{n+1}$. Here G acts trivially on S^1. Let $Sp(G)$ be the category of free G–spectra.

There are obvious forgetful functors $Sp(G) \to Sp^G$ and $Sp^G \to Sp$. A map of free G–spectra, or of spectra with G–action, is said to be a "strict", "metastable" or "stable equivalence" if the underlying map of spectra has the corresponding property. In particular, a stable equivalence of spectra with G–action is no more than a G–equivariant map that induces an isomorphism on all homotopy groups. This naive notion of stable equivalence permits the formation of homotopy orbits, but not (strict) fixed-points or orbits.

3 Excision conditions

A morphism $f\colon X_0 \to X_1$ in $\mathcal{S}$ is *k–connected* if for every choice of base point $x \in X_0$ the induced map $\pi_n(f)\colon \pi_n(X_0, x) \to \pi_n(X_1, f(x))$ is injective for $0 \leq n < k$ and surjective for $0 \leq n \leq k$. (No choice of base point is needed for $n = 0$, taking care of the case when X_0 is empty.) A *weak equivalence* is a map that is k–connected for every integer k.

Let c and κ be integers. A functor $F\colon \mathcal{S} \to \mathcal{S}$ is said to satisfy condition $E_1(c, \kappa)$ if for every k–connected map $X_0 \to X_1$ with $k \geq \kappa$ the map $F(X_0) \to F(X_1)$ is $(k - c)$–connected. A functor satisfying condition $E_1(c, \kappa)$ for some c and κ will be called *bounded below*. Such a functor necessarily takes weak equivalences to weak equivalences, ie, is a *weak homotopy functor*.

We form functorial homotopy limits and homotopy colimits of diagrams of simplicial sets as in [4]. A commutative square of simplicial sets

(3.1)
$$\begin{array}{ccc} X_0 & \longrightarrow & X_1 \\ \downarrow & & \downarrow \\ X_2 & \longrightarrow & X_3 \end{array}$$

is *k–cartesian* if the induced map $a\colon X_0 \to \mathrm{holim}(X_1 \to X_3 \leftarrow X_2)$ is k–connected. It is *cartesian* if a is a weak equivalence. The square is *k–cocartesian* if the induced map $b\colon \mathrm{hocolim}(X_1 \leftarrow X_0 \to X_2) \to X_3$ is k–connected. It is *cocartesian* if b is a weak equivalence. (Cf [6, 1.2].)

A functor $F\colon \mathcal{S} \to \mathcal{S}$ is said to satisfy condition $E_2(c;\kappa)$ if, for every cocartesian square as above for which $X_0 \to X_i$ is k_i–connected and $k_i \geq \kappa$ for $i = 1, 2$, the resulting square

$$\begin{array}{ccc} F(X_0) & \longrightarrow & F(X_1) \\ \downarrow & & \downarrow \\ F(X_2) & \longrightarrow & F(X_3) \end{array}$$

is $(k_1 + k_2 - c)$–cartesian. The functor F is called *stably excisive* if it satisfies condition $E_2(c;\kappa)$ for some integers c and κ. F is called *excisive* if it takes all cocartesian squares to cartesian squares. (Cf [6, 1.8].)

A morphism in one of the categories $\mathcal{S}/X$, $R(X)$ or $R(W;G)$ is said to be "k–connected", or a "weak equivalence", if the underlying morphism in $\mathcal{S}$ has that property. Similarly for k–cartesian, cartesian, k–cocartesian and cocartesian squares. The conditions $E_1(c;\kappa)$, "bounded below", "weak homotopy functor", $E_2(c;\kappa)$, "stably excisive" and "excisive" then also make sense for functors $\mathcal{S}/X \to \mathcal{S}$, $\mathcal{S}/X \to \mathcal{S}/Y$, $R(X) \to R(Y)$, $R(\ ;H) \to R(\ ;G)$, etc.

Proposition 3.2 *Let X be a simplicial set, $\Phi\colon \mathcal{S}/X \to \mathcal{S}$ a functor, $Y = \Phi(X)$ a simplicial set, and $\Psi\colon \mathcal{S}/Y \to \mathcal{S}$ a functor. Suppose that Φ and Ψ are bounded below and stably excisive. Then the composite functor $\Psi \circ \Phi\colon \mathcal{S}/X \to \mathcal{S}$ is also bounded below and stably excisive.*

Proof Suppose that Φ and Ψ satisfy $E_1(c;\kappa)$ and $E_2(c;\kappa)$, where we may assume that $c \geq 1$ and $\kappa \geq 0$. We claim that $\Psi \circ \Phi$ satisfies $E_1(2c;\kappa + c)$ and $E_2(3c+1;\kappa + c)$. The first claim is clear. For the second, consider a cocartesian diagram as in (3.1), with $X_0 \to X_i$ k_i–connected for $i = 1, 2$, and $k_i \geq \kappa + c$. Apply Φ to get a $(k_1 + k_2 - c)$–cartesian square

(3.3)
$$\begin{array}{ccc} \Phi(X_0) & \longrightarrow & \Phi(X_1) \\ \downarrow & & \downarrow \\ \Phi(X_2) & \longrightarrow & \Phi(X_3) \end{array}$$

with $\Phi(X_0) \to \Phi(X_i)$ $(k_i - c)$–connected for $i = 1, 2$. Let

$$PO = \operatorname{hocolim}(\Phi(X_1) \leftarrow \Phi(X_0) \to \Phi(X_2))$$

be the homotopy pushout in this square. By homotopy excision (cf [8, 4.23]) the cocartesian square

(3.4)
$$\begin{array}{ccc} (X_0) & \longrightarrow & (X_1) \\ \downarrow & & \downarrow \\ (X_2) & \longrightarrow & PO \end{array}$$

is $(k_1 + k_2 - 2c - 1)$–cartesian. It follows by comparison of (3.3) and (3.4) that the canonical map $PO \to (X_3)$ is $(k_1 + k_2 - 2c)$–connected (when $2c + 1 \quad c$). Applying Ψ to (3.4), we obtain a $(k_1 + k_2 - 3c)$–cartesian square. The map $\Psi(PO) \to \Psi(X_3)$ is $(k_1 + k_2 - 3c)$–connected (when 2), so the square obtained by applying Ψ to (3.3) is $(k_1 + k_2 - 3c - 1)$–cartesian. □

A weak homotopy functor $F\colon S \to S$ satisfies the *colimit axiom* if it preserves filtered homotopy colimits up to weak homotopy. This means that for any filtered diagram $X\colon D \to S$ the canonical map

(3.5)
$$\operatorname*{hocolim}_{d \in D} F(X_d) \to F(\operatorname*{hocolim}_{d \in D} X_d)$$

is a weak equivalence. Any simplicial set is weakly equivalent to the homotopy colimit of the filtered diagram of its finite sub-objects, where a simplicial set is *finite* if it has only finitely many non-degenerate simplices. Thus a functor satisfying the colimit axiom is determined by its restriction to the subcategory of finite simplicial sets. Such functors are therefore also said to be *finitary*.

Similarly, a functor $\colon R(W;G) \to S$ satisfies the *colimit axiom* if it preserves filtered homotopy colimits up to weak equivalence. An object of $R(W;G)$ is said to be *finite* if it can be obtained from W by attaching finitely many free G–cells. Again a functor satisfying the colimit axiom is determined by its restriction to the finite objects in $R(W;G)$.

The forgetful functor $u\colon S/X \to S$ preserves filtered homotopy colimits. Hence if $F\colon S \to S$ satisfies the colimit axiom, then so does the composite functor $= F \quad u\colon S/X \to S$.

Remark 3.6 Let $S\ /X$ be the full subcategory of S/X with objects the –connected maps $X' \to X$. The conditions $E_1(c; \ )$ and $E_2(c; \ )$ then make sense for functors $\colon S\ /X \to S$, and all of the results of this paper also apply to functors with such a restricted domain of definition. One could even consider *germs* of functors $S/X \to S$, ie, equivalence classes of functors $\colon S\ /X \to S$ defined for some integer , with two such functors and $'\colon S\ '/X \to S$ considered to be equivalent if there is a $''$ such that $|S\ ''/X = \ '|S\ ''/X$. For simplicity we will not include this extra generality in our notation.

4 Excisive approximation

If $X' \to X$ is an object of S/X, its *berwise (unreduced) cone* $C_X X'$ is the mapping cylinder $(X' \quad ^1) \cup_{X'} X$, and its *berwise (unreduced) suspension* $S_X X'$ is the union of two such mapping cylinders along X'. There is a cocartesian square of simplicial sets over X:

$$\begin{array}{ccc} X' & \longrightarrow & C_X X' \\ \downarrow & & \downarrow \\ C_X X' & \longrightarrow & S_X X' \end{array}$$

The functor S_X increases the connectivity of simplicial sets and maps by at least one.

Consider a weak homotopy functor $\ : S/X \to S$. Following Goodwillie [6, section 1], we associate to the weak homotopy functor $T \ : S/X \to S$ given by

$$(T \ )(X') = \text{holim}(\ (C_X X') \to \ (S_X X') \quad (C_X X')).$$

If satis es $E_1(c; \ )$ then T satis es $E_1(c; \ - 1)$. There is a natural map $t \ : \ \to T \ $. De ne $T^n \ : S/X \to S$ for $n \quad 0$ by iteration, and let the weak homotopy functor $P \ : S/X \to S$ be the homotopy colimit

$$(P \ )(X') = \underset{n}{\text{hocolim}}(T^n \ )(X').$$

Again there is a natural map $p \ : \ \to P \ $, as functors $S/X \to S$. (Cf [6, 1.10].) If satis es $E_1(c; \ )$ then P satis es $E_1(c; \ ')$ for all $'$.

We know that P lifts to a functor $S/X \to S/P \ (X)$, but we wish to modify it to a functor $S/X \to S/Y$, with $Y = \ (X)$. There is a commutative square

(4.1)
$$\begin{array}{ccc} (X') & \xrightarrow{p \ (X')} & P \ (X') \\ \downarrow & & \downarrow \\ (X) & \xrightarrow[\ ]{p \ (X)} & P \ (X) \end{array}$$

induced by the unique morphism $X' \to X$ in S/X. The lower horizontal map is a weak equivalence by inspection of the construction of P , using that was assumed to preserve weak equivalences. We set

$$P' \ (X') = \text{holim}(P \ (X') \to P \ (X) \quad (X))$$

equal to the homotopy limit (pullback) of the lower right hand part of the diagram. The commutative square (4.1) then extends to

$$\begin{array}{ccccc} (X') & \xrightarrow{p' \ (X')} & P' \ (X') & \xrightarrow{\ ,\ } & P \ (X') \\ \downarrow & & \downarrow & & \downarrow \\ (X) & \xrightarrow{=} & (X) & \xrightarrow{p \ ,} & P \ (X) ; \end{array}$$

where the right hand square only commutes up to homotopy. Thus we can view and P' as functors $S{=}X \to S{=}Y$, in which case there is a natural map $p' : \ \to P'$. Viewing and P' as functors to S, the natural map p factors as

(4.2) $$p \ : \ \xrightarrow{p'} P' \ \xrightarrow{,} P \ ;$$

where the right hand map is a natural weak equivalence. So P' is a weak homotopy functor, and if satisfies $E_1(c; \ )$ for some , then P' satisfies $E_1(c; \ ')$ for all $'$.

Remark 4.3 Note that $P' \ (X)$ is typically not equal to $(X) = Y$, although the canonical map $P' \ (X) \to Y$ is a weak equivalence, so $P' : S{=}X \to S{=}Y$ is not the canonical lift of its forgetful version $u \ P' : S{=}X \to S$.

Suppose now that $: S{=}X \to S$ satisfies condition $E_2(c; \ )$, hence is stably excisive. When the structure map $X' \to X$ is k-connected, for k , it follows immediately that the maps $t \ (X'): \ (X') \to T \ (X')$, $p \ (X'): \ (X') \to P \ (X')$ and $p' \ (X'): \ (X') \to P' \ (X')$ are all $(2k - c)$-connected.

We say that two functors $; \ ': S{=}X \to S$ satisfy condition $O(c; \ )$ along a natural map $f: \ \to \ '$ if whenever $X' \to X$ is k-connected and k , then $f(X'): \ (X') \to \ '(X')$ is $(2k - c)$-connected. If and $'$ satisfy condition $O(c; \ )$ for some integers c and , then we say that and $'$ *agree to first order along* f. (Cf [6, 1.13].)

Proposition 4.4 *Let X be a simplicial set, $: S{=}X \to S$ a stably excisive weak homotopy functor, and let $Y = \ (X)$. Then $P' : S{=}X \to S{=}Y$ is excisive, and and $P' : S{=}X \to S{=}Y$ agree to first order along p'. If is bounded below, then so is P'. If satisfies the colimit axiom, then so does P'.*

Proof Goodwillie proves in [6, 1.14] that the functor P is excisive and that Φ agrees with P to first order along p. In view of the weak equivalence in (4.2), the same applies to P'. We have noted above that if Φ is bounded below, then so are T, P and P'. If Φ satisfies the colimit axiom, then so does T, because filtered homotopy colimits commute with homotopy pullbacks, up to weak equivalence. (See theorem 1 on pages 215–216 in [12] for the corresponding statement for sets.) Hence also P and P' satisfy the colimit axiom, since the order of two homotopy colimits can be commuted. □

5 Goodwillie derivatives

Let X be a simplicial set, let $\Phi\colon S/X \to S$ be a weak homotopy functor, and let $Y = \Phi(X)$. As before, we may view Φ as a functor $S/X \to S/Y$ or $R(X) \to R(Y)$, without change in notation. Choose base points $x \in X$ and $y \in Y$.

There is a functor $i_0 = i_0(X; x)\colon S_* \to R(X)$ that takes a based simplicial set T to the retractive simplicial set

$$i_0(T) = X \vee_x T\,;$$

where $r\colon X \vee_x T \to X$ takes T to the base point $x \in X$, and $s\colon X \to X \vee_x T$ is the standard inclusion. This functor preserves cocartesian squares.

There is a second functor $j_0 = j_0(Y; y)\colon R(Y) \to S_*$ that takes a retractive simplicial set $(Y'; r; s)$ to the homotopy fiber

$$j_0(Y') = \mathrm{hofib}_y(r\colon Y' \to Y)\,;$$

(with the natural base point that maps to $s(y) \in Y'$). This functor preserves k–cartesian squares for all k, hence also cartesian squares.

We shall later consider equivariant improvements i and j of i_0 and j_0, respectively, which may justify the notation.

If Φ is an excisive weak homotopy functor, then the composite functor

$$L\colon S_* \xrightarrow{i_0} R(X) \xrightarrow{\Phi} R(Y) \xrightarrow{j_0} S_*$$

is an excisive weak homotopy functor that takes $*$ to $L(*) = \mathrm{hofib}_y(Y \to Y)$, which is contractible. We say that L is a *linear* functor. It corresponds to a generalized (reduced) homology theory given by $L_*(T) = \pi_*(L(T))$, with an associated coefficient spectrum $\mathbf{L} = \{n \mapsto L(S^n)\}$ (modulo a technical rectification, as in [6, 0.1]). There is a natural weak equivalence $\Omega^\infty(\mathbf{L} \wedge T) \to L(T)$, at least for finite simplicial sets T. (Cf [6] and proposition 10.4 below.)

Even if Φ is not excisive, we can still form the composite functor $j_0 \Phi i_0$ and assemble the based simplicial sets $(j_0 \Phi i_0)(S^n)$ into a spectrum. For any weak homotopy functor $\Phi\colon S/X \to S$ let

$$\partial_y^x \Phi(X)_n = \mathrm{hofib}_y(\Phi(X \vee_x S^n) \to \Phi(X)) \tag{5.1}$$

for $n \geq 0$. There is a natural chain of maps

$$\partial_y^x \Phi(X)_n \xleftarrow{\simeq} \mathrm{hofib}_y(\Phi(X \vee_x S^n) \to \Phi(X \vee_x CS^n))$$
$$\longrightarrow \mathrm{hofib}_y(\Phi(X \vee_x CS^n) \to \Phi(X \vee_x S^{n+1})) \xleftarrow{\simeq} \Omega \partial_y^x \Phi(X)_{n+1}$$

where the second map is the natural one between the horizontal homotopy fibers in the commutative diagram

$$\begin{array}{ccc} \Phi(X \vee_x S^n) & \longrightarrow & \Phi(X \vee_x CS^n) \\ \downarrow & & \downarrow \\ \Phi(X \vee_x CS^n) & \longrightarrow & \Phi(X \vee_x S^{n+1}) \end{array}$$

and the other two maps are natural weak equivalences derived from the weak equivalence $X \vee_x CS^n \to X$ and the Puppe sequence. We let

$$\partial_y^x \Phi(X) = \{n \mapsto \partial_y^x \Phi(X)_n\} \tag{5.2}$$

be the spectrum obtained from this sequence of based simplicial sets and (weak) adjoint structure maps by the functorial rectification procedure of [6, 0.1]. By definition, $\partial_y^x \Phi(X)$ is the *Goodwillie derivative* of Φ at X, with respect to the base points $x \in X$ and $y \in Y = \Phi(X)$. (Cf [6, 1.16].)

A natural map $f\colon \Phi \to \Phi'$ of functors $S/X \to S$ induces maps $\partial f\colon \partial_y^x \Phi(X)_n \to \partial_{y'}^x \Phi'(X)_n$ for all $n \geq 0$, and a spectrum map $\partial f\colon \partial_y^x \Phi(X) \to \partial_{y'}^x \Phi'(X)$. This presupposes that $\Phi'(X)$ is given the base point $y' = f(X)(y)$, where y is the chosen base point in $\Phi(X)$ and $f(X)\colon \Phi(X) \to \Phi'(X)$.

Proposition 5.3 *If Φ and $\Phi'\colon S/X \to S$ agree to first order along $f\colon \Phi \to \Phi'$, then f induces a meta-stable equivalence of spectra $\partial f\colon \partial_y^x \Phi(X) \to \partial_{y'}^x \Phi'(X)$.*

Proof This is basically [6, 1.17]. Suppose that Φ and Φ' satisfy $O(c, \kappa)$. The retraction $X \vee_x S^n \to X$ is n–connected, so for $n \geq \kappa$ the map $f(X \vee_x S^n)\colon \Phi(X \vee_x S^n) \to \Phi'(X \vee_x S^n)$ is $(2n - c)$–connected. In a similar way $f(X)\colon \Phi(X) \to \Phi'(X)$ is a weak equivalence. Hence the map of homotopy fibers $\partial f\colon \partial_y^x \Phi(X)_n \to \partial_{y'}^x \Phi'(X)_n$ is $(2n - c)$–connected. □

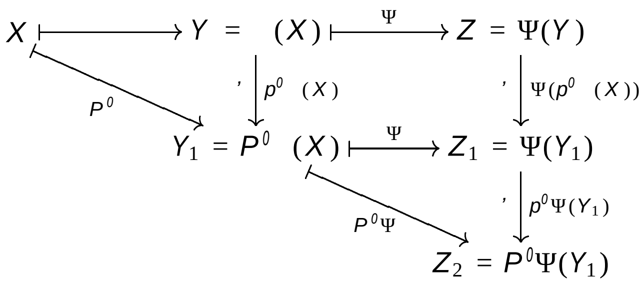
Let X be a simplicial set, $\Phi\colon S/X \to S$ a weak homotopy functor, $Y = \Phi(X)$, and choose base points $x \in X$ and $y \in Y$. Give $P^0 \Phi(X)$ (defined in section 4) the base point $y^0 = p^0 \Phi(X)(y)$.

Corollary 5.4 *If Φ is stably excisive, then $p^0\Phi$ induces a meta-stable equivalence of spectra $\partial(p^0\Phi)\colon \partial^x_y\Phi(X) \to \partial^x_{y'}(P^0\Phi)(X)$.*

When $F\colon S \to S$ is a weak homotopy functor, X a simplicial set, $Y = F(X)$, $x \in X$, $y \in Y$ and $\Phi = F \circ u$, we let $\partial^x_y F(X)_n = \partial^x_y \Phi(X)_n = \text{hofib}_y(F(X \vee_x S^n) \to F(X))$ and $\partial^x_y F(X) = \partial^x_y \Phi(X) = \{n \mapsto \partial^x_y F(X)_n\}$.

6 Composite functors

Let X be a simplicial set, $\Phi\colon S/X \to S$ a functor, $Y = \Phi(X)$, $\Psi\colon S/Y \to S$, and $Z = \Psi(Y)$. Suppose that Φ and Ψ are weak homotopy functors. Let $Y_1 = P^0\Phi(X)$, $Z_1 = \Psi(Y_1)$ and $Z_2 = P^0\Psi(Y_1)$. Choose base points $x \in X$, $y \in Y$ and $z \in Z$, and let $y_1 = p^0\Phi(X)(x) \in Y_1$, $z_1 = \Psi(p^0\Phi(X))(z) \in Z_1$ and $z_2 = p^0\Psi(Y_1)(z_1) \in Z_2$.



Proposition 6.1 *Suppose that Φ and Ψ are bounded below, stably excisive functors. Then the composite functors $\Psi \circ \Phi$ and $P^0\Psi \circ P^0\Phi\colon S/X \to S$ agree to first order along $p^0\Psi \circ \Psi(p^0\Phi) = P^0\Psi(p^0\Phi) \circ p^0\Psi$.*

Proof Assume that Φ and Ψ satisfy $E_1(c, \kappa)$ and $E_2(c, \kappa)$, for some sufficiently large integers c and κ. Let $X' \to X$ be a k-connected map, with $k \ge \kappa + c$. Then $p^0\Phi(X')\colon \Phi(X') \to P^0\Phi(X')$ is $(2k - c)$-connected, so $\Psi(p^0\Phi)(X')\colon (\Psi \circ \Phi)(X') \to (\Psi \circ P^0\Phi)(X')$ is $(2k - 2c)$-connected. Furthermore, $P^0\Phi(X') \to P^0\Phi(X)$ is $(k - c)$-connected, as noted after diagram (4.2), so $p^0\Psi(P^0\Phi(X'))\colon (\Psi \circ P^0\Phi)(X') \to (P^0\Psi \circ P^0\Phi)(X')$ is $(2k - 3c)$-connected. Thus $p^0\Psi \circ \Psi(p^0\Phi)$ satisfies $O(3c, \kappa + c)$, and $\Psi \circ \Phi$ and $P^0\Psi \circ P^0\Phi\colon S/X \to S$ agree to first order. □

Recall also that for Φ and Ψ bounded below and stably excisive the composite functor $\Psi \circ \Phi\colon S/X \to S$ is bounded below and stably excisive (proposition 3.2), hence agrees to first order with $P^0(\Psi \circ \Phi)\colon S/X \to S$ (proposition 4.4). We are therefore legitimately interested in its derivative $\partial^x_z(\Psi \circ \Phi)(X)$.

Proposition 6.2 *Suppose that* Φ *and* Ψ *are bounded below, stably excisive functors. Then there are natural meta-stable equivalences*

(1) $\partial(p'\Phi)\colon \partial^x_y \Phi(X) \simeq \partial^x_{y_1} P'\Phi(X)$,

(2) $\partial(p'\Psi)\colon \partial^{y_1}_{z_1} \Psi(Y_1) \simeq \partial^{y_1}_{z_2} (P'\Psi)(Y_1)$,

(3) $\partial(p'\Psi\Phi \Psi(p'\Phi))\colon \partial^x_z (\Psi\Phi)(X) \simeq \partial^x_{z_2} (P'\Psi P'\Phi)(X)$

and a strict equivalence

(4) $\partial^y_z \Psi(Y) \simeq \partial^{y_1}_{z_1} \Psi(Y_1)$.

Proof By propositions 4.4 and 6.1, the pairs of functors Φ and $P'\Phi$, Ψ and $P'\Psi$, and the functors $\Psi\Phi$ and $P'\Psi P'\Phi$ agree to first order, respectively. Hence their derivatives are meta-stably equivalent by proposition 5.3.

Case (4) remains. There is a commutative square in S/Y

$$\begin{array}{ccc} Y \vee_y S^n & \xrightarrow{\simeq} & Y_1 \vee_{y_1} S^n \\ \downarrow & & \downarrow \\ Y & \xrightarrow[\simeq]{p'\Phi(X)} & Y_1 \end{array}$$

where the vertical maps take S^n to the respective base points, and the upper horizontal map is the identity on S^n. The lower horizontal map is a weak equivalence, as in remark 4.3, hence so is the upper horizontal map. Applying Ψ and taking vertical homotopy fibers yields a weak equivalence of n-th spaces

$$\partial^y_z \Psi(Y)_n \to \partial^{y_1}_{z_1} \Psi(Y_1)_n .$$

Thus the associated spectra are strictly equivalent. □

Remark 6.3 It follows that for the purpose of expressing the derivative of $\Psi\Phi$ in terms of the derivatives of Φ and Ψ, we are free to replace the bounded below, stably excisive functors Φ and Ψ by their bounded below, excisive approximations $P'\Phi$ and $P'\Psi$, respectively. If Ψ satisfies the colimit axiom, then so does its replacement.

Equivalently, we may assume that Φ and Ψ are themselves bounded below, excisive functors. Furthermore, the derivatives only depend on the associated functors $\Phi\colon R(X) \to R(Y)$, $\Psi\colon R(X) \to R(Y)$ with composite $\Psi\Phi\colon R(X) \to R(Z)$.

7 Multiple connected components

We now reduce to the case when X, Y and Z are connected.

Let $\Phi\colon R(X) \to R(Y)$ and $\Psi\colon R(Y) \to R(Z)$ be weak homotopy functors, with $Y = \Phi(X)$ and $Z = \Psi(Y)$. Write $X = \coprod_{\alpha \in A} X_\alpha$, $Y = \coprod_{\beta \in B} Y_\beta$ and $Z = \coprod_{\gamma \in C} Z_\gamma$, where each X_α, Y_β and Z_γ is connected. So $A = \pi_0(X)$, $B = \pi_0(Y)$ and $C = \pi_0(Z)$. Choose base points $x_\alpha \in X_\alpha$, $y_\beta \in Y_\beta$ and $z_\gamma \in Z_\gamma$ for all α, β and γ.

Let $in_\alpha\colon R(X_\alpha) \to R(X)$ be given by pushout along $X_\alpha \subset X$, so $in_\alpha(X^0_\alpha) = X \cup_{X_\alpha} X^0_\alpha$. Similarly let $pr_\beta\colon R(Y) \to R(Y_\beta)$ be given by pullback along $Y_\beta \subset Y$, so $pr_\beta(Y^0) = Y_\beta \times_Y Y^0$. Let $\Phi_{\alpha\beta} = pr_\beta \circ \Phi \circ in_\alpha\colon R(X_\alpha) \to R(Y_\beta)$. Clearly in_α preserves k–connected maps and (k–)cocartesian squares, while pr_β preserves k–connected maps and (k–)cartesian squares. So if Φ is bounded below, excisive, stably excisive or satisfies the colimit axiom, then the same applies to $\Phi_{\alpha\beta}$.

Lemma 7.1 *There is a natural strict equivalence*

$$\partial^x_y \Phi(X) \simeq \partial^{x_\alpha}_{y_\beta} \Phi_{\alpha\beta}(X_\alpha).$$

Proof Let $X^0_\alpha = X_\alpha \vee_{x_\alpha} S^n$, $X^0 = in_\alpha(X^0_\alpha) = X \vee_x S^n$, $Y^0 = \Phi(X^0)$ and $Y^0_\beta = pr_\beta(Y^0)$, so that $Y^0_\beta = \Phi_{\alpha\beta}(X^0_\alpha)$. The pullback square

$$\begin{array}{ccc} Y^0_\beta & \longrightarrow & Y^0 \\ \downarrow & & \downarrow \\ Y_\beta & \longrightarrow & Y \end{array}$$

is cartesian, so there is a weak equivalence

$$\partial^x_y \Phi(X)_n = \mathrm{hofib}_y(Y^0 \to Y) \simeq \mathrm{hofib}_y(Y^0_\beta \to Y_\beta) = \partial^{x_\alpha}_{y_\beta} \Phi_{\alpha\beta}(X_\alpha)_n. \qquad \square$$

Let $\Psi_\beta = \Psi \circ in_\beta\colon R(Y_\beta) \to R(Z)$ and consider a base point $z \in Z$. For each Y^0 in $R(Y)$ and $\beta \in B$ let $Y^0_\beta = pr_\beta(Y^0)$ in $R(Y_\beta)$.

Proposition 7.2 *Let $\Psi\colon R(Y) \to R(Z)$ be a bounded below, excisive functor that satisfies the colimit axiom. Then the functors $Y^0 \mapsto \mathrm{hofib}_z(\Psi(Y^0) \to Z)$ and*

$$Y^0 \mapsto \prod_{\beta \in B} \mathrm{hofib}_z(\Psi_\beta(Y^0_\beta) \to Z)$$

agree to first order along a natural chain of maps.

Proof The retraction $Y' \to Y_\beta$ in $\mathcal{R}(Y')$ induces a retraction $\Psi(Y') \to \Psi_\beta(Y')$. More generally, for each finite subset $S \subset B$ let

$$Y_S = \coprod_{\beta \in S} Y' \cup \coprod_{\beta \notin S} Y_\beta .$$

There is then a strongly cocartesian (cf [7, 2.1]) cubical diagram

$$(T \subset S) \mapsto Y_{S \setminus T}$$

in $\mathcal{R}(Y)$. Applying the excisive functor Ψ yields a strongly cartesian cubical diagram, where each map admits a section. Hence there is a weak equivalence

$$\mathrm{hofib}_z(\Psi(Y_S) \to Z) \xrightarrow{\simeq} \prod_{\beta \in S} \mathrm{hofib}_z(\Psi_\beta(Y') \to Z).$$

Passing to homotopy colimits over $S \subset B$, and using that Ψ satisfies the colimit axiom, yields a weak equivalence

$$\mathrm{hofib}_z(\Psi(Y') \to Z) \xrightarrow{\simeq} \operatorname*{hocolim}_{S \subset B} \prod_{\beta \in S} \mathrm{hofib}_z(\Psi_\beta(Y') \to Z).$$

When Ψ satisfies $E_1(c, \kappa)$ and $Y' \to Y$ is k-connected, with $k \ge \kappa$, then $Y_\beta \to Y$ is k-connected and $\Psi_\beta(Y') \to Z$ is $(k-c)$-connected, for each β. So each space $\mathrm{hofib}_z(\Psi_\beta(Y') \to Z)$ is $(k-c-1)$-connected, each inclusion

$$\bigvee_{\beta \in S} \mathrm{hofib}_z(\Psi_\beta(Y') \to Z) \to \prod_{\beta \in S} \mathrm{hofib}_z(\Psi_\beta(Y') \to Z)$$

is $(2k-2c-1)$-connected, and the resulting map

$$\operatorname*{hocolim}_{S \subset B} \bigvee_{\beta \in S} \mathrm{hofib}_z(\Psi_\beta(Y') \to Z) \to \operatorname*{hocolim}_{S \subset B} \prod_{\beta \in S} \mathrm{hofib}_z(\Psi_\beta(Y') \to Z)$$

is $(2k-2c-1)$-connected. The source of this map is naturally equivalent to $\bigvee_{\beta \in B} \mathrm{hofib}_z(\Psi_\beta(Y') \to Z)$. □

Let $\Phi_\beta = \mathrm{in}_\beta \circ \Phi \colon \mathcal{R}(X) \to \mathcal{R}(Y)$ and $\Psi_y = \mathrm{pr}_y \circ \Psi \colon \mathcal{R}(Y) \to \mathcal{R}(Z_y)$.

Proposition 7.3 *Let $\Phi\colon \mathcal{R}(X) \to \mathcal{R}(Y)$ and $\Psi\colon \mathcal{R}(Y) \to \mathcal{R}(Z)$ be bounded below, and suppose that Ψ is excisive and satisfies the colimit axiom. There is a natural chain of meta-stable equivalences*

$$\partial^x_{z_y}(\Psi_y \circ \Phi)(X) \simeq \bigvee_{\beta \in B} \partial^x_{z_y}(\Psi_y \circ \Phi_\beta)(X).$$

Proof We keep the notation introduced in the last two proofs. Then

$$
\begin{aligned}
\partial^x_{z_y}(\Psi_y \Phi)(X)_n &= \text{hofib}_{z_y}((\Psi_y \Phi)(X') \to Z_y) \\
&= \text{hofib}_{z_y}(\Psi_y(Y') \to Z_y) \\
&\simeq \prod_{\beta\in B} \text{hofib}_{z_y}(\Psi_y(Y') \to Z_y) \\
&= \prod_{\beta\in B} \text{hofib}_{z_y}((\Psi_y \Phi)(X') \to Z_y) \\
&= \prod_{\beta\in B} \partial^x_{z_y}(\Psi_y \Phi)(X)_n
\end{aligned}
$$

where $\simeq$ denotes agreement up to first order as functors of Y', by proposition 7.2. Since Φ is bounded below, the map $Y' \to Y$ is $(n-c)$–connected for some constant c, and so this chain of maps is $(2n-c)$–connected for some (other) constant c. Hence the associated spectrum map is a meta-stable equivalence. □

8 The Kan loop group

By the results of section 6 we need only consider bounded below, excisive functors $\Phi\colon R(X) \to R(Y)$ and $\Psi\colon R(Y) \to R(Z)$, where X is a simplicial set, $Y = \Phi(X)$ and $Z = \Psi(Y)$, and by section 7 we may assume that X, Y and Z are all connected. Choose base points $x \in X$, $y \in Y$ and $z \in Z$, and recall the functors $i_0 = i_0(X;x)$ and $j_0 = j_0(Z;z)$ from section 5. We are interested in the coefficient spectrum of the composite functor:

$$S_* \xrightarrow{i_0} R(X) \xrightarrow{\Phi} R(Y) \xrightarrow{\Psi} R(Z) \xrightarrow{j_0} S_* \,.$$

We would like to be able to express this in terms of the coefficient spectra of the composites $j_0\Phi i_0\colon S_* \to R(X) \to R(Y) \to S_*$ and $j_0\Psi i_0\colon S_* \to R(Y) \to R(Z) \to S_*$. But, as consideration of the two special cases $\Phi = P_X(\text{id})$ and $\Psi = P_Y(\text{id})$ indicates, where id: $S_* \to S_*$ is the identity functor, this is not likely to be possible. Composition with the functors $j_0\colon R(Y) \to S_*$ and $i_0\colon S_* \to R(Y)$ does not retain enough information.

We shall instead replace the category $S_* = R(*)$ by the category $R(*;\Omega_y(Y))$, where $\Omega_y(Y)$ is the Kan loop group of $(Y;y)$, and replace the functors i_0 and j_0 by suitable inverse weak equivalences $i\colon R(*;\Omega_y(Y)) \to R(Y)$ and $j\colon R(Y) \to R(*;\Omega_y(Y))$, respectively. The Goodwillie derivative $\partial^x_y\Phi(X)$ then becomes a free left $\Omega_y(Y)$–spectrum. Furthermore, there is a model for the Goodwillie derivative $\partial^y_z\Psi(Y)$ that becomes a spectrum with right $\Omega_y(Y)$–action. It turns out that these extra simplicial group actions suffice to express the derivative $\partial_z^x(\Psi \circ \Phi)(X)$ in terms of these equivariant Goodwillie derivatives, leading to the chain rule.

Suppose now that Y is a connected simplicial set, with a chosen base point $y \in Y$. The *Kan loop group* of Y is then a functorially defined simplicial group $\Omega_y(Y)$. (Cf [15], where the Kan loop group $\Omega_y(Y)$ is denoted $G(Y)$.) There is a principal $\Omega_y(Y)$–bundle

$$\Omega_y(Y) \longrightarrow \tilde{\Gamma}(Y) \longrightarrow \Gamma(Y)$$

with $\tilde{\Gamma}(Y)$ weakly contractible, and a natural inclusion $\gamma\colon Y \to \Gamma(Y)$ which is a weak equivalence. There are natural base points in $\Gamma(Y)$ and $\tilde{\Gamma}(Y)$, and π and γ preserve these points. Let $\epsilon\colon \tilde{\Gamma}(Y) \to *$ be the (unique) weak equivalence.

Proposition 8.1 *There are natural pointed weak homotopy functors*

$$i\colon R(*;\Omega_y(Y)) \to R(Y)$$
$$j\colon R(Y) \to R(*;\Omega_y(Y));$$

such that $i = i(Y;y)$ preserves k–connected maps and (k–)cocartesian squares, and $j = j(Y;y)$ preserves k–connected maps and (k–)cartesian squares. There is a natural weak equivalence from the identity functor on $R(Y)$ to the composite $i \circ j\colon R(Y) \to R(Y)$.

Remark 8.2 This is closely related to [14, 2.1.4]. There is also a natural chain of weak equivalences from the composite $j \circ i\colon R(*;\Omega_y(Y)) \to R(*;\Omega_y(Y))$ to the identity functor on $R(*;\Omega_y(Y))$, but we shall not make any use of it in this paper.

Proof We construct i as a composite

$$i\colon R(*;\Omega_y(Y)) \longrightarrow R(\tilde{\Gamma}(Y);\Omega_y(Y)) \longrightarrow R(\Gamma(Y)) \xrightarrow{R\gamma} R(Y)$$

and j as a composite

$$j\colon R(Y) \xrightarrow{\gamma} R(\Gamma(Y)) \longrightarrow R(\tilde{\Gamma}(Y);\Omega_y(Y)) \longrightarrow R(*;\Omega_y(Y)).$$

The pointed functors $\epsilon_\#$ and ϵ^* are given by pushout and pullback along the map $\epsilon\colon \tilde{\Gamma}(Y) \to *$, respectively. These are weak homotopy functors because the structural section s is always a cofibration, and the map ϵ is a weak equivalence. On an object $(E;r;s)$ of $R(\tilde{\Gamma}(Y);\Omega_y(Y))$ the composite $\epsilon^*\epsilon_\#(E)$ equals

$\bar{\Gamma}(Y)$ ($[_{\bar{\Gamma}(Y)} E$), and there is a natural weak equivalence from the identity on $R(\bar{\Gamma}(Y); \Omega_y(Y))$ to in view of the commutative diagram:

$$\begin{array}{ccc} E & \xrightarrow{\quad , \quad} & \bar{\Gamma}(Y) \quad (\ [_{\bar{\Gamma}(Y)} E) \\ & {\scriptstyle ,}\searrow \qquad \swarrow{\scriptstyle \mathrm{pr}_2 ,} & \\ & [_{\bar{\Gamma}(Y)} E & \end{array}$$

Here the lower left arrow collapses the image of $s\colon \bar{\Gamma}(Y) \ ! \ E$ to a point, while the upper horizontal arrow has components $r\colon E \ ! \ \bar{\Gamma}(Y)$ and the map just mentioned.

The pointed functors $_!$ and $^!$ are given by passage to $\Omega_y(Y)${orbits and pullback along $: \bar{\Gamma}(Y) \ ! \ \Gamma(Y)$, respectively. These are weak homotopy functors because objects of $R(\bar{\Gamma}(Y); \Omega_y(Y))$ are free $\Omega_y(Y)${simplicial sets, and the bundle projection is a Kan bration. Let $(W; r; s)$ be an object of $R(\Gamma(Y))$. The composite value $_!\,{}^!(W) = (\bar{\Gamma}(Y) \ {}_{\Gamma(Y)} W)=\Omega_y(Y)$ is naturally isomorphic to W, so there is a natural isomorphism from the identity on $R(\Gamma(Y))$ to $_!\ {}^!$.

The pointed functor y is given by pushout along $y\colon Y \ ! \ \Gamma(Y)$. It is a weak homotopy functor because y is a co bration. The construction of the pointed functor Ry is a little more complicated, and we will do it in two steps. Let $(W; r; s)$ be an object of $R(\Gamma(Y))$. We rst de ne $y^{\#}(W)$ in $S=Y$ as the homotopy pullback

$$\begin{array}{ccc} y^{\#}(W) & \xrightarrow{\ ,\ } & W \\ \downarrow & & \downarrow{\scriptstyle r} \\ Y & \xrightarrow[\ ,\ ]{y} & \Gamma(Y) \end{array}$$

along y. This de nes a weak homotopy functor $y^{\#}\colon R(\Gamma(Y)) \ ! \ S=Y$, because we take homotopy pullback rather than pullback. By functoriality, $y^{\#}(W)$ contains $y^{\#}(\Gamma(Y))$ as a retract in $S=Y$. We next de ne $Ry\ (W)$ in $R(Y)$ as the pushout

$$\begin{array}{ccc} y^{\#}(\Gamma(Y)) & \xrightarrow{y^{\#}(s)} & y^{\#}(W) \\ \downarrow{\scriptstyle ,} & & \downarrow{\scriptstyle ,} \\ Y & \longrightarrow & Ry\ (W) \end{array}$$

in $S=Y$. This de nes a weak homotopy functor $Ry\ \colon R(\Gamma(Y)) \ ! \ R(Y)$, because the section $y^{\#}(s)$ is a (split) co bration. It is pointed by inspection.

We claim that there is a natural weak equivalence from the identity on $R(Y)$ to $Ry\ y$. Let $(Y';r;s)$ be an object of $R(Y)$ and write $W = y\,(Y') = \Gamma(Y) \cup_Y Y'$. The canonical inclusion $Y' \to W$ and the retraction $r\colon Y' \to Y$ taken together define a map from Y' to the (strict) pullback in the diagram defining $y^{\#}(W)$. Continuing by the canonical map from the pullback to the homotopy pullback defines a natural map $Y' \to y^{\#}(W)$. It is a weak equivalence, in view of the commutative diagram:

$$\begin{array}{ccc} Y' & \longrightarrow & y^{\#}(W) \\ & \searrow \quad \swarrow & \\ & W & \end{array}$$

A diagram chase shows that the composite weak equivalence $Y' \to y^{\#}(W) \to Ry\,(W)$ is a morphism in $R(Y)$. Hence this defines the desired natural weak equivalence.

It is clear by inspection that $\ ,\ _!$, $y^{\#}$ and Ry preserve k–connected maps and cartesian squares, and that $\ ,\ ^!$ and y preserve k–connected maps and cocartesian squares. □

Proposition 8.3 *The functor $i_0\colon S \to R(Y)$ factors up to a natural chain of weak equivalences as the composite*

$$S \xrightarrow{\Omega_y(Y)_+ \wedge (-)} R(\ ;\Omega_y(Y)) \xrightarrow{i} R(Y);$$

where the left hand functor takes T to $\Omega_y(Y)_+ \wedge T$.

The functor $j_0\colon R(Y) \to S$ factors up to a natural chain of weak equivalences as the composite

$$R(Y) \xrightarrow{j} R(\ ;\Omega_y(Y)) \xrightarrow{w} S\ ;$$

where the right hand functor forgets the $\Omega_y(Y)$–action.

Proof Let T be a based simplicial set. We interpret $\Omega_y(Y)_+ \wedge T$ as the pushout of

$$\Omega_y(Y) \qquad \to \Omega_y(Y) \quad T$$

and apply $_!$ to obtain the pushout square:

$$\begin{array}{ccc} \Gamma(Y) & \longrightarrow & \Gamma(Y) \quad T \\ \downarrow & & \downarrow \\ \Gamma(Y) & \longrightarrow & (\Gamma(Y) \quad (\Omega_y(Y)_+ \wedge T))/\Omega_y(Y) \end{array}$$

Using the weak equivalence $y\colon Y \to \Gamma(Y)$, and the inclusion of the canonical base point into $\bar\Gamma(Y)$, we obtain a natural map from the pushout square

$$\begin{array}{ccc} & \longrightarrow & T \\ \downarrow & & \downarrow \\ Y & \longrightarrow & i_0(T) \end{array}$$

to the square above. It is clearly a weak equivalence at the three upper or left hand corners, hence the pushout map

$$i_0(T) = Y \vee_y T \xrightarrow{\simeq} {}_! \, (\Omega_y(Y)_+ \wedge T):$$

is also a natural weak equivalence. The natural weak equivalences $y^{\#}(W) \to W$ and $y^{\#}(W) \to Ry\ (W)$ from the two diagrams de ning Ry , and the factorization $i = Ry \quad {}_!$, provide the remaining chain of natural weak equivalences linking $i_0(T)$ to $i(\Omega_y(Y)_+ \wedge T)$.

Next, let $(Y'; r; s)$ be a retractive simplicial set in $R(Y)$. We evaluate $j = \; {}^! y$ on Y' and obtain a commutative diagram in S :

$$\begin{array}{ccccccc} Y' & \xrightarrow{\simeq} & \Gamma(Y) \cup_Y Y' & \twoheadleftarrow & \bar\Gamma(Y) \; {}_{\Gamma(Y)} (\Gamma(Y) \cup_Y Y') & \xrightarrow{\simeq} & j(Y') \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ Y & \xrightarrow[y]{\simeq} & \Gamma(Y) & \twoheadleftarrow & \bar\Gamma(Y) & \xrightarrow{\simeq} & \end{array}$$

The left and right hand squares have horizontal weak equivalences, while the middle square is cartesian, using again that is a Kan bration. Hence the induced maps of vertical homotopy bers at the canonical base points of Y, $\Gamma(Y)$, $\bar\Gamma(Y)$ and de ne a natural chain of weak equivalences linking $j_0(Y') =$ ho $\mathrm{b}_y(Y' \to Y)$ to ho $\mathrm{b}(j(Y') \to \ )$, and thus to $j(Y')$, as based simplicial sets. □

Lemma 8.4 *Let* $\ : R(X) \to R(Y)$ *be a weak homotopy functor, and let* $\ = j \quad i\colon R(\ ; \Omega_x(X)) \to R(\ ; \Omega_y(Y))$. *If satis es the colimit axiom, then so does .*

Proof By a straightforward inspection of the de nitions, the functors $i = Ry \quad {}_!$ and $j = \quad {}^! \ y$ preserve ltered homotopy colimits up to weak equivalence. □

9 Equivariant derivatives

We now recast the Goodwillie derivative in an equivariant setting.

Let X be a simplicial set, let $\Phi\colon S/X \to S$ be a weak homotopy functor, and let $Y = \Phi(X)$. Consider Φ as a pointed functor $\Phi\colon R(X) \to R(Y)$. We choose base points $x \in X$ and $y \in Y$, and shall first assume that X and Y are connected. Let $H = \Omega_x(X)$ and $G = \Omega_y(Y)$ be the Kan loop groups of $(X; x)$ and $(Y; y)$, respectively.

Recall the functors $i = i(X; x)\colon R(*; H) \to R(X)$ and $j = j(Y; y)\colon R(Y) \to R(*; G)$. Let $\Psi\colon R(*; H) \to R(*; G)$ be the composite functor

$$\Psi\colon R(*; H) \xrightarrow{i} R(X) \xrightarrow{\Phi} R(Y) \xrightarrow{j} R(*; G). \tag{9.1}$$

By proposition 8.1, Ψ is also a pointed weak homotopy functor. (Here pointed means that $\Psi(*) = *$.)

The categories $R(*; H)$ and $R(*; G)$ are in fact *enriched* over the category S_* of based simplicial sets. Given objects U and V in $R(*; H)$, the (based) *simplicial mapping space* $\mathrm{Map}_H(U; V)$ is the based simplicial set with p–simplices the set of morphisms $\Delta^p_+ \wedge U \to V$ in $R(*; H)$. The usual set of morphisms $U \to V$ in $R(*; H)$ can be recovered as the 0–simplices of $\mathrm{Map}_H(U; V)$. Similarly, $\mathrm{Map}_G(-; -)$ is the simplicial mapping space in the category $R(*; G)$. A pointed *simplicial functor* $\Psi\colon R(*; H) \to R(*; G)$ comes equipped with a based map

$$\mathrm{Map}_H(U; V) \longrightarrow \mathrm{Map}_G(\Psi(U); \Psi(V))$$

which on 0–simplices takes a map $f\colon U \to V$ to the usual image $\Psi(f)\colon \Psi(U) \to \Psi(V)$.

Any (pointed) weak homotopy functor Ψ can be promoted to a weakly equivalent (pointed) simplicial weak homotopy functor Ψ', following [14, 3.1]. For a based, free H–simplicial set W, let $W^{\Delta^q} = \mathrm{Map}_H(H_+ \wedge \Delta^q_+; W)$. Then W^{Δ^q} is again a based, free H–simplicial set. The functor $[q] \mapsto \Psi(W^{\Delta^q})$ is a simplicial object in based, free G–simplicial sets. Let

$$\Psi'(W) = \mathrm{diag}([q] \mapsto \Psi(W^{\Delta^q})) = \coprod_{q \geq 0} (\Psi(W^{\Delta^q}) \times \Delta^q)/\sim$$

be the associated diagonal based, free G–simplicial set. Then Ψ' is naturally a simplicial functor. The required map

$$\mathrm{Map}_H(U; V) \longrightarrow \mathrm{Map}_G(\Psi'(U); \Psi'(V))$$

takes each p-simplex $f\colon \Delta^p_+ \wedge U \to V$ to a p-simplex $\Phi(f)\colon \Delta^p_+ \wedge \Phi(U) \to \Phi(V)$. The latter is the simplicial map given in degree q by the map

$$\Phi(f)_q\colon (\Delta^p_q)_+ \wedge \Phi(U^{\Delta^q}) = \bigvee_{\alpha\colon [q]\to[p]} \Phi(U^{\Delta^q}) \longrightarrow \Phi(V^{\Delta^q})$$

that on the wedge summand indexed by $\alpha\colon [q] \to [p]$ is the value of Φ applied to the composite map

$$U^{\Delta^q} \xrightarrow{(\alpha,\mathrm{id})} (\Delta^p_+ \wedge U)^{\Delta^q} \xrightarrow{f^{\Delta^q}} V^{\Delta^q}.$$

Each projection $\Delta^q \to *$ induces a weak equivalence $W \to W^{\Delta^q}$, and thus each q-fold degeneracy map $\Phi(W) \to \Phi(W^{\Delta^q})$ is a weak equivalence. Thus the inclusion of 0-simplices is a natural weak equivalence

$$\Phi \xrightarrow{\simeq} \hat{\Phi} \tag{9.2}$$

of weak homotopy functors $R(*;H) \to R(*;G)$, by the realization lemma. We can therefore replace $\Phi = j \circ \phi \circ i$ by $\hat{\Phi}$ without changing its weak homotopy type.

To each based, free H-simplicial set W we associate its *cone* $CW = W \wedge \Delta^1$, and its *suspension* $\Sigma W = CW \cup_W CW = W \wedge S^1$, where $S^1 = \Delta^1 \cup_{\partial\Delta^1} \Delta^1$. By iteration, $\Sigma^n W = W \wedge S^n$, where $S^n = S^1 \wedge \dots \wedge S^1$ (n copies of S^1).

Using that Φ is a pointed simplicial functor, we obtain a natural based map

$$\sigma\colon \Sigma\Phi(W) \to \Phi(\Sigma W) \tag{9.3}$$

in $R(*;G)$, as follows. The identity map $W \wedge S^1 = \Sigma W$ is left adjoint to a based map

$$S^1 \to \mathrm{Map}_H(W, \Sigma W).$$

Since Φ is pointed simplicial there is a natural based map

$$\mathrm{Map}_H(W, \Sigma W) \longrightarrow \mathrm{Map}_G(\Phi(W), \Phi(\Sigma W)).$$

The composite of these two maps is then right adjoint to the desired map σ.

We consider the sequence of based, free H-simplicial sets $\Sigma^n H_+ = H_+ \wedge S^n$ in $R(*;H)$. Applying Φ we obtain a sequence of based, free G-simplicial sets

$$\partial\Phi_n = \Phi(\Sigma^n H_+). \tag{9.4}$$

The natural map σ in the case of $W = \Sigma^n H_+$ then defines the structure map from $\Sigma\partial\Phi_n$ to $\partial\Phi_{n+1}$ in the free G-spectrum

$$\partial\Phi = \{n \mapsto \partial\Phi_n\}. \tag{9.5}$$

This is the *equivariant Goodwillie derivative* of . By proposition 8.3 and the weak equivalence (9.2), there is a natural chain of weak equivalences linking the underlying based simplicial set of $\partial \ _n$ to $\partial^x_y \ (X)_n$. Similarly the underlying non-equivariant spectrum of ∂ is strictly equivalent to the (non-equivariant) Goodwillie derivative $\partial^x_y \ (X)$. Hence ∂ provides a model for the Goodwillie derivative as a free left G-spectrum.

The simplicial enrichment ensures that the Goodwillie derivative ∂ also admits another simplicial group action, this time by H acting from the right. The multiplication on H defines a map $(\ ^n H_+) \wedge H_+ \to (\ ^n H_+)$, which is left adjoint to a based map

$$H_+ \longrightarrow \mathrm{Map}_H(\ ^n H_+; \ ^n H_+) .$$

Using that $\ : R(\ ; H) \to R(\ ; G)$ is a pointed simplicial functor, we obtain a map of based simplicial sets

$$\mathrm{Map}_H(\ ^n H_+; \ ^n H_+) \longrightarrow \mathrm{Map}_G(\ (\ ^n H_+); \ (\ ^n H_+)) .$$

The composite of these two based maps is then right adjoint to a map

$$(\ ^n H_+) \wedge H_+ \longrightarrow \ (\ ^n H_+);$$

that defines the desired right action of H on $(\ ^n H_+) = \partial \ _n$ in the category of based, free G-simplicial sets.

The structure maps of ∂ are likewise natural with respect to this right H-action, hence ∂ is also a spectrum with right H-action.

Turning to the general case, when X and Y are not necessarily connected, let X be the connected component of the base point x in X and let Y be the connected component of the base point y in Y. Define $\ : R(X \ ) \to R(Y \ )$ as in section 7, and let $\ = j \ \ \ i\colon R(\ ; \Omega_x(X \ )) \to R(\ ; \Omega_y(Y \ ))$. Then we have just seen that $\partial \ \simeq \partial^x_y \ (X \ )$, while $\partial^x_y \ (X \ ) \simeq \partial^x_y \ (X)$ by lemma 7.1.

We summarize this discussion in:

Definition 9.6 Let X be a simplicial set, $\ : S/X \to S$ a weak homotopy functor and $Y = \ (X)$. Let $(X \ ; x)$ and $(Y \ ; y)$ be based, connected components of X and Y, with Kan loop groups $\Omega_x(X \ )$ and $\Omega_y(Y \ )$, respectively. Let

$$= j \ \ \ i\colon R(\ ; \Omega_x(X \ )) \to R(\ ; \Omega_y(Y \ ));$$

with $i = i(X \ ; x)$ and $j = j(Y \ ; y)$, and let

$$(W) = j[q] \mapsto \ (W \ ^q)j$$

be the simplicial enrichment of . The *equivariant Goodwillie derivative* of is the free left $\Omega_y(Y)$–spectrum with right $\Omega_x(X)$–action:

$$\partial \; = \{n \mapsto \partial \;_n = \; (\;^n\Omega_x(X)_+)\}.$$

Its underlying spectrum is strictly equivalent to the (non-equivariant) Goodwillie derivative $\partial_y^x \; (X)$.

10 Equivariant Brown–Whitehead representability

Let H and G be simplicial groups. We say that a functor $\;: R(\;;H) \to R(\;;G)$ is *linear* if it is a pointed, excisive, weak homotopy functor. We show in this section that linear functors that satisfy the colimit axiom are classified by their equivariant Goodwillie derivative. Recall from section 3 that an object W of $R(\;;H)$ is finite if it can be obtained from by attaching finitely many free H–cells.

Proposition 10.1 *Let* $\;: R(\;;H) \to R(\;;G)$ *be a bounded below, linear functor. There is a free left* G*–spectrum*

$$\partial \; = \{n \mapsto \partial \;_n = \; (\;^nH_+)\}$$

with right H*–action, and a natural map*

$$(W)\colon \partial \; \wedge_H W \longrightarrow \{n \mapsto \; (\;^nW)\}$$

of free G*–spectra, which is a meta-stable equivalence for all finite* W *in* $R(\;;H)$*. If satisfies the colimit axiom, then the map is a stable equivalence for all* W *in* $R(\;;H)$*.*

Proof The group action map $H_+ \wedge W \to W$ suspends to a map $\;^nH_+ \wedge W \to \;^nW$, which is left adjoint to a based map

$$W \longrightarrow \mathrm{Map}_H(\;^nH_+;\;^nW)$$

of H–simplicial sets, where H acts on the simplicial mapping space by right multiplication in the domain. Since is pointed simplicial, there is a based map

$$\mathrm{Map}_H(\;^nH_+;\;^nW) \longrightarrow \mathrm{Map}_G(\;(\;^nH_+);\;(\;^nW))$$

of H–simplicial sets. The composite map is right adjoint to a map

$$(\;^nH_+) \wedge_H W \longrightarrow \;(\;^nW) \tag{10.2}$$

of based, free G–simplicial sets. These maps are compatible with the spectrum structure maps for varying n, hence define the natural map (W) of free G–spectra.

To finish the proof, we will use the following lemma:

Lemma 10.3 *Let*

$$\begin{array}{ccc} W_0 & \longrightarrow & W_1 \\ \downarrow & & \downarrow \\ W_2 & \longrightarrow & W_3 \end{array}$$

be a cocartesian square in $R(\ ;H)$. If (W_0), (W_1) and (W_2) are meta-stable equivalences, then so is (W_3).

Proof Applying (10.2) to the given cocartesian square yields a map from the square

$$\begin{array}{ccc} (\ ^nH_+)\wedge_H W_0 & \longrightarrow & (\ ^nH_+)\wedge_H W_1 \\ \downarrow & & \downarrow \\ (\ ^nH_+)\wedge_H W_2 & \longrightarrow & (\ ^nH_+)\wedge_H W_3 \end{array}$$

to the square:

$$\begin{array}{ccc} (\ ^nW_0) & \longrightarrow & (\ ^nW_1) \\ \downarrow & & \downarrow \\ (\ ^nW_2) & \longrightarrow & (\ ^nW_3) \end{array}$$

The rst square is cocartesian with $(n-c)${connected maps for some c, since $^nH_+$ is $(n-1)${connected, ' is bounded below and each W_i is H{free. By homotopy excision it is $(2n-c)${cartesian for some (other) constant c. The second square is cartesian since ' is excisive. By hypothesis the maps in the upper left, upper right and lower left hand corners are $(2n-c)${connected, for some constant c. It follows that the map in the lower right hand corner is also $(2n-c)${connected, for some constant c. Hence (W_3) is a meta-stable equivalence. □

By construction, the map (W) is the identity when $W = H_+$ and when $W =$. Hence it is a strict equivalence whenever W is weakly contractible. It follows that $(\ ^kH_+)$ is a meta-stable equivalence for all k 0, by induction on k, using lemma 10.3 applied to the pushout square:

$$\begin{array}{ccc} ^kH_+ & \longrightarrow & C\ ^kH_+ \\ \downarrow & & \downarrow \\ C\ ^kH_+ & \longrightarrow & ^{k+1}H_+ \end{array}$$

Likewise it follows that $\Phi(W)$ is a meta-stable equivalence for all finite objects W in $R(*;H)$, by induction on the number of free H–cells in W, using lemma 10.3 applied to the pushout square associated to the attachment of a free H–cell. If Φ satisfies the colimit axiom, then this implies that $\Phi(W)$ is a stable equivalence for any based, free H–simplicial set W. This completes the proof of 10.1. □

The following form of the Brown–Whitehead representability theorem is perhaps more familiar, although we shall not use it directly.

Proposition 10.4 *Let $\Phi\colon R(*;H) \to R(*;G)$ be a bounded below, linear functor that satisfies the colimit axiom. There is a natural chain of weak equivalences*

$$\Omega^\infty(\partial\Phi \wedge_H W) \simeq \Phi(W)$$

for all W in $R(*;H)$.

Proof The chain consists of the weak equivalence

$$\Omega^\infty(\partial\Phi \wedge_H W) \xrightarrow{\simeq} \Omega^\infty\{n \mapsto \Phi(\Sigma^n W)\}$$

of proposition 10.1, the weak equivalence

$$\Omega^\infty\{n \mapsto \Phi(\Sigma^n W)\} \xrightarrow{\simeq} \Omega^\infty\{n \mapsto \Phi(\Sigma^n W)\}$$

of (9.2), and the weak equivalence

$$\Phi(W) \xrightarrow{\simeq} \Omega^\infty\{n \mapsto \Phi(\Sigma^n W)\}$$

that follows since Φ is linear. □

11 The chain rule

Let $\Phi\colon R(X) \to R(Y)$ and $\Psi\colon R(Y) \to R(Z)$ be bounded below, excisive functors, with $Y = \Phi(X)$ and $Z = \Psi(Y)$, such that $(X;x)$, $(Y;y)$ and $(Z;z)$ are based connected simplicial sets. We form $\tilde\Phi = j \circ \Phi \circ i$ and $\tilde\Psi = j \circ \Psi \circ i$, as in (9.1).

$$\begin{array}{ccccc}
R(X) & \xrightarrow{\Phi} & R(Y) & \xrightarrow{\Psi} & R(Z) \\
\uparrow{\scriptstyle i} & & {\scriptstyle j}\downarrow\uparrow{\scriptstyle i} & & \downarrow{\scriptstyle j} \\
R(*;\Omega_x(X)) & \xrightarrow{\tilde\Phi} & R(*;\Omega_y(Y)) & \xrightarrow{\tilde\Psi} & R(*;\Omega_z(Z))
\end{array}$$

We let $\Gamma = \Psi \circ \Phi$ and $\tilde\Gamma = j \circ \Psi \circ \Phi \circ i$, so that $\partial\tilde\Gamma \simeq \partial^x_z(\Gamma)(X) = \partial^x_z(\Psi \circ \Phi)(X)$ is the equivariant Goodwillie derivative of the composite functor.

Proposition 11.1 *Suppose that Ψ satisfies the colimit axiom. Then there is a natural chain of stable equivalences of free left $\Omega_z(Z)$–spectra with right $\Omega_x(X)$–action*

$$\partial_z^x(\Psi\circ\Phi)(X) \simeq \partial_z^y\Psi(Y) \wedge_{\Omega_y(Y)} \partial_y^x\Phi(X).$$

Proof By proposition 8.1 there is a natural weak equivalence $\Psi\circ\Phi = j^*\Psi\Phi i_* \simeq j^*\Psi i_* j^*\Phi i_* = \Psi'\circ\Phi'$. Hence there is also a natural weak equivalence of simplicial functors $\Psi\Phi \simeq \Psi'\Phi'$, and a strict equivalence of equivariant Goodwillie derivatives $\partial\Psi\Phi \simeq \partial(\Psi'\Phi')$.

For brevity, let $H = \Omega_x(X)$ and $G = \Omega_y(Y)$.

The smash product of spectra is so constructed that in order to produce a stable equivalence of spectra $\partial\Psi \wedge_G \partial\Phi \longrightarrow \partial(\Psi\Phi)$ it suffices to define a stable equivalence of bi-spectra

$$\{m,n \mapsto \partial\Psi_m \wedge_G \partial\Phi_n\} \longrightarrow \{m,n \mapsto \partial(\Psi\Phi)_{m,n}\}$$

where

$$\partial\Psi_m \wedge_G \partial\Phi_n = \Psi(\Sigma^m G_+) \wedge_G \Phi(\Sigma^n H_+)$$
$$\partial(\Psi\Phi)_{m,n} = (\Psi\Phi)(\Sigma^m\Sigma^n H_+).$$

Such a stable equivalence is provided by the following composite:

$$\Psi(\Sigma^m G_+) \wedge_G \Phi(\Sigma^n H_+) \xrightarrow{a} \Psi(\Sigma^m\Phi(\Sigma^n H_+)) \xrightarrow{b} (\Psi\Phi)(\Sigma^m\Sigma^n H_+). \tag{11.2}$$

Here the first map a is a case of (10.2), which induces a stable equivalence (as $m\to\infty$) by proposition 10.1, in view of lemma 8.4. The second map b is applied to the map

$$\Sigma^m\Phi(\Sigma^n H_+) \longrightarrow \Phi(\Sigma^m\Sigma^n H_+),$$

which is (10.2) applied to the case $W = \Sigma^m H_+$. By proposition 10.1 again, this is a meta-stable equivalence (as $n\to\infty$). Since Ψ is bounded below, the second map b is also a meta-stable equivalence. ☐

Theorem 11.3 *Let $\Phi\colon S/X \to S$ and $\Psi\colon S/Y \to S$ be bounded below, stably excisive functors, with $Y = \Phi(X)$ and $Z = \Psi(Y)$, and suppose that Ψ satisfies the colimit axiom. Write $Y = \coprod_{\beta\in B} Y_\beta$ with each Y_β connected, and choose base points $x\in X$, $y_\beta\in Y_\beta$ and $z\in Z$. Let X_x be the connected component of x in X, and let Z_y be the connected component of z in Z. Then $\Psi\circ\Phi$ is also bounded below and stably excisive, and there is a natural chain of stable equivalences*

$$\partial_z^x(\Psi\circ\Phi)(X) \simeq \bigvee_{\beta\in B} \partial_z^{y_\beta}\Psi(Y) \wedge_{\Omega_{y_\beta}(Y_\beta)} \partial_{y_\beta}^x\Phi(X)$$

of free left $\Omega_z(Z_y)$–spectra with right $\Omega_x(X_x)$–action.

Proof The composite Ψ is bounded below and stably excisive by proposition 3.2. By propositions 4.4 and 6.2 we can replace and Ψ by the bounded below, excisive functors P^0 and $P^0\Psi$, respectively, without changing the derivatives of , Ψ and Ψ by more than a stable equivalence, and such that $P^0\Psi$ satisfies the colimit axiom. Hence we can assume from the beginning that and Ψ are excisive. By lemma 7.1 and proposition 7.3 there are stable equivalences

$$\partial_z^x(\Psi\ )(X) \simeq \partial_z^x(\Psi_y\ )(X\ ) \simeq \bigvee_{\in B} \partial_z^x(\Psi_y\ )(X\ ).$$

By proposition 11.1 the summand indexed by is stably equivalent to

$$\partial_z^y(\Psi_y)(Y\ ) \wedge_{\Omega_y(Y\ )} \partial_y^x(\ )(X\ ).$$

By lemma 7.1 each such term can be rewritten as $\partial_z^y \Psi(Y) \wedge_{\Omega_y(Y\ )} \partial_y^x (X)$. ☐

Theorem 11.4 *Let $E, F\colon \mathcal{S} \to \mathcal{S}$ be bounded below, stably excisive functors, with $Y = F(X)$ and $Z = E(Y)$, and suppose that E satisfies the colimit axiom. Write $Y = \coprod_{\in B} Y$ with each Y connected, and choose base points $x \in X$, $y \in Y$ and $z \in Z$. Let X be the connected component of x in X, and let Z_y be the connected component of z in Z. Then the composite $E\ F$ is bounded below and stably excisive, and there is a natural chain of stable equivalences*

$$\partial_z^x(E\ F)(X) \simeq \bigvee_{\in B} \partial_z^y E(Y) \wedge_{\Omega_y(Y\ )} \partial_y^x F(X)$$

of free left $\Omega_z(Z_y)$–spectra with right $\Omega_x(X\ )$–action.

Proof This is the special case of theorem 11.3 when $= F\ u$ and $\Psi = E\ u$. ☐

12 Topological spaces

Let $\mathcal{U}$ be the category of compactly generated topological spaces. The geometric realization functor $|-|\colon \mathcal{S} \to \mathcal{U}$ is left adjoint to the total singular simplicial set functor Sing: $\mathcal{U} \to \mathcal{S}$. There is a natural weak equivalence $|\mathrm{Sing}(X)| \to X$.

Given a weak homotopy functor $f\colon \mathcal{U} \to \mathcal{U}$ we get a weak homotopy functor $F\colon \mathcal{S} \to \mathcal{S}$ by setting $F(X) = \mathrm{Sing}(f(|X|))$. Let $Y = f(X)$, $Y^1 = \mathrm{Sing}(f(X))$ and $Y^2 = F(\mathrm{Sing}(X))$. There are natural weak equivalences $|Y^2| \to |Y^1| \to Y$. (The superscripts are simply labels, and do not mean powers or skeleta.)

Choose base points $x \in X$ and $y^2 \in Y^2$. Let $x \in \mathrm{Sing}(X)$, $y^1 \in Y^1$ and $y \in Y$ denote the corresponding base points, via the maps just mentioned.

The Goodwillie derivative of f at X with respect to the base points $x \in X$ and $y \in Y$ is the spectrum

$$\partial^x_y f(X) = \{n \mapsto \mathrm{hofib}_y(f(X \vee_x S^n) \to Y)\}. \tag{12.1}$$

It receives a natural strict equivalence from the spectrum

$$|\partial^x_{y^2} F(\mathrm{Sing}(X))| = \{n \mapsto |\mathrm{hofib}_{y^2}(F(\mathrm{Sing}\, X \vee_x S^n) \to Y^2|\}.$$

The latter spectrum has a free left action by $|\Omega_{y^2}(Y^2)|$, and a right action by $|\Omega_x(\mathrm{Sing}(X))|$, where X is the path component of x in X and Y^2 is the path component of y^2 in Y^2.

It will be convenient to forgo the condition that the left action is free. We thus apply the forgetful functor $Sp(G) \to Sp^G$ from free G–spectra to spectra with G–action, for $G = \Omega_{y^2}(Y^2)$. Hence we will consider $\partial^x_y f(X)$ up to strict equivalence as a spectrum with left $|\Omega_{y^2}(Y^2)|$–action and right $|\Omega_x(\mathrm{Sing}(X))|$–action. We emphasize that our weak equivalences of spectra with G–action are simply G–equivariant maps that are stable equivalences, so that no fixed-point information is retained.

We can always recover a strictly equivalent free G–spectrum by smashing with EG_+, where EG is a free, contractible G–space. For example we may take $EG = |\Gamma(Y^2)|$ as the geometric realization of the principal bundle introduced in section 8. Thus if L is a spectrum with right G–action and M is a free left G–spectrum, thought of as a spectrum with left G–action, the stable homotopy type of the G–orbit spectrum $L \wedge_G M$ can be recovered as the homotopy orbit spectrum

$$L \wedge_G (EG_+ \wedge M) = L \wedge_{hG} M. \tag{12.2}$$

We now switch to this notation.

Suppose that $e, f \colon \mathcal{U} \to \mathcal{U}$ are bounded below, stably excisive functors and that e satisfies the colimit axiom. Define $F(X) = \mathrm{Sing}(f(|X|))$ and $E(Y) = \mathrm{Sing}(e(|Y|))$. Let $Y \simeq f(X)$ and $Z = e(Y)$. Let $Y = \coprod_{\beta \in B} Y_\beta$, $Y^1 = \coprod_{\beta \in B} Y^1_\beta$ and $Y^2 = \coprod_{\beta \in B} Y^2_\beta$ be the decompositions into path components. (These simplicial sets are weakly equivalent, so the indexing sets B are equal.) Choose base points $x \in X$, $y^2 \in Y^2$ and $z \in E(Y^2) = (E \circ F)(\mathrm{Sing}(X))$. Let $x \in \mathrm{Sing}(X)$, $y^1 \in Y^1$, $y \in Y$ and $z \in Z$ denote their images under the natural weak equivalences.

The chain rule 11.4 for $E \circ F$ at $\mathrm{Sing}(X)$ then asserts that there is a stable equivalence

$$|\partial_z^x(E \circ F)(\mathrm{Sing}(X))| \simeq \bigvee_{\alpha \in B} |\partial_z^{y^2_\alpha} E(Y^2_\alpha)| \wedge_{h|\Omega_{y^2_\alpha}(Y^2_\alpha)|} |\partial^x_{y^2_\alpha} F(\mathrm{Sing}(X))| .$$

(Geometric realization commutes with homotopy orbits since bisimplicial sets can be realized in two stages, or at once.) There are natural weak equivalences

$$\begin{aligned}
|\partial^x_{y^2_\alpha} F(\mathrm{Sing}(X))| &\xrightarrow{\simeq} \partial^x_{y_\alpha} f(X)\\
|\partial_z^{y^2_\alpha} E(Y^2_\alpha)| &\xrightarrow{\simeq} \partial_z^{y^2_\alpha} e(Y^2_\alpha) \xrightarrow{\simeq} \partial_z^{y_\alpha} e(Y)\\
|\partial_z^x(E \circ F)(\mathrm{Sing}(X))| &\xrightarrow{\simeq} \partial_z^x(e \circ f)(X)\\
|\Omega_{y^2_\alpha}(Y^2_\alpha)| &\xrightarrow{\simeq} |\Omega_{y^1_\alpha}(Y^1_\alpha)| .
\end{aligned}$$

Hence we can summarize:

Theorem 12.3 *Let $e, f \colon \mathcal{U} \to \mathcal{U}$ be bounded below, stably excisive functors, with $Y = f(X)$ and $Z = e(Y)$, and suppose that e satisfies the colimit axiom. Let $Y = \coprod_{\alpha \in B} Y_\alpha$ with each Y_α path connected, and choose base points $x \in X$, $y_\alpha \in Y_\alpha$ and $z \in Z$. Let X be the path component of x in X, and let Z_y be the path component of z in Z. Also set $\Omega_{y_\alpha}(Y_\alpha) = |\Omega_{y_\alpha}(\mathrm{Sing}(Y_\alpha))|$. Then the composite $e \circ f$ is bounded below and stably excisive, and there is a stable equivalence*

$$\partial_z^x(e \circ f)(X) \simeq \bigvee_{\alpha \in B} \partial_z^{y_\alpha} e(Y) \wedge_{h\Omega_{y_\alpha}(Y_\alpha)} \partial^x_{y_\alpha} f(X)$$

of spectra with left $\Omega_z(Z_y)$–action and right $\Omega_x(X)$–action.

Remark 12.4 To be precise, this formula needs to be interpreted in line with the weak equivalences above. In particular, it is not $\Omega_{y_\alpha}(Y_\alpha) = |\Omega_{y^1_\alpha}(Y^1_\alpha)|$ that really acts, but the naturally weakly equivalent topological group $|\Omega_{y^2_\alpha}(Y^2_\alpha)|$. And the action is not really on the spectra $\partial_z^{y_\alpha} e(Y)$ and $\partial^x_{y_\alpha} f(X)$, but on the naturally weakly equivalent spectra $|\partial_z^{y^2_\alpha} E(Y^2_\alpha)|$ and $|\partial^x_{y^2_\alpha} F(\mathrm{Sing}(X))|$. However, all the constructions involved are weak homotopy invariant, so none of these adjustments have any homotopy-theoretic significance.

13 Examples

Example 13.1 Let $\mathrm{id}\colon \mathcal{U} \to \mathcal{U}$ be the identity functor. It is clearly bounded below, and is stably excisive by homotopy excision. For a path connected space X, choose non-degenerate base points $x \in X$ and $y \in \mathrm{id}(X) = X$. Let $P_y(X) = \{\gamma\colon I \to X \mid \gamma(0) = y\}$ and $P_y^x(X) = \{\gamma\colon I \to X \mid \gamma(0) = y, \gamma(1) = x\}$. There is a natural map

$$\begin{aligned}\mathrm{hofib}_y(X \vee_x S^n \to X) &= P_y(X) \cup_{P_y^x(X)} (P_y^x(X) \times S^n)\\ &\longrightarrow P_y^x(X)_+ \wedge S^n = \Sigma^n P_y^x(X)_+\end{aligned}$$

which is a homotopy equivalence. Hence there is a stable equivalence

$$\partial_y^x(\mathrm{id})(X) \simeq \Sigma^\infty P_y^x(X)_+ ,$$

with the natural left $\Omega_y(X)$–action and right $\Omega_x(X)$–action given by composition of paths.

Example 13.2 Let K be a finite CW complex and consider the mapping space functor $f(X) = X^K = \mathrm{Map}(K, X)$. Then f satisfies $E_1(d, \kappa)$ and $E_2(2d+1, \kappa)$ (by homotopy excision) for all κ, where $d = \dim(K)$. Choose a non-degenerate base point $x \in X$, and base $Y = X^K$ at the constant map y taking K to $\{x\}$. There are homotopy equivalences

$$\mathrm{hofib}_y((X \vee_x S^n)^K \to X^K) \simeq \mathrm{hofib}_x(X \vee_x S^n \to X)^K \simeq \mathrm{Map}(K, \Sigma^n \Omega_x(X)_+)$$

and thus a stable equivalence

$$\partial_y^x(X^K) \simeq \mathrm{Map}(K, \Sigma^\infty \Omega_x(X)_+).$$

When X^K is path connected, the group $\Omega_y(X^K) = \mathrm{Map}(K, \Omega_x(X))$ acts from the left by pointwise multiplication, while $\Omega_x(X)$ acts uniformly from the right.

Example 13.3 Let $e(Y) = Q(Y_+) = \mathrm{colim}_n\, \Omega^n(\Sigma^n Y_+)$ be the (unreduced) stable homotopy functor. This functor is bounded below and excisive, and satisfies the colimit axiom. Choose a base point $y \in Y$, and take any base point z of $Z = Q(Y_+)$. The pushout of $S^n \leftarrow (Y \vee_y S^n)_+ \to Y_+$ is $*$, so there is a natural cartesian square

$$\begin{array}{ccc} Q((Y \vee_y S^n)_+) & \longrightarrow & Q(S^n) \\ \downarrow & & \downarrow \\ Q(Y_+) & \longrightarrow & Q(*) \end{array}$$

and a natural weak equivalence $\partial^y_z Q(Y_+)_n \to Q(S^n)$. Thus there is a stable equivalence

$$\partial^y_z Q(Y_+) \simeq S,$$

where $S = \{n \mapsto Q(S^n)\}$ is the *sphere spectrum*. The left action of $\Omega_z(Q(Y_+))$ on $\partial^y_z Q(Y_+)_n$ pulls back from in the cartesian square above. Likewise the right action of $\Omega_y(Y)$ pulls back from the trivial action on $Q(S^n)$. Hence both of these actions are trivial, up to homotopy.

Example 13.4 Let $(e \circ f)(X) = Q(X^K_+)$ be the composite functor. By the chain rule 12.3, its derivative at X is

$$\begin{aligned}\partial^x_z Q(X^K_+) &\simeq \partial^y_z Q(Y_+) \wedge_{\Omega_y(Y)} \partial^x_y (X^K)\\ &\simeq S \wedge_{\mathrm{Map}(K,\Omega_x(X))} \mathrm{Map}(K, \Sigma^\infty \Omega_x(X)_+)\\ &\simeq \mathrm{Map}(K, \Sigma^\infty \Omega_x(X)_+)_{h\,\mathrm{Map}(K,\Omega_x(X))}.\end{aligned}$$

This assumes that X^K is path connected. The derivative of this functor $X \mapsto Q(X^K_+)$ was first computed as the spectrum of stable sections in a suitable Serre fibration in [6, section 2]. In the paper [10] the first author shows that the two descriptions of this derivative are indeed equivalent.

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