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Extended Bloch group and the Cheeger–Chern–Simons class

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Abstract

We define an extended Bloch group and show it is naturally isomorphic to $H_3(\mathrm{PSL}(2,\mathbb{C})^\delta;\mathbb{Z})$. Using the Rogers dilogarithm function this leads to an exact simplicial formula for the universal Cheeger–Chern–Simons class on this homology group. It also leads to an independent proof of the analytic relationship between volume and Chern–Simons invariant of hyperbolic 3–manifolds conjectured in [16] and proved in [24], as well as effective formulae for the Chern–Simons invariant of a hyperbolic 3–manifold.

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1 Introduction

There are several variations of the definition of the Bloch group in the literature; by [7] they differ at most by torsion and they agree with each other for algebraically closed fields. In this paper we shall use the following.

Definition 1.1 Let k be a field. The *pre-Bloch group* $\mathcal{P}(k)$ is the quotient of the free $\mathbb{Z}$–module $\mathbb{Z}(k-\{0,1\})$ by all instances of the following relation:

$$[x]-[y]+[\frac{y}{x}]-[\frac{1-x^{-1}}{1-y^{-1}}]+[\frac{1-x}{1-y}]=0, \tag{1}$$

This relation is usually called the *five term relation.* The *Bloch group* $\mathcal{B}(k)$ is the kernel of the map

$$\mathcal{P}(k)\to k^*\wedge_{\mathbb{Z}}k^*, \quad [z]\mapsto 2\big(z\wedge(1-z)\big).$$

(In [15] the additional relations

$$[x]=[1-\frac{1}{x}]=[\frac{1}{1-x}]=-[\frac{1}{x}]=-[\frac{x-1}{x}]=-[1-x]$$

were used. These follow from the five term relation when k is algebraically closed, as shown by Dupont and Sah [7]. Dupont and Sah use a different five term relation but it is conjugate to the one used here by $z\mapsto\frac{1}{z}$.)

There is an exact sequence due to Bloch and Wigner:

$$0\to\mathbb{Q}/\mathbb{Z}\to H_3(\mathrm{PSL}(2,\mathbb{C})^\delta;\mathbb{Z})\to\mathcal{B}(\mathbb{C})\to 0.$$

The superscript δ means "with discrete topology." We will omit it from now on.

$\mathcal{B}(\mathbb{C})$ is known to be uniquely divisible, so it has canonically the structure of a $\mathbb{Q}$–vector space (Suslin [21]). It's $\mathbb{Q}$–dimension is infinite and conjectured to be countable (the "Rigidity Conjecture," equivalent to the conjecture that $\mathcal{B}(\mathbb{C})=\mathcal{B}(\overline{\mathbb{Q}})$, where $\overline{\mathbb{Q}}$ is the field of algebraic numbers). In particular, the $\mathbb{Q}/\mathbb{Z}$ in the Bloch–Wigner exact sequence is precisely the torsion of $H_3(\mathrm{PSL}(2,\mathbb{C});\mathbb{Z})$, so any finite torsion subgroup is cyclic.

In the present paper we define an *extended Bloch group* $\widehat{\mathcal{B}}(\mathbb{C})$ by replacing $\mathbb{C}-\{0,1\}$ in the definition of $\mathcal{B}(\mathbb{C})$ by an abelian cover and appropriately lifting the five term relation (1). Our main results are that we can lift the Bloch–Wigner map $H_3(\mathrm{PSL}(2,\mathbb{C});\mathbb{Z})\to\mathcal{B}(\mathbb{C})$ to an isomorphism

$$\lambda\colon\ H_3(\mathrm{PSL}(2,\mathbb{C});\mathbb{Z})\xrightarrow{\cong}\widehat{\mathcal{B}}(\mathbb{C})$$

Moreover, the "Roger's dilogarithm function" (see below) gives a natural map

$$R\colon \widehat{\mathcal{B}}(\mathbb{C}) \to \mathbb{C}/\pi^2\mathbb{Z}.$$

We show that the composition

$$R\circ\lambda\colon H_3(\mathrm{PSL}(2,\mathbb{C});\mathbb{Z}) \to \mathbb{C}/\pi^2\mathbb{Z}$$

is the Cheeger–Chern–Simons class (cf [3]), so it can also be described as $i(\mathrm{vol}+i\,\mathrm{cs})$, where cs is the universal Chern–Simons class. It has been a long-standing problem to provide such a computation of the Chern–Simons class. Dupont in [5] gave an answer modulo $\pi^2\mathbb{Q}$ and our computation is a natural lift of his.

We show that any complete hyperbolic 3–manifold M of finite volume has a natural "fundamental class" in $H_3(\mathrm{PSL}(2,\mathbb{C});\mathbb{Z})$. For compact M the existence of this class is easy and well known: $M=\mathbb{H}^3/\Gamma$ is a $K(\Gamma,1)$–space, so the inclusion $\Gamma\to\mathrm{Isom}^+(\mathbb{H}^3)=\mathrm{PSL}(2,\mathbb{C})$ induces $H_3(M)=H_3(\Gamma)\to H_3(\mathrm{PSL}(2,\mathbb{C}))$, and the class in question is the image of the fundamental class $[M]\in H_3(M)$. For non-compact M the existence is shown in Section 14. The Cheeger–Chern–Simons class applied to this class gives $i(\mathrm{vol}(M)+i\,\mathrm{cs}(M))$ where $\mathrm{vol}(M)$ is hyperbolic volume and $\mathrm{cs}(M)$ is the Chern–Simons invariant. This Chern–Simons invariant is defined modulo $\pi^2\mathbb{Z}$ and for compact M it arises here as the Chern–Simons invariant for the flat $PSL(2,\mathbb{C})$–bundle over M. According to J. Dupont, this is the same as the Chern–Simons invariant for the Riemannian connection over M (private communication; this is proved modulo 6–torsion in [6] and our results also provide confirmation—see [4] for discussion). However, the Chern–Simons invariant for the Riemannian connection carries slightly more information, since it is defined modulo $2\pi^2$ rather than modulo π^2. For non-compact M we show that the Chern–Simons invariant is the one defined by Meyerhoff [11], which is naturally only defined modulo π^2.

This fundamental class of M in $H_3(\mathrm{PSL}(2,\mathbb{C});\mathbb{Z})$ determines an element $\hat\beta(M)\in\widehat{\mathcal{B}}(\mathbb{C})$. Our results describe $\hat\beta(M)$ directly in terms of an ideal triangulation of M, as a lift of the Bloch invariant $\beta(M)\in\mathcal{B}(\mathbb{C})$ defined in [15]. We need a "true" ideal triangulation, which is more restrictive than the "degree 1" ideal triangulations that sufficed in [15]. The ideal triangulations resulting from Dehn filling that are used by the programs Snappea [22] and Snap [10] (see also [4]) are not true, but we describe $\hat\beta(M)$ in terms of these "Dehn filling triangulations" in Theorems 14.5 and 14.7, and deduce practical simplicial formulae for Chern–Simons invariants of hyperbolic 3–manifolds.

The formula of Theorem 14.5 exhibits directly the analytic relationship, conjectured in [16] and proved in [24], between volume and Chern–Simons invariant of hyperbolic manifolds, and gives an independent proof of it. A similar formula for Chern–Simons invariant was derived from [16, 24] in [13], but that formula was only accurate up to an unknown constant on any given Dehn surgery space (the constant was conjectured to be a multiple of $\pi^2/6$, and our results now confirm this). Snappea and versions of Snap prior to 1.10.2 use versions of that formula, and use a bootstrapping procedure to discover the constant and obtain the Chern–Simons invariant for many manifolds. Thanks to Oliver Goodman, Snap now implements the formula of Theorem 14.7 and can thus compute the Chern–Simons invariant for any hyperbolic manifold.

More generally, any flat $\mathrm{PSL}(2,\mathbb{C})$ bundle over a closed oriented 3–manifold M determines a class in $H_3(\mathrm{PSL}(2,\mathbb{C});\mathbb{Z})$ and our results give a simplicial computation of this class as an element of $\widehat{\mathcal{B}}(\mathbb{C})$.

The main results of this paper were announced in [14] and partial proofs were in the preliminary preprint [17]. This paper corrects the tentative statement in [14] (also in [17]) that our map λ is not an isomorphism but has a cyclic kernel of order 2. There is a related error in Section 6B of [4]: it is wrongly stated that the Cheeger–Chern–Simons class takes values in $\mathbb{C}/2\pi^2\mathbb{Z}$ rather than $\mathbb{C}/\pi^2\mathbb{Z}$ (and therefore that the fundamental class in $H_3(\mathrm{PSL}(2,\mathbb{C});\mathbb{Z})$ of a cusped hyperbolic 3–manifold has a mod 2 ambiguity).

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2 The extended Bloch group

We shall need a $\mathbb{Z}\times\mathbb{Z}$ cover $\widehat{\mathbb{C}}$ of $\mathbb{C}-\{0,1\}$ which can be constructed as follows. Let P be $\mathbb{C}-\{0,1\}$ split along the rays $(-\infty,0)$ and $(1,\infty)$. Thus each real number r outside the interval $[0,1]$ occurs twice in P, once in the upper half plane of $\mathbb{C}$ and once in the lower half plane of $\mathbb{C}$. We denote these two occurrences of r by $r+0i$ and $r-0i$. We construct $\widehat{\mathbb{C}}$ as an identification space from $P\times\mathbb{Z}\times\mathbb{Z}$ by identifying

$$\begin{aligned}(x+0i,p,q)&\sim(x-0i,p+2,q)\quad\text{for each } x\in(-\infty,0)\\(x+0i,p,q)&\sim(x-0i,p,q+2)\quad\text{for each } x\in(1,\infty).\end{aligned}$$

We will denote the equivalence class of (z, p, q) by $(z; p, q)$. $\widehat{\mathbb{C}}$ has four components:

$$\widehat{\mathbb{C}} = X_{00} \cup X_{01} \cup X_{10} \cup X_{11}$$

where $X_{\epsilon_0\epsilon_1}$ is the set of $(z; p, q) \in \widehat{\mathbb{C}}$ with $p \equiv \epsilon_0$ and $q \equiv \epsilon_1 \pmod 2$.

We may think of X_{00} as the Riemann surface for the multivalued function

$$\mathbb{C} - \{0, 1\} \to \mathbb{C}^2, \qquad z \mapsto \big(\log z, -\log(1-z)\big).$$

Taking the branch $(\log z + 2p\pi i, -\log(1-z) + 2q\pi i)$ of this function on the portion $P \times \{(2p, 2q)\}$ of X_{00} for each $p, q \in \mathbb{Z}$ defines an analytic function from X_{00} to $\mathbb{C}^2$. In the same way, we may think of $\widehat{\mathbb{C}}$ as the Riemann surface for the multivalued function $(\log z + p\pi i, -\log(1-z) + q\pi i)$, $p, q \in \mathbb{Z}$, on $\mathbb{C} - \{0, 1\}$.

Consider the set

$$\mathrm{FT} := \left\{ \left(x, y, \frac{y}{x}, \frac{1 - x^{-1}}{1 - y^{-1}}, \frac{1-x}{1-y} \right) : x \neq y, x, y \in \mathbb{C} - \{0, 1\} \right\} \subset (\mathbb{C} - \{0,1\})^5$$

of 5–tuples involved in the five term relation (1). An elementary computation shows:

Lemma 2.1 *The subset FT^+ of $(x_0, \dots, x_4) \in \mathrm{FT}$ with each x_i in the upper half plane of $\mathbb{C}$ is the set of elements of FT for which y is in the upper half plane of $\mathbb{C}$ and x is inside the triangle with vertices $0, 1, y$. Thus FT^+ is connected (even contractible).* □

Definition 2.2 Let $V \subset (\mathbb{Z} \times \mathbb{Z})^5$ be the subspace

$$\begin{aligned} V := \{ \big((p_0, q_0), (p_1, q_1), (p_1 - p_0, q_2), (p_1 - p_0 + q_1 - q_0, q_2 - q_1), \\ (q_1 - q_0, q_2 - q_1 - p_0) \big) : p_0, p_1, q_0, q_1, q_2 \in \mathbb{Z} \}. \end{aligned}$$

Let $\widehat{\mathrm{FT}}_0$ denote the unique component of the inverse image of FT in $\widehat{\mathbb{C}}^5$ which includes the points $\big((x_0; 0, 0), \dots, (x_4; 0, 0)\big)$ with $(x_0, \dots, x_4) \in \mathrm{FT}^+$, and define

$$\widehat{\mathrm{FT}} := \widehat{\mathrm{FT}}_0 + V = \{ \mathbf{x} + \mathbf{v} : \mathbf{x} \in \widehat{\mathrm{FT}}_0 \text{ and } \mathbf{v} \in V \},$$

where we are using addition to denote the action of $(\mathbb{Z} \times \mathbb{Z})^5$ by covering transformations on $\widehat{\mathbb{C}}^5$. (Although we do not need it, one can show that the action of $2V$ takes $\widehat{\mathrm{FT}}_0$ to itself, so $\widehat{\mathrm{FT}}$ has 2^5 components, determined by the parities of p_0, p_1, q_0, q_1, q_2.)

Define $\widehat{\mathcal{P}}(\mathbb{C})$ as the free $\mathbb{Z}$–module on $\widehat{\mathbb{C}}$ factored by all instances of the relations:

$$\sum_{i=0}^{4}(-1)^i(x_i;p_i,q_i)=0 \quad \text{with } \big((x_0;p_0,q_0),\dots,(x_4;p_4,q_4)\big)\in\widehat{\mathrm{FT}} \tag{2}$$

and

$$(x;p,q)+(x;p',q')=(x;p,q')+(x;p',q) \quad \text{with } p,q,p',q'\in\mathbb{Z} \tag{3}$$

We shall denote the class of $(z;p,q)$ in $\widehat{\mathcal{P}}(\mathbb{C})$ by $[z;p,q]$

We call relation (2) the *lifted five term relation.* We shall see that its precise form arises naturally in several contexts. In particular, we give it a geometric interpretation in Section 3.

We call relation (3) the *transfer relation.* It is almost a consequence of the lifted five term relation, since we shall see that the effect of omitting it would be to replace $\widehat{\mathcal{P}}(\mathbb{C})$ by $\widehat{\mathcal{P}}(\mathbb{C})\oplus\mathbb{Z}/2$, with $\mathbb{Z}/2$ generated by an element $\kappa := [x;1,1]+[x;0,0]-[x;1,0]-[x;0,1]$ which is independent of x.

Lemma 2.3 *There is a well-defined homomorphism*

$$\nu\colon\ \widehat{\mathcal{P}}(\mathbb{C})\to\mathbb{C}\wedge_{\mathbb{Z}}\mathbb{C}$$

defined on generators by $[z;p,q]\mapsto(\log z+p\pi i)\wedge(-\log(1-z)+q\pi i)$.

Proof We must verify that ν vanishes on the relations that define $\widehat{\mathcal{P}}(\mathbb{C})$. This is trivial for the transfer relation (3). We shall show that the lifted five term relation is the most general lift of the five term relation (1) for which ν vanishes. If one applies ν to an element $\sum_{i=0}^4(-1)^i[x_i;p_i,q_i]$ with $(x_0,\dots,x_4)=(x,y,\dots)\in\mathrm{FT}^+$ one obtains after simplification:

$$\begin{aligned}\big((q_0-p_2-q_2+p_3+q_3)\log x+(p_0-q_3+q_4)\log(1-x)+(-q_1+q_2-q_3)\log y+\\ +(-p_1+p_3+q_3-p_4-q_4)\log(1-y)+(p_2-p_3+p_4)\log(x-y)\big)\wedge\pi i.\end{aligned}$$

An elementary linear algebra computation shows that this vanishes identically if and only if $p_2=p_1-p_0$, $p_3=p_1-p_0+q_1-q_0$, $q_3=q_2-q_1$, $p_4=q_1-q_0$, and $q_4=q_2-q_1-p_0$, as in the lifted five term relation. The vanishing of ν for the general lifted five term relation now follows by analytic continuation. ☐

Definition 2.4 Define $\widehat{\mathcal{B}}(\mathbb{C})$ as the kernel of $\nu\colon\ \widehat{\mathcal{P}}(\mathbb{C})\to\mathbb{C}\wedge\mathbb{C}$.

Define

$$R(z;p,q)=\mathcal{R}(z)+\frac{\pi i}{2}(p\log(1-z)+q\log z)-\frac{\pi^2}{6}$$

where $\mathcal{R}$ is the Rogers dilogarithm function

$$\mathcal{R}(z)=\frac{1}{2}\log(z)\log(1-z)-\int_0^z\frac{\log(1-t)}{t}dt.$$

Then we have:

Proposition 2.5 *R gives a well defined map $R\colon\widehat{\mathbb{C}}\to\mathbb{C}/\pi^2\mathbb{Z}$. The relations which define $\widehat{\mathcal{P}}(\mathbb{C})$ are functional equations for R modulo π^2 (the lifted five term relation is in fact the most general lift of the five term relation (1) with this property). Thus R also gives a homomorphism $R\colon\widehat{\mathcal{P}}(\mathbb{C})\to\mathbb{C}/\pi^2\mathbb{Z}$.*

Proof If one follows a closed path from z that goes anti-clockwise around the origin it is easily verified that $R(z;p,q)$ is replaced by $R(z;p,q)+\pi i\log(1-z)-q\pi^2=R(z;p+2,q)-q\pi^2$. Similarly, following a closed path clockwise around 1 replaces $R(z;p,q)$ by $R(z;p,q+2)+p\pi^2$. Thus R modulo π^2 is well defined on $\widehat{\mathbb{C}}$ (and R modulo $2\pi^2$ is well defined on X_{00}; in fact R itself is well defined on a $\mathbb{Z}$ cover of X_{00} which is the universal nilpotent cover of $\mathbb{C}-\{0,1\}$).

It is well known that $L(z):=\mathcal{R}(z)-\frac{\pi^2}{6}$ satisfies the functional equation

$$L(x)-L(y)+L\Big(\frac{y}{x}\Big)-L\Big(\frac{1-x^{-1}}{1-y^{-1}}\Big)+L\Big(\frac{1-x}{1-y}\Big)=0$$

for $0<y<x<1$. Since the 5–tuples involved in this equation are on the boundary of FT^+, the functional equation

$$\sum(-1)^iR(x_i;0,0)=0$$

is valid by analytic continuation on the whole of FT^+. Now

$$\sum(-1)^iR(x_i;p_i,q_i)$$

differs from this by

$$\frac{\pi i}{2}\sum(-1)^i(p_i\log(1-x_i)+q_i\log x_i).$$

For $(x_0,\dots,x_4)=(x,y,\dots)\in\mathrm{FT}^+$ this equals

$$\frac{\pi i}{2}\big((q_0-p_2-q_2+p_3+q_3)\log x+(p_0-q_3+q_4)\log(1-x)+(-q_1+q_2-q_3)\log y+ \\ +(-p_1+p_3+q_3-p_4-q_4)\log(1-y)+(p_2-p_3+p_4)\log(x-y)\big)$$

and as in the proof of Lemma 2.3, this vanishes identically on FT^+ if and only if the p_i and q_i are as in the lifted five term relation. Thus the lifted five-term relation gives a functional equation for R when $(x_0,\dots,x_4)\in\mathrm{FT}^+$. By analytic continuation, it is a functional equation for R mod π^2 in general. The transfer relation is trivially a functional equation for R. □

Our main result is the following:

Theorem 2.6 *There exists an isomorphism* $\lambda\colon H_3(\mathrm{PSL}(2,\mathbb{C});\mathbb{Z})\to\widehat{\mathcal{B}}(\mathbb{C})$ *such that the composition* $\lambda\circ R\colon H_3(\mathrm{PSL}(2,\mathbb{C});\mathbb{Z})\to\mathbb{C}/\pi^2\mathbb{Z}$ *is the characteristic class given by* $i(\mathrm{vol}+i\,\mathrm{cs})$.

To describe the map λ we must give a geometric interpretation of $\widehat{\mathbb{C}}$.

3 Parameters for ideal hyperbolic simplices

In this section we shall interpret $\widehat{\mathbb{C}}$ as a space of parameters for what we call "combinatorial flattenings" of ideal hyperbolic simplices. We need this to define the above map λ. It also gives a geometric interpretation of the lifted five term relation.

We shall denote the standard compactification of $\mathbb{H}^3$ by $\overline{\mathbb{H}}^3=\mathbb{H}^3\cup\mathbb{C}P^1$. An ideal simplex Δ with vertices $z_0,z_1,z_2,z_3\in\mathbb{C}P^1$ is determined up to congruence by the cross-ratio

$$z=[z_0:z_1:z_2:z_3]=\frac{(z_2-z_1)(z_3-z_0)}{(z_2-z_0)(z_3-z_1)}.$$

Permuting the vertices by an even (ie, orientation preserving) permutation replaces z by one of

$$z,\quad z'=\frac{1}{1-z},\quad\text{or}\quad z''=1-\frac{1}{z}.$$

The parameter z lies in the upper half plane of $\mathbb{C}$ if the orientation induced by the given ordering of the vertices agrees with the orientation of $\mathbb{H}^3$. But we allow simplices whose vertex ordering does not agree with their orientation. We also allow degenerate ideal simplices whose vertices lie in one plane, so the parameter z is real. However, we always require that the vertices are distinct. Thus the parameter z of the simplex lies in $\mathbb{C}-\{0,1\}$ and every such z corresponds to an ideal simplex.

There is another way of describing the cross-ratio parameter $z = [z_0 : z_1 : z_2 : z_3]$ of a simplex. The group of orientation preserving isometries of $\mathbb{H}^3$ fixing the points z_0 and z_1 is isomorphic to $\mathbb{C}^*$ and the element of this $\mathbb{C}^*$ that takes z_2 to z_3 is z (equivalently: if we position z_0, z_1 at $0, \infty$ in the upper half-space model of $\mathbb{H}^3$ then $z_3 = zz_2$). Thus the cross-ratio parameter z is associated with the edge z_0z_1 of the simplex[1]. The parameter associated in this way with the other two edges z_0z_3 and z_0z_2 out of z_0 are z' and z'' respectively, while the edges z_2z_3, z_1z_2, and z_1z_3 have the same parameters z, z', and z'' as their opposite edges. See Figure 1.

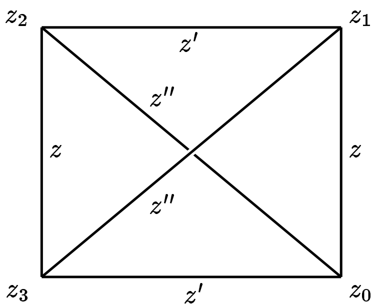

Figure 1

Note that $zz'z'' = -1$, so the sum

$$\log z + \log z' + \log z''$$

is an odd multiple of πi, depending on the branches of log used. In fact, if we use standard branch of log then this sum is πi or $-\pi i$ depending on whether z is in the upper or lower half plane.

The imaginary parts of $\log z$, $\log z'$, and $\log z''$ are the dihedral angles of the ideal simplex (resp. their negatives if the vertex ordering does not agree with the orientation the simplex inherits from $\mathbb{H}^3$). We now consider certain "adjustments" of these angles by multiples of π.

Definition 3.1 We shall call any triple of the form

$$\mathbf{w} = (w_0, w_1, w_2) = (\log z + p\pi i, \log z' + q\pi i, \log z'' + r\pi i)$$

with

$$p, q, r \in \mathbb{Z} \quad \text{and} \quad w_0 + w_1 + w_2 = 0$$

[1]It is associated with the *unoriented* edge since there is an orientation preserving symmetry of the simplex exchanging z_0 and z_1 (it also exchanges z_2 and z_3 and acts on $\mathbb{C}^*$ by $z \mapsto 1/z$).

a *combinatorial flattening* for our simplex.

Each edge E of Δ is assigned one of the components w_i of $\mathbf{w}$, with opposite edges being assigned the same component. We call w_i the *log-parameter* for the edge E and denote it $l_E(\Delta,\mathbf{w})$.

This combinatorial flattening can be written

$$\ell(z;p,q) := (\log z + p\pi i, -\log(1-z) + q\pi i, \log(1-z) - \log z - (p+q)\pi i),$$

and ℓ is then a map of $\widehat{\mathbb{C}}$ to the set of combinatorial flattenings of simplices.

Lemma 3.2 *This map ℓ is a bijection, so $\widehat{\mathbb{C}}$ may be identified with the set of all combinatorial flattenings of ideal tetrahedra.*

Proof We must show that we can recover $(z;p,q)$ from $(w_0,w_1,w_2)=\ell(z;p,q)$. It clearly suffices to recover z. But $z = \pm e^{w_0}$ and $1-z = \pm e^{-w_1}$, and the knowledge of both z and $1-z$ up to sign determines z. □

We can give a geometric interpretation of the choice of parameters in the five term relation (2). If $z_0,\dots,z_4$ are five distinct points of $\partial\overline{\mathbb{H}}^3$, then each choice of four of five points $z_0,\dots,z_4$ gives an ideal simplex. We denote the simplex which omits vertex z_i by Δ_i. The cross ratio parameters $x_i = [z_0 : \dots : \hat{z}_i : \dots : z_4]$ of these simplices can be expressed in terms of $x := x_0$ and $y := x_1$ as follows:

$$\begin{aligned}
x_0 &= [z_1 : z_2 : z_3 : z_4] =: x\\
x_1 &= [z_0 : z_2 : z_3 : z_4] =: y\\
x_2 &= [z_0 : z_1 : z_3 : z_4] = \frac{y}{x}\\
x_3 &= [z_0 : z_1 : z_2 : z_4] = \frac{1-x^{-1}}{1-y^{-1}}\\
x_4 &= [z_0 : z_1 : z_2 : z_3] = \frac{1-x}{1-y}
\end{aligned}$$

The lifted five term relation has the form

$$\sum_{i=0}^{4}(-1)^i(x_i;p_i,q_i) = 0 \tag{4}$$

with certain relations on the p_i and q_i. We will give a geometric interpretation of these relations.

Using the map of Lemma 3.2, each summand in this relation (4) represents a choice $\ell(x_i;p_i,q_i)$ of combinatorial flattening for one of the five ideal simplices.

For each 1–simplex E connecting two of the points z_i we get a corresponding linear combination

$$\sum_{i=0}^{4}(-1)^i l_E(\Delta_i, \ell(x_i;p_i,q_i)) \tag{5}$$

of log-parameters (Definition 3.1), where we put $l_E(\Delta_i, \ell(x_i;p_i,q_i)) = 0$ if the line E is not an edge of Δ_i. This linear combination has just three non-zero terms corresponding to the three simplices that meet at the edge E. One easily checks that the real part is zero and the imaginary part can be interpreted (with care about orientations) as the sum of the "adjusted angles" of the three flattened simplices meeting at E.

Definition 3.3 We say that the $(x_i;p_i,q_i)$ satisfy the *flattening condition* if each of the above linear combinations (5) of log-parameters is equal to zero. That is, the adjusted angle sum of the three simplices meeting at each edge is zero.

Lemma 3.4 *Relation (4) is an instance of the lifted five term relation (2) if and only if the $(x_i;p_i,q_i)$ satisfy the flattening condition.*

Proof We first consider the case that $(x_0,\dots,x_4)\in \mathrm{FT}^+$. Recall this means that each x_i is in $\mathbb{H}$. Geometrically, this implies that each of the above five tetrahedra is positively oriented by the ordering of its vertices. This implies the configuration of Figure 2 with z_1 and z_3 on opposite sides of the plane of the triangle $z_0z_2z_4$ and the line from z_1 to z_3 passing through the interior

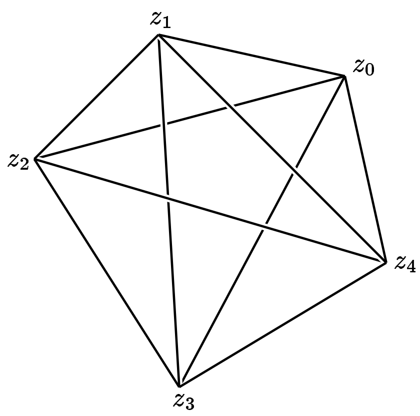

Figure 2

of this triangle. Denote the combinatorial flattening of the i^{th} simplex by $\ell(x_i;p_i,q_i)$. If we consider the log-parameters at the edge z_3z_4 for example, they are $\log x + p_0\pi i$, $\log y + p_1\pi i$, and $\log(y/x) + p_2\pi i$ and the condition is that $(\log x + p_0\pi i) - (\log y + p_1\pi i) + (\log(y/x) + p_2\pi i) = 0$. This implies $p_2 = p_1 - p_0$. Similarly the other edges lead to other relations among the p_i and q_i, namely:

$$\begin{array}{llll}
z_0z_1\colon & p_2 - p_3 + p_4 = 0 & z_0z_2\colon & -p_1 + p_3 + q_3 - p_4 - q_4 = 0\\
z_1z_2\colon & p_0 - q_3 + q_4 = 0 & z_1z_3\colon & -p_0 - q_0 + q_2 - p_4 - q_4 = 0\\
z_2z_3\colon & q_0 - q_1 + p_4 = 0 & z_2z_4\colon & -p_0 - q_0 + p_1 + q_1 - p_3 = 0\\
z_3z_4\colon & p_0 - p_1 + p_2 = 0 & z_3z_0\colon & p_1 + q_1 - p_2 - q_2 + q_4 = 0\\
z_4z_0\colon & -q_1 + q_2 - q_3 = 0 & z_4z_1\colon & q_0 - p_2 - q_2 + p_3 + q_3 = 0.
\end{array}$$

Elementary linear algebra verifies that these relations are equivalent to the equations $p_2 = p_1 - p_0$, $p_3 = p_1 - p_0 + q_1 - q_0$, $q_3 = q_2 - q_1$, $p_4 = q_1 - q_0$, and $q_4 = q_2 - q_1 - p_0$, as in the lifted five term relation (2). The lemma thus follows for $(x_0,\dots,x_4) \in \mathrm{FT}^+$. It is then true in general by analytic continuation. □

We mention here a lemma that will be useful later.

Lemma 3.5 *If the flattenings (x_i,p_i,q_i) are specified for a subset of the above five ideal simplices and the sum of adjusted angles is zero around each edge E that lies on three simplices of this subset, then one can specify flattenings on the remaining simplices so that the flattening condition holds (sum of adjusted angles is zero for all edges E). Moreover, once the flattenings are specified on three of the simplices, the flattenings on the final two are uniquely determined.*

Proof The lemma does not depend on the ordering of $z_0,\dots,z_4$ since the flattening condition at an edge is purely geometric. We assume therefore that the specified flattenings are $(x_i;p_i,q_i)$ for $i = 0,\dots,k$ with $0 \le k < 4$. The above proof shows that if $k = 0$ or 1 there is no restriction on the flattenings, while if $k = 2$ the the condition at the edge z_3z_4 implies $p_2 = p_1 - p_0$ if $(x_0,\dots,x_4) \in \mathrm{FT}^+$, or the appropriate analytic continuation of this in general. In each of these cases the previous lemma says how the flattenings on the remaining simplices may be chosen, and moreover, that this choice is unique if $k = 2$. If $k = 3$ then the conditions from the edges z_0z_4 and z_2z_4 determine $p_3 = p_1 - p_0 + q_1 - q_0$ and $q_3 = q_2 - q_1$ if $(x_0,\dots,x_4) \in \mathrm{FT}^+$, or the approriate analytic continuation in general, and the previous lemma then determines the flattening $(x_4;p_4,q_4)$. □

4 Definition of λ

Let $G = \mathrm{PSL}(2,\mathbb{C})$ (with the discrete topology). In this section we describe the map $\lambda\colon H_3(G;\mathbb{Z}) \to \widehat{\mathcal{P}}(\mathbb{C})$.

We first describe the combinatorial representation of elements of $H_3(G;\mathbb{Z})$ that we will use. As we will describe, a special case is to give a closed oriented triangulated 3–manifold with a flat G–bundle on it. Any element of $H_3(G;\mathbb{Z})$ can be represented this way, but we do not want to restrict to this type of representation.

We need to clarify our terminology for simplicial complexes.

Definition 4.1 We use the usual concept of *simplicial complex* K except that we do not require that distinct simplices have different vertex sets (but we do require that closed simplices embed in $|K|$, ie, vertices of a simplex are distinct in K). The *(open) star* of a 0–simplex v of K is the union of $\tau - \partial\tau$ over simplices that have v as a vertex. It is an open neighborhood of v and is the open cone on a simplicial complex L_v called the *link of* v. Note that L_v immerses, but does not necessarily embed in K as the boundary of the star of v.

We really only need quasi-simplicial complexes — cell complexes whose cells are simplices with simplicial attaching maps, but no requirement that closed simplices embed. We discuss this later; our more restrictive hypothesis is first needed near the end of section 10, but is eliminated again by Proposition 11.2.

4.1 Representing elements of $H_n(G;\mathbb{Z})$

Definition 4.2 An *ordered n–cycle* will be a compact n–dimensional simplicial complex K such that the complement $|K| - |K^{(n-3)}|$ of its $(n-3)$–skeleton is an oriented n–manifold, together with an ordering of the vertices of each n–simplex of K so that these orderings agree on common faces. The ordering orients each n–simplex Δ_i of K, and this orientation may or may not agree with the orientation of $|K| - |K^{(n-3)}|$. Let $\epsilon_i = +1$ or -1 accordingly. Then the n–cycle $\sum_i \epsilon_i \Delta_i$ represents a homology class $[K] \in H_n(|K|;\mathbb{Z})$ called the *fundamental class*.

We will also require that the link L_v of each 0–simplex is connected. This can always be achieved by duplicating 0–simplices if necessary.

One can represent any element of $H_n(G;\mathbb{Z})$ by giving an ordered n–cycle K and labeling the vertices of each simplex Δ of K by elements of G so that:

- Two G–labelings $(g_0,\dots,g_k)$ and $(g'_0,\dots,g'_k)$ of an ordered k–simplex are considered equivalent if there is a $g\in G$ with $gg_i=g'_i$ for each i;
- The G–labeling of any face of any simplex Δ is equivalent to the G–labeling induced from Δ.

We will also require that the labels for the vertices of any n–simplex are distinct (we can do this because G is infinite).

We describe two ways of seeing how any element of $H_n(G,\mathbb{Z})$ can be described by such data. The first is algebraic, and is taken from [17].

4.1.1 Algebraic description

We recall a standard chain complex for homology of $G=\mathrm{PSL}(2,\mathbb{C})$, the chain complex of "homogeneous simplices for G." We will, however, diverge from the standard by using only non-degenerate simplices, ie, simplices with distinct vertices — we may do this because G is infinite.

Let $C_n(G)$ denote the free $\mathbb{Z}$–module on all ordered $(n+1)$–tuples $\langle g_0,\dots,g_n\rangle$ of distinct elements of G. Define $\delta\colon C_n\to C_{n-1}$ by

$$\delta\langle g_0,\dots,g_n\rangle=\sum_{i=0}^{n}(-1)^i\langle g_0,\dots,\hat{g}_i,\dots,g_n\rangle.$$

Then each C_n is a free $\mathbb{Z}G$–module under left-multiplication by G. Since G is infinite the sequence

$$\cdots\to C_2\to C_1\to C_0\to Z\to 0$$

is exact, so it is a $\mathbb{Z}G$–free resolution of $\mathbb{Z}$. Thus the chain complex

$$\cdots\to C_2\otimes_{\mathbb{Z}G}\mathbb{Z}\to C_1\otimes_{\mathbb{Z}G}\mathbb{Z}\to C_0\otimes_{\mathbb{Z}G}\mathbb{Z}\to 0$$

computes the homology of G. Note that $C_n\otimes_{\mathbb{Z}G}\mathbb{Z}$ is the free $\mathbb{Z}$–module on symbols $\langle g_0:\ \dots:\ g_n\rangle$, where the g_i are distinct elements of G and $\langle g_0:\ \dots:\ g_n\rangle=\langle g'_0:\ \dots:\ g'_n\rangle$ if and only if there is a $g\in G$ with $gg_i=g'_i$ for $i=0,\dots,n$ (we call these *homogeneous n–simplices for G*).

Thus an element of $\alpha\in H_n(G;\mathbb{Z})$ is represented by a sum

$$\sum\epsilon_i\langle g_0^{(i)}:\ \dots:\ g_n^{(i)}\rangle$$

of homogeneous n–simplices for G and their negatives (here each ϵ_i is ± 1). The fact that this is a cycle means that the $(n-1)$–faces of these homogeneous simplices cancel in pairs. We choose some specific way of pairing canceling faces and form a geometric quasi-simplicial complex K by taking a n–simplex Δ_i for each homogeneous n–simplex of the above sum and gluing together $(n-1)$–faces of these Δ_i that correspond to $(n-1)$–faces of the homogeneous simplices that have been paired with each other.

As already mentioned, a quasi-simplicial complex actually suffices for our purposes. But we can obtain a simplicial complex by replacing K by its barycentric subdivision if necessary (label the barycenter of each simplex by an arbitrary element of G that differs from the G–labels of the barycenters of all proper faces of that simplex). A standard argument shows that this does not change the homology class represented by K. We thus get a representation of the homology class in the promised form.

4.1.2 Topological description

Given a space X, any element $\alpha \in H_n(X;\mathbb{Z})$ can be represented as $f_*[K]$ for some map $f\colon |K| \to X$ of an ordered n–cycle to X. If $X = BG$ is a classifying space for G then this map is determined up to homotopy by the flat G–bundle f^*EG over $|K|$, so this flat bundle determines the homology class $\alpha \in H_n(BG;\mathbb{Z}) = H_n(G;\mathbb{Z})$

Flat G–bundles over $|K|$ are determined by G–valued 1–cocycles on K up to the coboundary action of G–valued 0–cochains (the 1–cocycle is the obstruction to extending over $|K^{(1)}|$ a chosen section of the flat bundle over $K^{(0)}$). We recall the basic definitions.

Let $S_q(K)$ be the set of ordered q–simplices of K. A G–valued 1–cocyle on K is a map $\sigma\colon S_1(K) \to G$ with the cocyle property:

$$\sigma\langle v_0, v_2\rangle = \sigma\langle v_0, v_1\rangle\sigma\langle v_1, v_2\rangle \qquad \text{for } \langle v_0, v_1, v_2\rangle \in S_2(K)\,.$$

For convenience we extend the definition of σ to reverse-ordered simplices by

$$\sigma\langle v_1, v_0\rangle = \sigma\langle v_0, v_1\rangle^{-1} \qquad \text{for } \langle v_0, v_1\rangle \in S_1(K)\,.$$

If $\tau\colon S_0(K) \to G$ is a 0–cochain, then its coboundary action on 1–cocycles is to replace σ by

$$\langle v_0, v_1\rangle \mapsto \tau(v_0)^{-1}\sigma\langle v_0, v_1\rangle\tau(v_1)\,.$$

A G–valued 1–cocycle σ determines a G–labeling of simplices: label a simplex $\langle v_0, \dots, v_k\rangle$ by $(1, \sigma\langle v_0, v_1\rangle, \dots, \sigma\langle v_0, v_k\rangle)$. Conversely, a G–labeling of simplices determines a 1–cocycle: assign to a 1–simplex with label (g_1, g_2) the element $g_1^{-1}g_2 \in G$. These correspondences are clearly mutually inverse. When a 1–cocyle σ is changed by the coboundary action of a 0–cochain τ, the corresponding G–labeling $(g_0, \dots, g_k)$ of a k–simplex $\langle v_0, \dots, v_k\rangle$ is replaced by $(g_0\tau(v_0), \dots, g_k\tau(v_k))$.

Thus, we get our desired representative as in subsection 4.1 for a homology class $\alpha \in H_n(G;\mathbb{Z})$ by representing α by a flat G bundle over an ordered n–cycle K, representing that by a G–valued 1–cycle on K, and then taking the corresponding G–labeling of K. We want labels of the vertices of any n–simplex to be distinct, which means that the 1–cocycle should never take the value 1. Since G is infinite, this can be achieved by modifying by a coboundary if necessary.

4.2 Definition of λ: $H_3(G;\mathbb{Z}) \to \widehat{\mathcal{P}}(\mathbb{C})$

Let K be an ordered 3–cycle. We call a closed path γ in $|K|$ a *normal* path if it meets no 0– or 1–simplices of K and crosses all 2–faces that it meets transversally. When such a path passes through a 3–simplex Δ_i, entering and departing at different faces, there is a unique edge E of the 3–simplex between these faces. We say the path *passes* this edge E.

(In the following it will often be necessary to distinguish between a 1–simplex of K and the various edges of 3–simplices that are identified with this 1–simplex. To avoid excess notation, we will often use the same symbol for a 1 simplex of K and the edges of 3–simplices that are identified with it, but we will refer to "edges" or "1–simplices" to flag which we mean.)

Consider a choice of combinatorial flattening $\mathbf{w}_i$ for each simplex Δ_i. Then for each edge E of a simplex Δ_i of K we have a log-parameter $l_E = l_E(\Delta_i, \mathbf{w}_i)$ assigned. Recall that this log-parameter has the form $\log z + s\pi i$ where z is the cross-ratio parameter associated to the edge E of simplex Δ_i and s is some integer. We call (s mod 2) the *parity parameter* at the edge E of Δ_i and denote it $\delta_E = \delta_E(\Delta_i, \mathbf{w}_i)$.

Definition 4.3 Suppose γ is a normal path in $|K|$. The *parity* along γ is the sum

$$\sum_E \delta_E \text{ modulo } 2$$

of the parity parameters of all the edges E that γ passes. Moreover, if γ runs in the star of some fixed 0–simplex v of K, then the *log-parameter along the* *path* is the sum

$$\sum_E \pm\epsilon_{i(E)} l_E,$$

summed over all edges E that γ passes, where:

- $i(E)$ is the index i of the simplex Δ_i that the edge E belongs to and $\epsilon_{i(E)}$ is the coefficient ± 1 that encodes whether the ordering of the vertices of $\Delta_{i(E)}$ matches the orientation of $|K|$ or not.
- the extra sign $\pm$ is $+$ or $-$ according as the edge E is passed in a counterclockwise or clockwise fashion as viewed from the vertex.

If γ just encircles a 1–simplex (so the extra signs are all $+$ or all $-$) we speak of the parity or log-parameter "around the 1–simplex."

We assume we have represented an element $\alpha \in H_3(G;\mathbb{Z})$ by a G–labeled ordered 3–cycle K as in subsection 4.1. So to each 3–simplex Δ_i of K is associated a 4–tuple $\langle g_0^{(i)} : \dots : g_3^{(i)} \rangle$, defined up to left-multiplication by elements of G, and $\epsilon_i = \pm 1$ encodes whether the ordering of the vertices of Δ_i matches the orientation of K or not. In the following definition we identify $G = \mathrm{PSL}(2,\mathbb{C})$ with the isometry group $\mathrm{Isom}^+(\mathbb{H}^3)$.

Definition 4.4 *Flattening of K* Choose $z \in \partial\overline{\mathbb{H}}^3$ so $g_0^{(i)}z$, $g_1^{(i)}z$, $g_2^{(i)}z$, $g_3^{(i)}z$ are distinct points of $\partial\overline{\mathbb{H}}^3$ for each i (this excludes finitely many points $z \in \partial\overline{\mathbb{H}}^3$). We then have an ideal hyperbolic simplex shape for each simplex Δ_i of K and an associated cross ratio $x_i = [g_0^{(i)}z : g_1^{(i)}z : g_2^{(i)}z : g_3^{(i)}z]$. A *flattening of K* consists of a choice of combinatorial flattenings $\mathbf{w}_i = \ell(x_i; p_i, q_i)$ of the simplices of K such that the parity along any normal path in K is zero and the log-parameter around each edge of K is zero. If log-parameter along any normal path in the star of each 0–simplex of K is zero we call it a *strong flattening of K*.

Theorem 4.5 *For each choice of z as in the above definition, a strong flattening of K exists.*

Recall that our G–labelled ordered 3–cycle K was chosen to represent an element $\alpha \in H_3(G;\mathbb{Z})$ ($G = \mathrm{PSL}(2,\mathbb{C})$). The following theorem finally gives the definition of the map $\lambda\colon H_3(G;\mathbb{Z}) \to \widehat{\mathcal{B}}(\mathbb{C})$.

Theorem 4.6 *For any flattening of K the element $\sum_i \epsilon_i[x_i; p_i, q_i] \in \widehat{\mathcal{P}}(\mathbb{C})$ only depends on the homology class $\alpha \in H_3(\mathrm{PSL}(2,\mathbb{C});\mathbb{Z})$. We denote it $\lambda(\alpha)$. Moreover, $\lambda(\alpha) \in \widehat{\mathcal{B}}(\mathbb{C})$ and $\lambda\colon H_3(\mathrm{PSL}(2,\mathbb{C});\mathbb{Z}) \to \widehat{\mathcal{B}}(\mathbb{C})$ is a homomorphism.*

In Remark 10.5 we point out that instead of choosing a single z in the definition of flattening, we may choose a different z for each vertex of K, and Theorem 4.6 remains true. This is useful in practice (Section 14).

We now give a brief overview of the proofs.

If the extended Bloch group $\widehat{\mathcal{B}}(\mathbb{C})$ is to be isomorphic to $H_3(G;\mathbb{Z})$ then it must fit in the same Bloch–Wigner short exact sequence that was given for $H_3(G;\mathbb{Z})$ in the Introduction. This is proved in section 7, and section 8 is a digression describing a related group. These sections are independent of the proofs of Theorems 4.5 and 4.6, which take up Sections 9 through 11.

First some basic tools are developed in Sections 5 and 6. Section 5 describes a developing map $\tilde{K} \to \overline{\mathbb{H}}^3$ which is helpful for visualizing the flattening condition. It is used to prove some preliminary lemmas. Section 6 proves a consequence of the five term relations, a general "cycle relation" which is used frequently later.

Section 9 proves Theorem 4.5, the existence of a strong flattening of K. The argument of this section is combinatorial and depends heavily on [13].

The combinatorial argument is continued at the beginning of the next section (Section 10) to show that the resulting element of $\widehat{\mathcal{P}}(\mathbb{C})$ is independent of the choice of strong flattening and lies in $\widehat{\mathcal{B}}(\mathbb{C})$. It then remains to show that the element of $\widehat{\mathcal{B}}(\mathbb{C})$ only depends on the homology class $\alpha \in H_3(G;\mathbb{Z})$ and not on the G–labeled ordered 3-cycle K used to represent it. In the rest of Section 10 we first show that this element of $\widehat{\mathcal{B}}(\mathbb{C})$ is unchanged if the G–labeling of K is changed by a coboundary (so it just depends on the flat G–bundle and not on the 1–cocycle used to represent it) and next that it is invariant under alteration of the triangulation of $|K|$ by Pachner moves. Thus, by the end of this section we know that we have an element of $\widehat{\mathcal{B}}(\mathbb{C})$ that only depends on the space $|K|$ and the flat G–bundle over it.

This section uses a strong flattening of K and, moreover, it needs to assume that K is a simplicial rather than quasi-simplicial complex and that the vertex orderings of its simplices are induced by a global ordering of the 0–simplices of K. These assumptions will be removed one by one in the next section, so Theorem 4.6 eventually applies to flattenings that are not necessarily strong, and also to quasi-simplicial complexes for which the vertex orderings of the simplices need not be globally induced.

Section 11 completes the proof of Theorem 4.6. First it is shown that the singularities of $|K|$ can be "resolved" to make $|K|$ into a manifold. Since $H_3(G;\mathbb{Z})$

can be represented as the bordism group of 3–manifolds with flat G–bundles, the proof is completed by showing that the element of $\widehat{\mathcal{B}}(\mathbb{C})$ is unchanged by elementary bordisms (ie, surgery). Flattenings and strong flattenings are the same thing when K is a manifold, so the requirement of strong flattenings dissolves, while retriangulation using Pachner moves is used to relax the requirements on K to require only that it be a quasi-simplicial complex with vertex-ordered simplices.

Finally, the main Theorem 2.6, stating that $\lambda\colon H_3(\mathrm{PSL}(2,\mathbb{C});\mathbb{Z}) \to \widehat{\mathcal{B}}(\mathbb{C})$ is an isomorphism and that Cheeger–Chern–Simons class is given by Rogers dilogarithm, is proved in section 12. At this point it follows easily, using work of Dupont.

Section 13 describes the weaker conclusions obtained if orderings of simplices are not used, confirming a conjecture in [13].

The next two sections then apply the results to invariants of 3–manifolds, and a very brief final section describes what happens if $\mathbb{C}$ is replaced by a number-field.

5 Developing map

Suppose we have a G–labeled 3–cycle K. Choose a generic point $z \in \partial\overline{\mathbb{H}}^3$ as in the definition of flattenings (Definition 4.4). Then each simplex Δ_i of K corresponds to a non-degenerate ideal simplex $\langle g_0^{(i)}z, g_1^{(i)}z, g_2^{(i)}z, g_3^{(i)}z\rangle$ in $\overline{\mathbb{H}}^3$. This simplex is determined up to isometry, since $(g_0^{(i)},\dots,g_3^{(i)})$ is well defined up to left multiplication by elements of $G = \mathrm{PSL}(2,\mathbb{C}) = \mathrm{Isom}^+(\mathbb{H}^3)$. This associates a geometry as an ideal simplex to each simplex of K, and hence, by lifting, also to each simplex of the universal cover $\tilde{K}$.

We would like to use this geometry to construct a *developing map* $|\tilde{K}| \to \overline{\mathbb{H}}^3$. The next lemma says that this is possible. Note that the ideal simplex corresponding to a 3–simplex of K inherits an orientation from $\mathbb{H}^3$ that will not in general agree with the orientation of the simplex in the manifold $|K - K^{(0)}|$, so the developing map may be "folded" in that some simplices may be mapped reversing orientation. In particular, adjacent simplices will map to the same side of a common 2–face whenever the ideal simplex orientation of one of the simplices agrees with its orientation in $|K|$ and for the other simplex it differs.

Lemma 5.1 *There is a map $D\colon |\tilde{K}| \to \overline{\mathbb{H}}^3$ (unique up to isometries of $\mathbb{H}^3$) which maps each simplex of the universal cover $\tilde{K}$ of K isometrically and* *preserving orientation with respect to its ideal simplex shape just described. We call it the "developing map." The map is equivariant with respect to an action of $\pi_1(K)$ on $\overline{\mathbb{H}}^3$.*

Proof Consider the corresponding 1–cocyle σ (subsection 4.1.2) whose value on an unordered 1–simplex $\langle v_1, v_2\rangle$ with label (g_1, g_2) is $\sigma(\langle v_1, v_2\rangle) = g_1^{-1}g_2$. Any edge path $\langle v_0, v_1\rangle\langle v_1, v_2\rangle \dots \langle v_{k-1}, v_k\rangle$ determines an element

$$\sigma(\langle v_0, v_1\rangle)\sigma(\langle v_1, v_2\rangle) \dots \sigma(\langle v_{k-1}, v_k\rangle) \in G$$

and this gives a homomorphism ϕ_K from the groupoid of edge paths to G. In particular, it restricts to a group homomorphism $\pi_1(K, v_0) \to G$. We thus get an action of $\pi_1(K, v_0)$ on $\overline{\mathbb{H}}^3$.

Choose a 0–simplex $\tilde v_0$ as a basepoint in the universal cover $\tilde K$ of K. We can then G–label the vertices of $\tilde K$ by labeling vertex $\tilde v$ by the element $\phi_K(\gamma)$ where γ is the image of any edge path from $\tilde v_0$ to $\tilde v$ in $\tilde K$. This gives us a lift of the G–labeling of K to a labeling of $\tilde K$ in which labels are absolute rather than just well defined up to left multiplication by G. Using this labeling, the desired mapping of $|\tilde K|$ to $\overline{\mathbb{H}}^3$ maps a simplex of $\tilde K$ with labels $(g_1, \dots, g_4)$ to the ideal simplex $\langle g_1 z, \dots, g_4 z\rangle$.

To show the equivariance of this map it helps to describe it more explicitly. For any two 0–simplices $w_1, w_2 \in \tilde K$, let $\phi[w_1, w_2]$ be ϕ_K of the image in K of a simplicial path $[w_1, w_2]$ from w_1 to w_2 in $\tilde K$. $\phi[w_1, w_2]$ only depends on w_1 and w_2 since any two paths from w_1 to w_2 are homotopic. With this notation the developing map is

$$D(w) = \phi[\tilde v_0, w]z \quad \text{for any } w \in \tilde K^{(0)}.$$

The action of $\pi_1(K, v_0)$ by covering transformations is such that the lift of a closed path representing $\gamma \in \pi_1(K, v_0)$ is a path $[\tilde v_0, \gamma\tilde v_0]$.

Thus for any $w \in \tilde K^{(0)}$ and $\gamma \in \pi_1(K, v_0)$ we have

$$\begin{aligned}\phi_K(\gamma)D(w) = \phi_K(\gamma)\phi[\tilde v_0, w]z &= \phi[\tilde v_0, \gamma\tilde v_0]\phi[v_0, w]z = \\ =&\phi[\tilde v_0, \gamma\tilde v_0]\phi[\gamma v_0, \gamma w]z = \phi[v_0, \gamma w]z = D(\gamma w)\end{aligned}$$

This proves the equivariance. □

The developing map gives a convenient way to visualize what it means to satisfy the flattening condition in the star neighborhood N of a 0–simplex v of K. Since N is contractible, it lifts homeomorphically to the star $\tilde N$ of a lift $\tilde v$ of v in $\tilde K$. Use the upper half-space model of $\mathbb{H}^3$ and choose the developing map

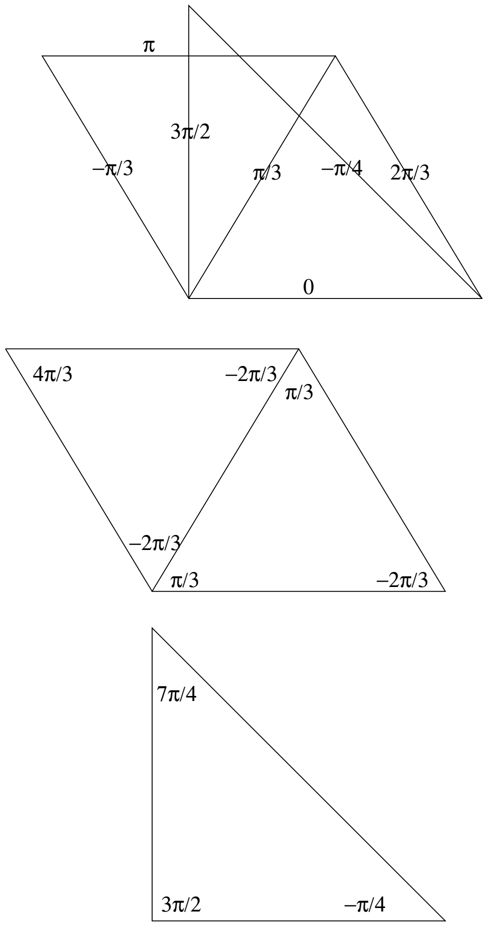

Figure 3: Three triangles in developing image of N in $\mathbb{C}$ showing rotation levels of edges and resulting adjusted angles. One triangle is from a "folded" simplex.

$D\colon |\tilde{K}| \to \overline{\mathbb{H}}^3$ so $D(\tilde{v}) = \infty$. Thus for each 3–simplex of the closure of $\tilde{N}$, the other three vertices of the simplex map to points in $\mathbb{C} \subset \mathbb{C} \cup \{\infty\} = \partial\overline{\mathbb{H}}^3$, and thus determine a triangle in $\mathbb{C}$. Each 2–simplex incident to v determines a 1–simplex in L_v and hence a line segment in $\mathbb{C}$. For each such segment S, the angle of this line segment from horizontal is well defined in $\mathbb{R}/\pi\mathbb{Z}$; choose a specific lift $a(S)$ of this number in $\mathbb{R}$ and call it the *rotation level* for S. Then differences of rotation levels for segments corresponding to adjacent 2–faces of a 3–simplex determine a flattening in the star neighborhood of N (see Figure 3). Specifically:

Suppose we have two adjacent 2–faces incident to v of a 3–simplex Δ_i of N.

Let S_1 and S_2 be the corresponding segments in the plane $\mathbb{C}$, taken in the order specified by the orientation Δ_i inherits from K (so S_1 and S_2 are in clockwise order around the triangle in $\mathbb{C}$ determined by Δ if the developing map D preserves orientation of Δ and anticlockwise order otherwise). Specify the adjusted angle between these 2–faces of Δ_i to be $\epsilon_i(a(S_2)-a(S_1))$ (where, as usual, $\epsilon_i=\pm 1$ is determined by vertex-order of Δ_i).

This clearly specifies a flattening for the 3–simplices of N and the flattening conditions are satisfied at v. Conversely, such a flattening for N plus a rotation level $a(S)$ for one of the line segments determines $a(S)$ for every segment, so the flattening determines the rotation level function a up to a multiple of π.

We will use the developing map to prove a preliminary proposition towards Theorem 4.5. That theorem says a strong flattening of K exists. Since changing flattenings changes log parameters by multiples of πi, it must be true that if we make an *arbitrary* choice of flattenings of the individual ideal simplices then the sum of log parameters along any normal path in the star of a 0–simplex is a multiple of πi. We will show, in fact:

Proposition 5.2 *Notation as in Definition 4.4. If we use flattenings $\ell(x_i;0,0)$ for the simplices of K then the log parameter along any normal path in the star of a 0–simplex is a multiple of $2\pi i$.*

The proof of this will involve the following related proposition, which we prove first.

Proposition 5.3 *Notation as in Definition 4.4. If we use flattenings $\ell(x_i;0,0)$ for the simplices of K then the parity along any normal path in K is zero.*

Proof As we follow a normal path the contribution to the parity as we pass an edge of a simplex is 0 if the edge is the 01, 03, 12, or 23 edge of the simplex and the contribution is ± 1 if it is the 02 or 13 edge. Consider the orientations of the triangular faces we cross as we follow the path, where the orientation is the one induced by the ordering of its vertices. As we pass a 02 or 13 edge this orientation changes while for the other edges it does not. Since K is oriented, we must have an even number of orientation changes as we traverse the normal path, proving the claim. □

Proof of Proposition 5.2 Consider a normal path γ in the star N of a 0–simplex v of K. As above, use the upper half-space model of $\mathbb{H}^3$ and choose the developing map $D\colon |\tilde{K}| \to \overline{\mathbb{H}}^3$ so $D(\tilde{v})=\infty$, where $\tilde{v}$ is a lift of v.

Denote the lift of γ to $\tilde{N}$ by $\tilde{\gamma}$. The path $D \circ \tilde{\gamma}$ has a "shadow" in $\mathbb{C}$. As γ passes an edge of a 3–simplex of $\tilde{N}$, the shadow passes the corresponding vertex of the corresponding triangle in $\mathbb{C}$, entering at one edge of the triangle and exiting at another. The cross-ratio parameter corresponding to the 3–simplex edge (or its inverse, depending on vertex ordering and how we are passing the edge) is the derivative of the linear map of $\mathbb{C}$ that takes the entering edge of the triangle to the exiting edge, fixing the common vertex. Since the path $D \circ \tilde{\gamma}$ ends up where it started, the product of these cross-ratio parameters (each raised to an appropriate power ± 1) is 1, so the appropriately signed sum of their logs is a multiple of $2\pi i$. This sum is precisely the log-parameter along the path, except for the adjustments by multiples of πi that are involved in forming the log-parameters from the logarithms of cross-ratio parameters. These adjustments add up to an even multiple of πi since the parity along the path γ to be zero by Proposition 5.3. □

The following corollary of Propositions 5.2 and 5.3 is useful. It is immediate from the definition of parity.

Corollary 5.4 *For a normal path in the star of a vertex, if the flattening condition for log-parameters is satisfied, then so is the parity condition.* □

6 The cycle relation

In this section we prove a general relation called the "cycle relation," that holds in $\widehat{\mathcal{P}}(\mathbb{C})$. We will use it repeatedly later. It is, in fact, a consequence of the five-term relation (2) alone and does not involve the transfer relation (3).

Let K be a simplicial complex obtained by gluing 3–simplices $\Delta_1, \dots, \Delta_n$ together in sequence around a common 1–simplex E. Thus, for each index j modulo n, Δ_j is glued to each of Δ_{j-1} and Δ_{j+1} along one of the two faces of Δ_j incident to E. Suppose, moreover, that the vertices of each Δ_j are ordered such that orderings agree on the common 2–faces of adjacent 3–simplices.

There is then a sequence $\epsilon_1 = \pm 1, \dots, \epsilon_n = \pm 1$ such that the 2–faces used for gluing all cancel in the boundary of the 3–chain $\sum_{j=1}^{n} \epsilon_j \Delta_j$. (Proof: choose $\epsilon_1 = 1$ and then for $i = 2, \dots, n$ choose ϵ_i so the common face of Δ_{i-1} and Δ_i cancels. The common face of Δ_n and Δ_1 must then cancel since otherwise that face occurs with coefficient ± 2 in $\partial \sum_{j=1}^{n} \epsilon_j \Delta_j$, and E occurs with coefficient ± 2 in $\partial\partial \sum_{j=1}^{n} \epsilon_j \Delta_j$.)

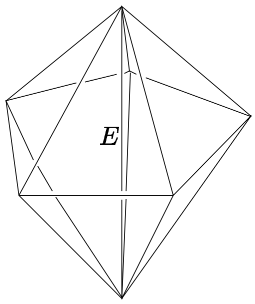

Figure 4: Configuration for the cycle relation

Suppose now further that a combinatorial flattening $\mathbf{w}_i$ has been chosen for each Δ_j such that the "signed sum" of log parameters around the 1–simplex E vanishes and the same for parity parameters:

$$\sum_{j=1}^{n} \epsilon_j l_E(\Delta_j, \mathbf{w}_j) = 0, \quad \sum_{j=1}^{n} \epsilon_j \delta_E(\Delta_j, \mathbf{w}_j) = 0. \tag{6}$$

We think of the 1–simplex E as being vertical, so that we can label the two 1–simplices other than E of the common triangle of Δ_j and Δ_{j+1} as T_j and B_j (for "top" and "bottom"). Let $\mathbf{w}'_j$ be the flattening obtained from $\mathbf{w}_j$ by adding $\epsilon_j \pi i$ to the log parameter at T_j and its opposite edge in Δ_j and subtracting $\epsilon_j \pi i$ from the log parameter at B_j and its opposite edge in Δ_j. If we do this for each j then the total log parameter (and, by Corollary 5.4, also parity parameter) around any 1–simplex of the complex K is not changed (we sum log-parameters with the appropriate sign ϵ_j): — at E no log-parameter has changed while at every other 1–simplex πi has been added at one of the two adjacent 3–simplices and subtracted at the other.

Lemma 6.1 (Cycle relation about E) *With the above notation,*

$$\sum_{j=1}^{n} \epsilon_j [\mathbf{w}_j] = \sum_{j=1}^{n} \epsilon_j [\mathbf{w}'_j] \quad \in \widehat{\mathcal{P}}(\mathbb{C}),$$

where we are using $[\mathbf{w}]$ as a shorthand for $[\ell^{-1}\mathbf{w}]$ (ie, $[\mathbf{w}]$ means $[z;p,q]$ where $\ell(z;p,q) = \mathbf{w}$; see Lemma 3.2).

Proof Each of the 3–simplices Δ_i has an associated ideal hyperbolic structure compatible with the combinatorial flattenings $\mathbf{w}_j$. This ideal hyperbolic structure is also compatible with the flattening $\mathbf{w}'_j$. Choose a realization of Δ_1

as an ideal simplex in $\overline{\mathbb{H}}^3$. We think of this as a mapping of Δ_1 to $\overline{\mathbb{H}}^3$. We can extend this to a mapping of K to $\overline{\mathbb{H}}^3$ which maps each Δ_j to an ideal simplex with shape appropriate to its combinatorial flattening. Adjacent simplices will map to the same side of their common face in $\overline{\mathbb{H}}^3$ if either their orientations or the signs ϵ_j do not match and will be on opposite sides otherwise. The fact that the signed sums of log and parity parameters around the 1–simplex E are zero guarantees that the identifications match up as we go once around E.

Note that K has $n+2$ vertices. We first consider the special case that $n=3$ and there is an ordering $v_0,\dots,v_4$ of the five vertices of K that restricts to the given vertex ordering for each simplex. We also assume the five vertices of K map to distinct points $z_0,\dots,z_4$ of $\partial\mathbb{H}^3$.

Each 3–simplex Δ_j for $j=1,2,3$ has vertices obtained by omitting one of the five vertices $v_0,\dots,v_4$. Denote by Δ_4 and Δ_5 the 3–simplices obtained by omitting each of the other two vertices. The fact that the common 2–faces of the Δ_j cancel when taking boundary of the chain $\epsilon_1\Delta_1+\epsilon_2\Delta_2+\epsilon_3\Delta_3$ means that, up to sign this sum corresponds to three summands of the chain $\partial\langle v_0,\dots,v_4\rangle=\sum(-1)^i\langle v_o,\dots,\hat{v_i},\dots,v_4\rangle$. Choose ϵ_4 and ϵ_5 so that $\sum_{j=1}^5\epsilon_j\Delta_j$ is $\pm\partial\langle v_0,\dots,v_4\rangle$.

By Lemma 3.5, we can choose unique combinatorial flattenings $\mathbf{w}_4$ and $\mathbf{w}_5$ of Δ_4 and Δ_5 so that the signed sum of log parameters and parity parameters around any 1–simplex of $K\cup\Delta_4\cup\Delta_5$ is zero. Note that $\mathbf{w}_4$ and $\mathbf{w}_5$ do not change if we replace $\mathbf{w}_1,\dots,\mathbf{w}_3$ by $\mathbf{w}'_1,\dots,\mathbf{w}'_3$. By Lemma 3.4 we then have

$$\begin{aligned}\epsilon_1\mathbf{w}_1+\epsilon_2\mathbf{w}_2+\epsilon_3\mathbf{w}_3&=-(\epsilon_4\mathbf{w}_4+\epsilon_5\mathbf{w}_5)\\ \epsilon_1\mathbf{w}'_1+\epsilon_2\mathbf{w}'_2+\epsilon_3\mathbf{w}'_3&=-(\epsilon_4\mathbf{w}_4+\epsilon_5\mathbf{w}_5),\end{aligned}$$

proving this case.

We next consider the case that for some index j modulo n the images of Δ_j and Δ_{j+1} in $\overline{\mathbb{H}}^3$ do not coincide, so their union has five distinct vertices. By cycling our indices we may assume $j=1$. Since the orderings of the vertices of Δ_1 and of Δ_2 agree on the three vertices they have in common, there is an ordering of all five vertices compatible with both Δ_1 and Δ_2. Let Δ_0 be the simplex determined by the common 1–simplex E and the two vertices that Δ_1 and Δ_2 do not have in common. Then there is an $\epsilon_0=\pm1$ such that the common faces of Δ_0, Δ_1, and Δ_2 cancel in the boundary of the chain $\epsilon_0\Delta_0+\epsilon_1\Delta_1+\epsilon_2\Delta_2$. Choose a flattening $\mathbf{w}_0$ of Δ_0 such that $\epsilon_0 l_E(\Delta_0,\mathbf{w}_0)+\epsilon_1 l_E(\Delta_1,\mathbf{w}_1)+\epsilon_2 l_E(\Delta_2,\mathbf{w}_2)=0$. Then the relation of the lemma has already been proved for $\mathbf{w}_0$, $\mathbf{w}_1$, $\mathbf{w}_2$, and by subtracting this relation from the relation to be proved for $\mathbf{w}_1,\dots,\mathbf{w}_n$ we obtain a case of the lemma with one fewer simplices. Thus, if we assume the lemma proved for $n-1$ simplices then this case is also proved.

The above induction argument fails only for the case that there are $2m$ simplices that alternately "fold back on each other" so that their images in $\overline{\mathbb{H}}^3$ all have the same four vertices. The above induction eventually reduces us to this case (usually with $m=1$). We must therefore deal with this situation to complete the proof. We first consider the case that $m=1$ so $n=2$. We then have four vertices $z_0,\dots,z_3$ in $\partial\overline{\mathbb{H}}^3$. We assume the 1–simplex E is z_0z_1. Then the ordering of the vertices of the faces $z_0z_1z_2$ and $z_0z_1z_3$ is the same in each of Δ_1 and Δ_2. Choose a new point z_4 in $\partial\overline{\mathbb{H}}^3$ distinct from $z_0,\dots,z_3$ and consider the ordered simplex with vertices z_0,z_1,z_2 ordered as above followed by z_4. Call this Δ_3. Similarly make Δ_4 using z_0,z_1,z_3 ordered as above followed by z_4. Choose flattenings of Δ_3 and Δ_4 so that the signed sum of log parameters for $\Delta_1,\Delta_3,\Delta_4$ around E is zero. Then we obtain a three simplex relation of the type already proved for $\Delta_1,\Delta_3,\Delta_4$ and another for $\Delta_2,\Delta_3,\Delta_4$, and the difference of these two relations gives the desired two-simplex relation.

More generally, if we are in the above "folded" case with $m>1$ we can use an instance of the three-simplex relation to replace one of the $2m$ simplices by two. We then use the induction step to replace one of these new simplices together with an adjacent old simplex by one simplex and then repeat for the other new simplex. In this way we reduce to a relation involving $2m-1$ simplices, completing the proof. □

7 Computation of $\widehat{\mathcal{B}}(\mathbb{C})$

In this section we work out the relationship of the extended Bloch group with the usual one. The result is Theorem 7.5.

We consider the case $n=3$ of the cycle relation (Lemma 6.1). Denote the five vertices involved $v_0\dots,v_4$ and assume the three 3–simplices meet along the edge $E=v_3v_4$. If we order $v_0,\dots,v_4$ in this order and give the simplices the inherited vertex orders then the cycle relation can be written (with the appropriate relationship among p_0,p_1,p_2):

$$\begin{aligned}&[x;p_0,q_0]-[y;p_1,q_1]+[y/x;p_2,q_2]=\\&[x;p_0,q_0-1]-[y;p_1,q_1-1]+[y/x;p_2,q_2-1].\end{aligned}\tag{7}$$

This is true for any choice of q_0,q_1,q_2 so long as p_0,p_1,p_2 satisfy the appropriate relation. Thus if we just change q_0 and subtract the resulting equation from the above we get

$$[x;p_0,q_0]-[x;p_0,q_0']=[x;p_0,q_0-1]-[x;p_0,q_0'-1].$$

From the versions of the three-simplex cycle relation with different orderings of the vertices $v_0,\dots,v_4$ we can similarly derive three versions of this relation:

$$\begin{aligned}[x;p,q]-[x;p,q'] &= [x;p,q-1]-[x;p,q'-1]\\ [x;p,q]-[x;p',q] &= [x;p-1,q]-[x;p'-1,q]\\ [x;p,q]-[x;p+s,q-s] &= [x;p+1,q-1]-[x;p+s+1,q-s-1]\end{aligned} \tag{8}$$

From these we obtain:

Lemma 7.1 $[x;p,q]=pq[x;1,1]-(pq-p)[x;1,0]-(pq-q)[x;0,1]+(pq-p-q+1)[x;0,0]$.

Proof The first of the relations (8) implies $[x;p,q]=[x;p,q-1]+[x;p,1]-[x;p,0]$ and applying this repeatedly shows

$$[x;p,q]=q[x;p,1]-(q-1)[x;p,0]. \tag{9}$$

The second equation of (8) implies similarly that $[x;p,q]=p[x;1,q]-(p-1)[x;0,q]$ and using this to expand each of the terms on the right of (9) gives the desired equation. □

Up to this point we have only used consequences of the five-term relation and not used the transfer relation (3). We digress briefly to show that the transfer relation almost follows from the five term relation.

Proposition 7.2 *If $\widehat{\mathcal{P}}'(\mathbb{C})$ and $\widehat{\mathcal{B}}'(\mathbb{C})$ are defined like $\widehat{\mathcal{P}}(\mathbb{C})$ and $\widehat{\mathcal{B}}(\mathbb{C})$ but without the transfer relation, then in $\widehat{\mathcal{P}}'(\mathbb{C})$ the element $\kappa:=[x;1,1]+[x;0,0]-[x;1,0]-[x;0,1]$ is independent of x and has order 2. Moreover, $\widehat{\mathcal{P}}'(\mathbb{C})=\widehat{\mathcal{P}}(\mathbb{C})\times C_2$ and $\widehat{\mathcal{B}}'(\mathbb{C})=\widehat{\mathcal{B}}(\mathbb{C})\times C_2$, where C_2 is the cyclic group of order 2 generated by κ.*

Proof If we subtract equation (7) with $p_0=p_1=q_0=q_1=q_2=1$ from the same equation with $p_0=p_1=0$, $q_0=q_1=q_2=1$ we obtain $[x;1,1]-[y;1,1]-[x;0,1]+[y;0,1]=[x;1,0]-[y;1,0]-[x;0,0]+[y;0,0]$, which rearranges to show that κ is independent of x. The last of the equations (8) with $p=q=0$ and $s=-1$ gives $2[x;0,0]=[x;1,-1]+[x;-1,1]$ and expanding the right side of this using Lemma 7.1 gives $2[x;0,0]=-2[x;1,1]+2[x;1,0]+2[x;0,1]$, showing that κ has order dividing 2.

To show κ has order exactly 2 we note that there is a homomorphism $\epsilon\colon \widehat{\mathcal{P}}'(\mathbb{C}) \to \mathbb{Z}/2$ defined on generators by $[z;p,q] \mapsto (pq \bmod 2)$. Indeed, it is easy to check that this vanishes on the lifted five-term relation, and is thus well defined on $\widehat{\mathcal{P}}'(\mathbb{C})$. Since $\epsilon(\kappa)=1$ we see κ is non-trivial. Finally, Lemma 7.1 implies that the effect of the transfer relation is simply to kill the element κ, so the final sentence of the proposition follows. $\square$

Lemma 7.3 *For any* $[x;p,q] \in \widehat{\mathcal{P}}(\mathbb{C})$ *one has* $[x;p,q]+[1-x;-q,-p] = 2[1/2;0,0]$.

Proof Assume first that $0<y<x<1$. Then, as remarked in the proof of Proposition 2.5,

$$[x;p_0,q_0]-[y,p_1,q_1]+[\frac{y}{x};p_1-p_0,q_2]-[\frac{1-x^{-1}}{1-y^{-1}};p_1-p_0+q_1-q_0,q_2-q_1]+ [\frac{1-x}{1-y};q_1-q_0,q_2-q_1-p_0]=0$$

is an instance of the lifted five term relation. Replacing y by $1-x$, x by $1-y$, p_0 by $-q_1$, q_1 by $-q_0$, q_0 by $-p_1$, q_1 by $-p_0$, and q_2 by $q_2-q_1-p_0$ replaces this relation by exactly the same relation except that the first two terms are replaced by $[1-y;-q_1,-p_1]-[1-x;-q_0,-p_0]$. Thus subtracting the two relations gives:

$$[x;p_0,q_0]-[y;p_1,q_1]-[1-y;-q_1,-p_1]+[1-x;-q_0,-p_0]=0.$$

Putting $[y;p_1,q_1]=[1/2;0,0]$ now proves the lemma for $1/2<x<1$. But since we have shown this as a consequence of the lifted five term relation, we can analytically continue it over the whole of $\widehat{\mathbb{C}}$. $\square$

Proposition 7.4 *The following sequence is exact:*

$$0 \to \mathbb{C}^* \xrightarrow{\chi} \widehat{\mathcal{P}}(\mathbb{C}) \to \mathcal{P}(\mathbb{C}) \to 0$$

where $\widehat{\mathcal{P}}(\mathbb{C}) \to \mathcal{P}(\mathbb{C})$ *is the natural map and* $\chi(z) := [z;0,1]-[z;0,0]$ *for* $z\in\mathbb{C}^*$.

Proof Denote $\{z,p\} := [z;p,q]-[z;p,q-1]$ which is independent of q by the first equation of (8). By Lemma 7.3 we have $[z;p,q]-[z;p-1,q]=-\{1-z,-q\}$. It follows that elements of the form $\{z,p\}$ generate $\operatorname{Ker}\big(\widehat{\mathcal{P}}(\mathbb{C})\to\mathcal{P}(\mathbb{C})\big)$. Computing $\{z,p\}$ using Lemma 7.1 and the transfer relation, one finds $\{z,p\}=\{z,0\}$ which only depends on z. Thus the elements $\{z,0\}=\chi(z)$ generate $\operatorname{Ker}\big(\widehat{\mathcal{P}}(\mathbb{C})\to\mathcal{P}(\mathbb{C})\big)$. If we take equation (7) with even p_i and subtract the same equation with the q_i reduced by 1 we get an equation that says that $\chi\colon \mathbb{C}^* \to \operatorname{Ker}\big(\widehat{\mathcal{P}}(\mathbb{C}) \to \mathcal{P}(\mathbb{C})\big)$ is a homomorphism. We have just shown that it is surjective, and it is injective because $R \circ \chi$ is the map $\mathbb{C}^* \to \mathbb{C}/\pi^2$ defined by $z \mapsto \frac{\pi i}{2} \log z$. □

We can now describe the relationship of our extended groups with the "classical" ones.

Theorem 7.5 *There is a commutative diagram with exact rows and columns*

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \mu^* \arrow[r] \arrow[d, "\chi|\mu^*"'] & \mathbb{C}^* \arrow[r] \arrow[d, "\chi"'] & \mathbb{C}^*/\mu^* \arrow[r] \arrow[d, "\beta"'] & 0 \arrow[d] & \\
0 \arrow[r] & \widehat{\mathcal{B}}(\mathbb{C}) \arrow[r] \arrow[d] & \widehat{\mathcal{P}}(\mathbb{C}) \arrow[r, "\nu"] \arrow[d] & \mathbb{C} \wedge \mathbb{C} \arrow[r] \arrow[d, "\epsilon"'] & K_2(\mathbb{C}) \arrow[r] \arrow[d, "="'] & 0 \\
0 \arrow[r] & \mathcal{B}(\mathbb{C}) \arrow[r] \arrow[d] & \mathcal{P}(\mathbb{C}) \arrow[r, "\nu'"] \arrow[d] & \mathbb{C}^* \wedge \mathbb{C}^* \arrow[r] \arrow[d] & K_2(\mathbb{C}) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 & 0 &
\end{tikzcd}$$

Here μ^* is the group of roots of unity and the labeled maps defined as follows:

$$\begin{aligned}
\chi(z) &= [z;0,1] - [z;0,0] \in \widehat{\mathcal{P}}(\mathbb{C});\\
\nu[z;p,q] &= (\log z + p\pi i) \wedge (-\log(1-z) + q\pi i);\\
\nu'[z] &= 2\big(z \wedge (1-z)\big);\\
\beta[z] &= \log z \wedge \pi i;\\
\epsilon(w_1 \wedge w_2) &= -2(e^{w_1} \wedge e^{w_2});
\end{aligned}$$

and the unlabeled maps are the obvious ones.

Proof The top horizontal sequence is trivially exact while the other two are exact at their first two non-trivial groups by definition of $\widehat{\mathcal{B}}$ and $\mathcal{B}$. The bottom row is exact also at its other two places by Milnor's definition of K_2. The exactness of the third vertical sequence is elementary and the second one has just been proved. The commutativity of all but the top left square is elementary. A diagram chase confirms that χ maps μ^* to $\widehat{\mathcal{B}}(\mathbb{C})$ and that the left vertical sequence is also exact. Another confirms exactness of the middle row. □

8 The more extended Bloch group

This section is a digression. We describe a slightly more natural looking variant of the extended Bloch group, based on the universal abelian cover X_{00} of $\mathbb{C}-\{0,1\}$. It turns out to be a $\mathbb{Z}/2$ extension of $\widehat{\mathcal{B}}(\mathbb{C})$. We are not sure of its significance, so we describe it briefly.

Recall that $\widehat{\mathbb{C}}$ consists of four components X_{00}, X_{01}, X_{10}, and X_{11}, of which X_{00} is naturally the universal abelian cover of $\mathbb{C}-\{0,1\}$.

Let $\widehat{\mathrm{FT}}_{00}$ be $\widehat{\mathrm{FT}}\cap(X_{00})^5$, so $\widehat{\mathrm{FT}}_{00}=\widehat{\mathrm{FT}}_0+2V$ in the notation of Definition 2.2. As mentioned earlier, $\widehat{\mathrm{FT}}_{00}$ is, in fact, equal to $\widehat{\mathrm{FT}}_0$, but we do not need this.

Define $\mathcal{EP}(\mathbb{C})$ to be the free $\mathbb{Z}$–module on X_{00} factored by all instances of the relation (we do not need a "transfer relation"):

$$\sum_{i=0}^{4}(-1)^i(x_i;2p_i,2q_i)=0 \quad\text{with } \big((x_0;2p_0,2q_0),\dots,(x_4;2p_4,2q_4)\big)\in\widehat{\mathrm{FT}}_{00}. \tag{10}$$

As before, we have a well-defined map

$$\nu\colon\ \mathcal{EP}(\mathbb{C})\to\mathbb{C}\wedge\mathbb{C},$$

given by $\nu[z;2p,2q]=(\log z+2p\pi i)\wedge(-\log(1-z)+2q\pi i)$, and we define

$$\mathcal{EB}(\mathbb{C}):=\operatorname{Ker}\nu. \tag{11}$$

The proof of Proposition 2.5 shows:

Proposition 8.1 *The function $R(z;2p,2q):=\mathcal{R}(z)+\pi i(p\log(1-z)+q\log z)-\frac{\pi^2}{6}$ gives a well defined map $X_{00}\to\mathbb{C}/2\pi^2\mathbb{Z}$ and induces a homomorphism $R\colon\ \mathcal{EP}(\mathbb{C})\to\mathbb{C}/2\pi^2\mathbb{Z}$.* □

We can repeat the computations in Section 6 word-for-word, replacing anything of the form $[x;p,q]$ by $[x;2p,2q]$, to show that $\operatorname{Ker}\big(\mathcal{EP}(\mathbb{C})\to\mathcal{P}(\mathbb{C})\big)$ is generated by elements of the form $\hat{\chi}(z):=[z;0,2]-[z;0,0]$. We thus get:

Proposition 8.2 *The following sequence is exact:*

$$0\to\mathbb{C}^*\xrightarrow{\hat{\chi}}\mathcal{EP}(\mathbb{C})\to\mathcal{P}(\mathbb{C})\to 0$$

where $\widehat{\mathcal{P}}(\mathbb{C})\to\mathcal{P}(\mathbb{C})$ is the natural map and $\hat{\chi}(z):=[z;0,2]-[z;0,0]$ for $z\in\mathbb{C}^*$.

Proof The only thing to prove is the injectivity of $\hat{\chi}$ which follows by noting that $R\big(\hat{\chi}(z)\big) = \pi i \log z \in \mathbb{C}/2\pi^2$. □

Corollary 8.3 *We have a commutative diagram with exact rows and columns:*

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & \mathbb{Z}/2 \arrow[r, "="] \arrow[d] & \mathbb{Z}/2 \arrow[d] & & \\
0 \arrow[r] & \mathbb{C}^* \arrow[r, "\hat{\chi}"] \arrow[d, "z\mapsto z^2"] & \mathcal{EP}(\mathbb{C}) \arrow[r] \arrow[d] & \mathcal{P}(\mathbb{C}) \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & \mathbb{C}^* \arrow[r, "\chi"] \arrow[d] & \widehat{\mathcal{P}}(\mathbb{C}) \arrow[r] \arrow[d] & \mathcal{P}(\mathbb{C}) \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}
$$

and analogously for the Bloch group:

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & \mathbb{Z}/2 \arrow[r, "="] \arrow[d] & \mathbb{Z}/2 \arrow[d] & & \\
0 \arrow[r] & \mu^* \arrow[r, "\hat{\chi}"] \arrow[d, "z\mapsto z^2"] & \mathcal{EB}(\mathbb{C}) \arrow[r] \arrow[d] & \mathcal{B}(\mathbb{C}) \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & \mu^* \arrow[r, "\chi"] \arrow[d] & \widehat{\mathcal{B}}(\mathbb{C}) \arrow[r] \arrow[d] & \mathcal{B}(\mathbb{C}) \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}
$$

□

Question Is $\mathcal{EB}(\mathbb{C})$ related to $H_3(\mathrm{SL}(2,\mathbb{C});\mathbb{Z})$? Do $\mathrm{SL}(2,\mathbb{C})$–labeled ordered 3–cycles have flattenings using only X_{00}?

9 Proof of Theorem 4.5

In this section we will prove Theorem 4.5, which says that a strong flattening of any G–labeled ordered 3–cycle K exists.

This proof, and the first two lemmas in the next section, depend heavily on [13]. We must first therefore recall some notation and results from there. (The results of [13] do not require as strong conditions on K as we have here: there, the ordering is not needed and K is only required to be a quasi-simplicial complex.)

Recall that K is an ordered 3–cycle: the vertices of each 3–simplex are ordered with orderings agreeing on common faces. The underlying space $|K|$ is an oriented 3–manifold except possibly at 0–simplices, where it is topologically the cone on a connected oriented surface, L_v, the link of v (see Definitions 4.1 and 4.2).

To an oriented 3–simplex Δ of K we associate a 2–dimensional bilinear space J_Δ over $\mathbb{Z}$ as follows. As a $\mathbb{Z}$–module J_Δ is generated by the six edges $e_0, \dots, e_5$ of Δ (see Figure 5) with the relations:

$$e_i - e_{i+3} = 0 \quad \text{for } i = 0, 1, 2.$$
$$e_0 + e_1 + e_2 = 0.$$

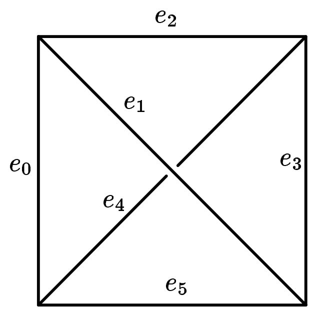

Figure 5

Thus, opposite edges of Δ represent the same element of J_Δ, so J_Δ has three "geometric" generators, and the sum of these three generators is zero. The bilinear form on J_Δ is the non-singular skew-symmetric form given by

$$\langle e_0, e_1\rangle = \langle e_1, e_2\rangle = \langle e_2, e_0\rangle = -\langle e_1, e_0\rangle = -\langle e_3, e_1\rangle = -\langle e_0, e_2\rangle = 1.$$

Let J be the direct sum $\coprod J_\Delta$, summed over the oriented 3–simplices of K. For $i = 0, 1$ let C_i be the free $\mathbb{Z}$–module on the *unoriented* i–simplices of K. Define homomorphisms

$$\alpha\colon C_0 \longrightarrow C_1 \quad \text{and} \quad \beta\colon C_1 \longrightarrow J$$

as follows. α takes a vertex to the sum of the incident 1–simplices (with a 1–simplex counted twice if both endpoints are at the given vertex). The J_Δ component of β takes a 1–simplex E of K to the sum of those edges e_i in the edge set $\{e_0, e_1, \dots, e_5\}$ of Δ which are identified with E in K.

The natural basis of C_i gives an identification of C_i with its dual space and the bilinear form on J gives an identification of J with its dual space. With respect to these identifications, the dual map

$$\alpha^*\colon C_1 \longrightarrow C_0$$

is easily seen to map a 1–simplex E of K to the sum of its endpoints, and the dual map

$$\beta^*\colon J \longrightarrow C_1$$

can be described as follows. To each 3–simplex Δ of K we have a map $j = j_\Delta$ of the edge set $\{e_0, e_1, \dots, e_5\}$ of Δ to the set of 1–simplices of K: put $j(e_i)$ equal to the 1–simplex that e_i is identified with in K. For e_i in J_Δ we have

$$\beta^*(e_i) = j(e_{i+1}) - j(e_{i+2}) + j(e_{i+4}) - j(e_{i+5}) \quad \text{(indices mod 6)}.$$

Let K_0 be the result of removing a small open cone neighborhood of each 0–simplex v of K, so ∂K_0 is the disjoint union of the links L_v of the vertices of K.

Theorem 9.1 ([13], Theorem 4.2) *The sequence*

$$\mathcal{J}\colon \quad 0 \longrightarrow C_0 \xrightarrow{\alpha} C_1 \xrightarrow{\beta} J \xrightarrow{\beta^*} C_1 \xrightarrow{\alpha^*} C_0 \longrightarrow 0$$

is a chain complex. Its homology groups $H_i(\mathcal{J})$ (indexing the non-zero groups of $\mathcal{J}$ from left to right with indices $5,4,3,2,1$) are

$$H_5(\mathcal{J}) = 0, \quad H_4(\mathcal{J}) = \mathbb{Z}/2, \quad H_1(\mathcal{J}) = \mathbb{Z}/2,$$
$$H_3(\mathcal{J}) = \mathcal{H} \oplus H^1(K;\mathbb{Z}/2), \quad H_2(\mathcal{J}) = H_1(K;\mathbb{Z}/2),$$

where $\mathcal{H} = \operatorname{Ker}(H_1(\partial K_0;\mathbb{Z}) \to H_1(K_0;\mathbb{Z}/2))$. Moreover, the isomorphism $H_2(\mathcal{J}) \to H_1(K;\mathbb{Z}/2)$ results by interpreting an element of $\operatorname{Ker}(\alpha^*) \subset C_1$ as an unoriented 1-cycle in K. □

Proof of Theorem 4.5: strong flattening exists Give each simplex Δ_i of the complex K the flattening $\mathbf{w}_i^{(0)} := \ell(x_i;0,0)$. This choice will not in general satisfy the conditions for a strong flattening of K, so we need to describe how to modify it so that the conditions are satisfied.

Recall that to each 3–simplex Δ_i of K is associated a sign $\epsilon_i = \pm 1$ that says whether the vertex-ordering of Δ_i agrees or disagrees with the orientation Δ_i inherits from the orientation of K.

If Δ is an ideal simplex and $\mathbf{w} = (w_0, w_1, w_2)$ is a flattening of it, then denote

$$\xi(\mathbf{w}) := w_1 e_0 - w_0 e_1 \in J_\Delta \otimes \mathbb{C}.$$

This definition is only apparently unsymmetrical since $w_1e_0 - w_0e_1 = w_2e_1 - w_1e_2 = w_0e_2 - w_2e_0$. Denote by ω the element of $J \otimes \mathbb{C}$ whose Δ_i–component is $\epsilon_i\xi(\mathbf{w}_i^{(0)})$ for each i. That is, the Δ_i–component of ω is $-\epsilon_i\big(\log(1-x_i)e_0 + \log(x_i)e_1\big)$.

Lemma 9.2 *$\frac{1}{\pi i}\beta^*(\omega)$ is an integer class in the kernel of α^*, so it represents an element of the homology group $H_2(\mathcal{J})$. Moreover this element in $H_2(\mathcal{J})$ vanishes, so $\frac{1}{\pi i}\beta^*(\omega) = \beta^*(\delta)$ for some $\delta \in J$.*

Proof Let $\overline{J}_\Delta$ be defined like J_Δ but without the relation $e_0 + e_1 + e_2 = 0$, so it is generated by the six edges $e_0, \dots, e_5$ of Δ with relations $e_i = e_{i+3}$ for $i = 0,1,2$. Let $\overline{J}$ be the direct sum $\coprod \overline{J}_\Delta$ over 3–simplices Δ of K.

The map $\beta^*\colon J \to C_2$ factors as

$$\beta^*\colon\ J \xrightarrow{\beta_1} \overline{J} \xrightarrow{\beta_2} C_2,$$

with β_1 and β_2 defined on each component by:

$$\left.\begin{aligned} \beta_1(e_i) &= e_{i+1} - e_{i+2} \\ \beta_2(e_i) &= j(e_i) + j(e_{i+3}) \end{aligned}\right\} \text{ for } i = 0,1,2.$$

Note that $\beta_1(\xi(\mathbf{w})) = w_0e_0 + w_1e_1 + w_2e_2 \in \overline{J}_\Delta \otimes \mathbb{C}$. Thus if E is a 1–simplex of K then the E–component of $\beta^*(\omega) = \beta_2\beta_1(\omega)$ is the signed sum of the log-parameters for E in the ideal simplices of K around E and is hence a multiple of $2\pi i$ by Proposition 5.2. That is,

$$\frac{1}{\pi i}\beta^*(\omega) \in 2C_2. \tag{12}$$

The lemma follows, since the isomorphism $H_2(\mathcal{J}) \to H_1(K;\mathbb{Z}/2)$ is the map which interprets an element of $\mathrm{Ker}(\alpha^*)$ as an unoriented 1-cycle in K, and equation (12) says this 1-cycle is zero modulo 2. □

Let $\omega' := \omega - \pi i \delta \in J \otimes \mathbb{C}$ with δ as in the lemma, so $\beta^*(\omega') = 0$. The Δ_i–component of ω' is $\epsilon_i \xi(\mathbf{w}_i)$, where $\mathbf{w}_i = \ell(x_i; p_i, q_i)$ with the integers p_i, q_i determined by the coefficients occurring in the element $\delta \in J$. The element $\delta \in J$ is only determined by the lemma up to elements of $\mathrm{Ker}(\beta^*)$. We want to show that for suitable choice of δ, the $\mathbf{w}_i$ satisfy the parity and log-parameter conditions of the definition of strong flattening (Definition 4.4). We will need to review a computation of $H_3(\mathcal{J})$ from [13].

We define a map $\gamma'\colon H_3(\mathcal{J}) \to H^1(\partial K_0; \mathbb{Z}) = \mathrm{Hom}(H_1(\partial K_0), \mathbb{Z})$ as follows. Given elements $a \in H_3(\mathcal{J})$ and $c \in H_1(\partial K_0)$ we wish to define $\gamma'(a)(c)$. It is enough to do this for a class c which is represented by a normal path C in the link of some vertex of K. Represent a by an element $A \in J$ with $\beta^*(A) = 0$ and consider the element $\beta_1(A) \in \overline{J}$. This element has a coefficient for each edge of each simplex of K. To define $\gamma'(a)(c)$ we consider the coefficients of $\beta_1(A)$ corresponding to edges of simplices that C passes and sum these using the orientation conventions of Definition 4.3. It is easy to see that the result only depends on the homology class of C.

We can similarly define a map $\gamma_2'\colon H_3(\mathcal{J}) \to H^1(K_0; \mathbb{Z}/2) = \mathrm{Hom}(H_1(K_0), \mathbb{Z}/2)$ by using normal paths in K_0 and taking modulo 2 sum of coefficients of $\beta_1(A)$.

Lemma 9.3 ([13], Theorem 5.1) *The sequence*

$$0 \to H_3(\mathcal{J}) \xrightarrow{(\gamma', \gamma_2')} H^1(\partial K_0; \mathbb{Z}) \oplus H^1(K_0; \mathbb{Z}/2) \xrightarrow{r - i^*} H^1(\partial K_0; \mathbb{Z}/2) \to 0$$

is exact, where $r\colon H^1(\partial K_0; \mathbb{Z}) \to H^1(\partial K_0; \mathbb{Z}/2)$ is the coefficient map and the map $i^*\colon H^1(K_0; \mathbb{Z}/2) \to H^1(\partial K_0; \mathbb{Z}/2)$ is induced by the inclusion $\partial K_0 \to K_0$. □

Returning to the choice of δ above, assume we have made a choice so that the resulting flattenings $\mathbf{w}_i$ do not lead to zero log-parameters and parities for normal paths. Taking $\frac{1}{\pi i}$ times the log-parameters of normal paths leads as above to an element $c \in H^1(\partial K_0; \mathbb{Z})$. Similarly, parities of normal paths leads to an element of $c_2 \in H^1(K_0; \mathbb{Z}/2)$. These elements satisfy $r(c) = i^*(c_2)$. The lemma thus gives an element of $H_3(\mathcal{J})$ that maps to (c, c_2). Subtracting a representative for this element from δ gives the desired correction of δ so the log-parameters and parities of normal paths with respect to the corresponding changed $\mathbf{w}_i$'s are zero.

This completes the proof that a strong flattening of K exists: Theorem 4.5. □

10 Start of the proof of Theorem 4.6

Lemma 10.1 *The choice of strong flattening of K does not affect the resulting element $\sum_i \epsilon_i[x_i;p_i,q_i] \in \widehat{\mathcal{P}}(\mathbb{C})$.*

Proof If we have a different choice of flattenings $\mathbf{w}_i$ satisfying the parity and log-parameter conditions for a strong flattening of K then Lemma 9.3 implies that the difference between the corresponding elements δ represents 0 in $H_3(\mathcal{J})$. It is thus in the image of β. For $E \in C_2$ the effect of changing δ by $\beta(E)$ is to change the element $\sum_i \epsilon_i[x_i;p_i,q_i] \in \widehat{\mathcal{P}}(\mathbb{C})$ by the cycle relation about E of Lemma 6.1. Since this is a consequence of the lifted five term relations, the element in $\widehat{\mathcal{P}}(\mathbb{C})$ is unchanged. □

Ultimately we will want to see that $\sum_i \epsilon_i[x_i;p_i,q_i]$ is independent of flattening rather than strong flattening. This will follow in Section 11. The following lemma only needs flattening and not strong flattening.

Lemma 10.2 *Given a flattening of K, the element $\sum_i \epsilon_i[x_i;p_i,q_i] \in \widehat{\mathcal{P}}(\mathbb{C})$ lies in $\widehat{\mathcal{B}}(\mathbb{C})$.*

Proof For any $\mathbb{Q}$–vector-space V, the skew-symmetric bilinear form $\langle\ \rangle$ on J induces a symmetric bilinear map

$$B\colon\ (J\otimes V)\otimes(J\otimes V)\to V\wedge V\,,\quad (a\otimes v)\otimes(b\otimes w)\mapsto\langle a,b\rangle\, v\wedge w\,.$$

Theorem 4.1 of [13] says that after tensoring with $\mathbb{Q}$, the bilinear form $\langle,\rangle$ on $\operatorname{Ker}(\beta^*\colon\ J\otimes\mathbb{Q}\to C_1\otimes\mathbb{Q})$ induces two times the intersection form on $H_1(\partial K_0;\mathbb{Q}) = \operatorname{Ker}\beta^*/\operatorname{Im}\beta$. Hence, on $(\operatorname{Ker}\beta^*/\operatorname{Im}\beta)\otimes V = H_1(\partial K_0)\otimes V = H_1(\partial K_0;V)$, the above bilinear map B induces the map

$$B'\colon\ (H_1(\partial K_0)\otimes V)\otimes(H_1(\partial K_0)\otimes V)\to V\wedge V$$

given by

$$([a]\otimes v)\otimes([b]\otimes w)\mapsto 2([a]\cdot[b])\, v\wedge w$$

where $[a]\cdot[b]$ is intersection form.

We take $V=\mathbb{C}$. For our element $w' = w-\pi i\delta\in J\otimes\mathbb{C}$ we have

$$B(w',w') = 2\sum\nolimits_i \epsilon_i(\log x_i + p_i\pi i)\wedge(-\log(1-x_i)+q_i\pi i)\in\mathbb{C}\wedge\mathbb{C}.$$

Thus, we want to show that $B(w',w')=0$. If we have a strong flattening, w' is in $\operatorname{Ker}(\beta^*)\otimes\mathbb{C}$ and represents zero in $H_1(\partial K_0;\mathbb{C})$. Thus $B(w',w') =$

$B'([w'],[w']) = B'(0,0) = 0$, as desired. If the flattening is not strong, then rather than w' representing zero in $H_1(\partial K_0;\mathbb{C})$, it represents πi times an integral homology class α say, and we still have $B(w',w') = B'([w'],[w']) = (\alpha\cdot\alpha)\pi i \wedge \pi i = 0$. □

At this point we have shown that the element $\sum_i \epsilon_i[x_i;p_i,q_i]$ determined by a strong flattening of K lies in $\widehat{\mathcal{B}}(\mathbb{C})$ and is independent of choice of strong flattening. In the remainder of this section we show that it depends only on $|K|$ and the flat G–bundle over it.

Lemma 10.3 *Changing the choice of the point $z \in \partial\overline{\mathbb{H}}^3$ in the definition of strong flattening (Definition 4.4) does not change the element $\sum_i \epsilon_i[x_i;p_i,q_i] \in \widehat{\mathcal{B}}(\mathbb{C})$.*

Proof This is a special case of the following lemma, since if we change the 1–cocycle corresponding to a G–labeling by the coboundary of the constant 0–cochain with value g, the effect is the same as replacing z by gz (see subsection 4.1.2). □

Lemma 10.4 *If we change the G–labeling of K by changing the corresponding 1–cocycle by a coboundary (subsection 4.1.2) then the element $\sum_i \epsilon_i[x_i;p_i,q_i] \in \widehat{\mathcal{B}}(\mathbb{C})$ does not change.*

Proof The corresponding result is known for the element in $\mathcal{B}(\mathbb{C})$, so the change is in $\mathrm{Ker}(\widehat{\mathcal{B}}(\mathbb{C}) \to \mathcal{B}(\mathbb{C})) = \mathbb{Q}/\mathbb{Z}$. Thus, if we knew that the coboundary action of 0–cochains was continuous using the standard topology of $G = \mathrm{PSL}(2,\mathbb{C})$ the Lemma would follow. This continuity seems "self-evident" but we do not know an easier proof than what follows, which directly proves the local constancy of the element of $\widehat{\mathcal{B}}(\mathbb{C})$ under this action.

It suffices to prove the lemma for the change given by the coboundary action of a 0–cochain τ that takes the value 1 on all 0–simplices except one. Denote that one 0–simplex by v. The effect is that the ideal simplex corresponding to a simplex Δ of K is unchanged if v is not one of its vertices, while if it has vertices v, v_1, v_2, v_3 with G–labels g, g_1, g_2, g_3 before the change, then the corresponding ideal simplex has vertices gz, g_1z, g_2z, g_3z before the change and $g\tau(v)z, g_1z, g_2z, g_3z$ after. We assume $\tau(v)$ is sufficiently close to 1 that none of these latter simplices are degenerate (two vertices equal).

Since the flattening condition on K is a discrete condition (it can only fail by multiples of πi), it will stay valid if we vary strong flattenings of simplices continuously as $\tau(v)\in G$ varies (considering $G=\mathrm{PSL}(2,\mathbb{C})$ with its usual topology, rather than as a discrete group). Thus we get a continuous family of strong flattenings of K as $\tau(v)$ varies in a neighborhood of $1\in G$. We must show that any two of them lead to equal elements of $\widehat{\mathcal{B}}(C)$, ie, they can be related using lifted five-term relations. Since only simplices in the star N of v are affected, it suffices to show that

$$\sum_{\Delta_i\subset N}\epsilon_i([x_i;p_i,q_i]-[x'_i;p'_i,q'_i])=0\in\widehat{\mathcal{P}}(\mathbb{C}),\tag{13}$$

where $\ell([x_i;p_i,q_i])$ and $\ell([x'_i;p'_i,q'_i])$ are the flattenings of Δ_i before and after changing the G–label of v by $\tau(v)$.

Denote the 0–simplices in the simplicial link of v by $v_1,\dots,v_m$, and let their G–labels be $g_1,\dots,g_m$. For each ordered 2–simplex $\langle v_i,v_j,v_k\rangle$ in the simplicial link of v we wish to give a lifted five-term relation based on the five points gz, $g\tau(v)z$, g_iz, g_jz, g_kz so that when we sum these relations (with appropriate signs) we get the left side of equation (13).

We already have flattenings of the two simplices

$$\langle gz,g_1z,g_2z,g_3z\rangle\quad\text{and}\quad\langle g\tau(v)z,g_1z,g_2z,g_3z\rangle.$$

We wish to find flattenings of the other three simplices

$$\langle gz,g\tau(v)z,g_jz,g_kz\rangle,\quad\langle gz,g\tau(v)z,g_iz,g_kz\rangle,\quad\langle gz,g\tau(v)z,g_iz,g_jz\rangle$$

so that we have an instance of the five-term relation.

Consider the appropriate signed sum of the three adjusted angles about the 1–simplex $\langle g_jz,g_kz\rangle$ in the the flattening condition of the five-term relation. This sum is a multiple of π which should be zero. The two adjusted angles at this edge in $\langle gz,g_iz,g_jz,g_kz\rangle$ and $\langle g\tau(v)z,g_iz,g_jz,g_kz\rangle$ almost cancel in the sum. Since we may assume $\tau(v)$ is close to 1, the angle at the edge $\langle g_jz,g_kz\rangle$ in the simplex $\langle gz,g\tau(v)z,g_jz,g_kz\rangle$ will be small. Thus the adjustment to this angle must be zero. Similarly for the $\langle g_iz,g_kz\rangle$ edge of $\langle z,\tau(v)z,g_iz,g_kz\rangle$ and the $\langle g_iz,g_jz\rangle$ edge of $\langle z,\tau(v)z,g_iz,g_jz\rangle$. In each of these three simplices the adjusted angle $\langle z,\tau(v)z\rangle$ equals the adjusted angle at the opposite edge and is hence small, so the sum of these adjusted angles must be 0 (since it is a multiple of π) so the flattening condition holds also at the edge $\langle z,\tau(v)z\rangle$.

It remains to choose the angle adjustments at edges of the form $\langle\tau(v)z,g_iz\rangle$ of these simplices. We use the description of flattenings in terms of rotation levels on edges described at the end of section 5. So we position the developing map restricted to N so that vertex v goes to $gz=\infty$. Then $g\tau(v)z$ is close to ∞, ie, very far from the points $g_\nu z$ for $\nu = 1, \ldots, m$. Thus the segments joining $g\tau(v)z$ to the points $g_\nu z$ will be almost mutually parallel, so we can choose the rotation levels of these segments to be almost equal to each other. With such a choice it is clear by inspection that the remaining flattening conditions for the desired five-term relations hold. When we sum these five-term relations with appropriate signs, each flattened simplex with vertices of the form $gz = \infty, g\tau(v)z, g_i z, g_j z$ occurs in two of the relations and cancels and we are left with the desired relation (13).

This proves the existence of the desired lifted five-term relations when $\tau(v)$ is close to 1. This shows that the element of $\widehat{\mathcal{B}}(\mathbb{C})$ is locally constant under the coboundary action of 0–cochains, and since the space $G^{K^{(0)}}$ of 0–cochains is connected, the Lemma follows. $\square$

Remark 10.5 The above lemma can be interpreted to say that, instead of choosing a single $z \in \partial\mathbb{H}^3$ in Definition 4.4 to obtain the ideal simplex shapes for the simplices of K, we can choose a different z for each 0–simplex of K, and the element of $\widehat{\mathcal{B}}(\mathbb{C})$ is unaffected. This will be important when discussing the applications to 3–manifolds in Section 14.

We have been assuming that K is a simplicial rather than quasi-simplicial complex, but so far we have not really used this. We will initially prove Theorem 4.6 under the following assumption, which we will eliminate again in Proposition 11.2.

Assumption K is a simplicial complex and the vertex orderings of the simplices of K are inherited from an ordering of the set of 0–simplices of K.

Lemma 10.6 *Changing the ordering of the 0–simplices of K does not change the element $\sum_i \epsilon_i[x_i; p_i, q_i] \in \widehat{\mathcal{B}}(\mathbb{C})$.*

Proof The full permutation group on $K^{(0)}$ is generated by transpositions on adjacent elements with respect to the ordering, so we need only consider such transpositions. If the two 0–simplices are not joined by a 1–simplex then the the transposition has no effect on the vertex-ordering of any 3–simplex so there is nothing to prove. Assume therefore that v and w are two 0–simplices that are adjacent in the ordering of $K^{(0)}$ and are joined by a 1–simplex. Only the 3–simplices that have $\langle v, w\rangle$ as an edge have their vertex-ordering changed by the transposition, so we just consider these simplices. The configuration of these simplices is as in Lemma 6.1 and the same inductive argument used in the proof of that lemma lets us deduce the lemma from the case that there are just three 3–simplices about this 1–simplex and their vertex orderings are induced from an ordering of the five 0–simplices involved. In the proof of Lemma 6.1 it is shown that there is then a lifted five-term relation which replaces these three simplices by two. After doing so, v and w no longer joined by an edge, so the transposition no longer has an effect on the vertex-order of any three-simplex, so the lemma follows. □

Finally for this section, we show that we can change the triangulation of K by Pachner moves without changing the element of $\widehat{\mathcal{B}}(\mathbb{C})$. These are the moves ($1 \to 4$, $2 \to 3$, $3 \to 2$ and $4 \to 1$ on numbers of simplices) that replace a union of 3–simplices of K that is combinatorially equivalent to part of the boundary of a 4–simplex by the union of the complementary set of 3–simplices in the boundary of the 4–simplex. Only the $1 \to 4$ Pachner move adds a new 0–simplex. In this case, we insert the new 0–simplex anywhere in the ordering of 0–simplices and, if the old simplex had G–labels (g_0, g_1, g_2, g_3), we give the four new simplices G–labels (g_1, g_2, g_3, g), (g_0, g_2, g_3, g), (g_0, g_1, g_3, g), and (g_0, g_1, g_2, g), for some $g \neq g_0, g_1, g_2, g_3$. We are again using the assumption of a global ordering of the 0–simplices of K: if we only had local vertex-orderings of simplices, we could not guarantee that the two 3–simplices produced by a $3 \to 2$ Pachner move have vertex-orderings compatible with adjacent simplices.

Lemma 10.7 *A change of triangulation of K by Pachner moves does not change the represented element $\sum_i \epsilon_i[x_i; p_i, q_i] \in \widehat{\mathcal{B}}(\mathbb{C})$.*

Proof We want to leave unchanged the flattenings on the unaltered simplices of K and put flattenings on the new simplices to get a flattening of the new complex K'. We can do this if and only if the flattenings of the changed simplices (the ones in K that have been replaced and the ones they have been replaced by) give a flattening of the boundary of the 4–simplex (in the sense that log and parity parameters around edges are zero). In particular, the change in the element of $\widehat{\mathcal{B}}(\mathbb{C})$ is then an instance of the lifted five-term relation, and hence zero. We thus need to know that any flattening defined on part of the boundary of a 4–simplex can be extended over the whole 4–simplex. This is the content of Lemma 3.5. □

It follows from Pachner [19] that any two simplicial triangulations of $|K|$ are related by Pachner moves, so at this point we know that the element of $\widehat{\mathcal{B}}(\mathbb{C})$ is determined just by $|K|$ and the flat G–bundle over it. We will actually only need this in the case that $|K|$ is a manifold, which is explicit in Pachner's work. In the next section we reduce to the case that $|K|$ is a manifold and use this to complete the proof of Theorem 4.6.

11 Completion of proof of Theorem 4.6

In this section we show that we can always resolve the singularities of $|K|$ to replace it by a triangulated 3–manifold and then discuss surgery on this 3–manifold to complete the proof of Theorem 4.6.

The underlying space of K is a 3–manifold except at 0–simplices, where it may look locally like the cone on a closed surface of genus ≥ 1. Suppose this occurs at some 0–simplex v. Choose a non-separating simple closed simplicial curve C in the link L_v. This curve is determined by an open disk D in the star of v. We slice $|K|$ open along D and splice in a complex as in Figure 6, gluing the top and the bottom of that complex to the two copies of D resulting from slicing $|K|$. This introduces a new 0–simplex v' in K. We give v' a G–label distinct from that of each adjacent 0–simplex. We order 0–simplices by inserting v' anywhere (eg, as a new maximal element). We must check that we can put flattenings on the inserted 3–simplices so that the conditions for a strong flattening of our complex still hold.

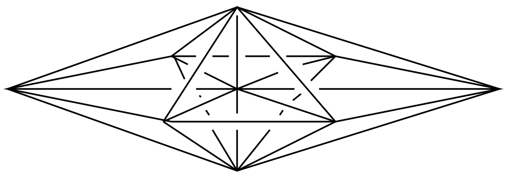

Figure 6: Double cone on a disc

We first show that it suffices to satisfy the log-parameter flattening conditions. Indeed, by Corollary 5.4, if we satisfy the logarithmic flattening conditions, the parity conditions hold around edges. Parity along normal paths then gives a homomorphism from $H_1(|K - K^{(0)}|)$ to $\mathbb{Z}/2$. If we denote the new complex by K' and the curve in K' given by the two vertical 1–simplices by C, then $H_1(|K - K^{(0)}|) = H_1(|K' - K^{(0)}| - C)$ which surjects to $H_1(|K' - (K')^{(0)}|)$ by the inclusion, so the parity condition will hold for all normal paths.

We satisfy the log-parameter strong flattening condition as follows. The added 3–simplices come in pairs with identical G–labelings, adjacent to each other above and below the central disk in the picture. We will give the two simplices of a pair identical flattenings so they cancel in the sum $\sum_i \epsilon_i[x_i, p_i, q_i] \in \widehat{\mathcal{B}}(\mathbb{C})$ (they appear with opposite signs in this sum). Flatten all the added simplices in the top layer in order to satisfy the flattening conditions at the top edges in the picture (there are n degrees of freedom in doing this, where n is the number of 1–simplices around the curve C). We then need only verify the flattening condition around the central vertical 1–simplex. Since the sum of the three log-parameters of a flattened simplex is zero, the appropriately signed sum of log parameters around the central edge is the negative of the signed sum of the $2n$ log-parameters of these simplices at the top edges in the picture. This, in turn, is equal to the log-parameter along a normal path in the star of v determined by a path in the link L_v that runs parallel to C on one side. It is thus zero by the strong flattening condition for such normal paths.

(An alternate argument is to look at the developing image centered at v as in section 5. The flattenings before we change K give us rotation levels on the edges of the image in $\mathbb{C}$. After changing K we have a new point in $\mathbb{C}$ at the image of v' and n segments incident on it, corresponding to the 2–simplices of the central horizontal disk in the picture. Assign any rotation levels to these n segments. As in section 5, this determines flattenings of the 3–simplices so that the flattening conditions are satisfied.)

We call the above procedure "blowing up" at the 0–simplex v. Note that it reduces the genus of the link of v. By repeated blowing up we can thus make $|K|$ into a 3–manifold. The following lemma improves on this.

Lemma 11.1 *If K is just flattened (rather than strongly flattened) then we can blow up to make $|K|$ into a 3–manifold without changing the represented element in $\widehat{\mathcal{B}}\mathbb{C}$. In particular, Theorem 4.6 holds for flattenings if it holds for strong flattenings.*

Proof To apply the above argument to to blow up and simplify the link of a vertex we need a separating curve C in the link so that the log-parameter along a parallel normal curve in the link L_v is zero. Suppose the log-parameter flattening conditions are satisfied around edges, but not necessarily along normal curves. Log-parameters along normal curves then give a homomorphism $H_1(L_v;\mathbb{Z}) \to \mathbb{Z}\pi i \subset \mathbb{C}$. It is well known that for any closed oriented surface S and non-trivial homomorphism $H_1(S;\mathbb{Z}) \to \mathbb{Z}$ there exist elements in the kernel represented by non-separating simple closed curves (in fact, the mapping class group acts transitively on the set of surjective homomorphisms $H_1(S;\mathbb{Z}) \to \mathbb{Z}$). Thus, there exists a curve to use for blowing up, so the lemma follows. □

At this point we may assume our element in $H_3(G;\mathbb{Z})$ is represented by a triangulated 3–manifold with flat G–bundle over it, or equivalently, a 3–manifold $|K|$ with a homotopy class of mappings $|K| \to BG$. Since $H_3(BG;\mathbb{Z}) = \Omega_3(BG)$ (oriented bordism), and bordism is generated by homotopy (ie, maps of $|K| \times I$) and handle addition, it suffices to show that when we modify K by surgery resulting from a handle-addition to $|K| \times I$ the represented element in $\widehat{\mathcal{B}}(\mathbb{C})$ is not changed.

It suffices to consider adding 1–handles and 2–handles, since any bordism of connected 3–manifolds is the composition of bordisms that add 1, 2, and 3 handles in that order, and the latter are inverses of bordisms that add 1–handles.

We first consider adding a 1–handle. Combinatorially, homotoping $|K|$ in BG means changing the G–valued 1–cocycle by the coboundary action. We make such a change so that the handle addition can be realized by gluing a 4–simplex Δ^4 to K by gluing two of the 3–faces of Δ^4 to disjoint 3–simplices Δ^3_1 and Δ^3_2 in K. The common 2–simplex of the two 3–simplices in $\partial\Delta^4$ determines 2–faces of Δ^3_1 and Δ^3_2 that will be glued together in this construction. Our initial change of the G–valued 1–cocycle arranges for the cocycle to match on these two simplices.

The resulting surgery of K is the result of first removing the interiors of Δ^3_1 and Δ^3_2, then gluing the resulting boundary components along the 2–simplices mentioned above to give a manifold with a single 2–sphere boundary triangulated as the suspension of a triangle, and finally gluing in a 3–ball triangulated as three 3–simplices meeting along a common 1–simplex. Call the resulting triangulated manifold K'.

We must make the construction compatible with orderings and provide suitable flattenings. By Lemma 10.6 we may order the vertices of K as $v_1, v_2, \dots$ so that the vertices of Δ^3_1 are v_1, v_3, v_5, v_7 and of Δ^3_2 are v_2, v_4, v_6, v_8 and the 2–simplices that are identified are given by the first three vertices of each in the given order. Then the vertices of K' are $v_1 = v_2, v_3 = v_4, v_5 = v_6, v_7, v_8, \dots$ and the common 3–simplices of K and K' have not changed their vertex orderings.

Each of the two 3–simplices Δ^3_1 and Δ^3_2 had a flattening before the surgery, and we use lemma 3.5 to define the flattening on the three new 3–simplices. The log-parameter part of the flattening condition is then satisfied in K', but the parity condition may not be. The surgery has added a generator to $H_1(K,\mathbb{Z})$ represented by any normal path that passes once through the 2–simplex that we glued. We must check the parity condition along one such path. If it fails, then before performing the surgery we change flattenings of K by the cycle relation of Lemma 6.1 about the 1–simplex v_1v_3. After doing so the parity part of the flattening condition will be satisfied for K'.

Finally, we must consider adding a 2–handle along a simplicial curve in K. We can realize this by doing the reverse of a blow-up (a "blow-down") to collapse the curve to a single vertex v with link a torus, and then blowing up again using a longitude of the curve we collapsed. We need to show, therefore, that if K contains a curve that maps to the trivial element in G under the homomorphism determined by the G–valued 1–cocycle, then we can (after adjusting the triangulation) find a flattening that lets us perform a blow-down to collapse the curve, followed by a blow-up using a longitude of the curve.

By retriangulating we may assume the curve C in question is length 2 and the 3–simplices that have a 1–simplex in common with it are configured as in Figure 6 (with, for concreteness, 6 simplices around each edge of C, as in the figure). In particular, the bottom and top vertices in the figure are the same 0–simplex v in K, and, since the 1–cocyle along the curve gives the trivial element of G, if we G–label the simplices of the figure all with the same label for v' then each pair of vertically adjacent 3–simplices has the same labeling. The rest of the star of v will be the cone on an annulus. In particular, the 1–simplices radiating from v on the top surface of the figure are distinct from the 1–simplices radiating from v on the bottom surface.

We must verify that we can modify the flattenings so that vertically adjacent 3–simplices have the same flattening, in which case we can perform the desired blow-down.

Denote the 1–simplices radiating from v in the bottom surface of the figure $e_1,\dots,e_6$ in order, and the 1–simplices above these in the central disk $e'_1,\dots,e'_6$ (with e'_i above e_i). Denote by Δ_i the 3–simplex in the bottom layer with edges e_i and e_{i+1} (indices mod 6). By applying suitable multiples of the cycle relation of Lemma 6.1 around 1–simplices e_1 and e_2 we can adjust the flattening of Δ_1 arbitrarily, so we can make it match the 3–simplex above it. The flattening condition at 1–simplex e'_2 now implies that the log-parameter of Δ_2 at edge e'_2 matches the simplex above it, and by applying the cycle relation around e_3 we can fix up the flattening of Δ_2 at edge e'_3. Continuing this way, we fix $\Delta_3,\dots,\Delta_5$, at which point the flattening conditions about e'_5 and e'_6 imply that Δ_6 also has its desired flattening.

We can now do the blow-down by removing the interior of the complex shown in the figure from K and then gluing the top and bottom surfaces. The result is a complex K' in which the link of 0–simplex v is a torus, and if we can blow up using a complementary curve in this torus then our handle addition is complete.

However, to blow up we need the log-parameter along the appropriate normal curve to be zero, which need not be the case. We describe how to remedy this.

Suppose we are in the situation just before we do the blow-down, so the flattening have been matched on pairs of 3–simplices in the figure. If we apply the cycle relation about the lower vertical 1–simplex joining v to v' we destroy this matching. The procedure described above to make flattenings match again then applies the cycle relation once about each of the 1–simplices $e_1, \dots, e_6$. We claim that the end result is to change the log-parameter along the normal curve we are interested in by $2\pi i$.

Indeed, recall that the full star neighborhood of the curve C consists of the complex of Figure 6 glued to the cone on an annulus. In Figure 7 we show a possible triangulation of the annulus (we may assume it is triangulated this way, since we may choose the triangulation). The figure also shows how log-parameters are changed by the above procedure. The normal curve that interests us is the vertical curve, and the log-parameter along it has been changed by $2\pi i$.

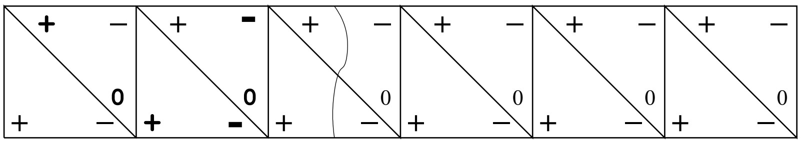

Figure 7: Left and right sides are identified to form an annulus. Signs indicate change of flattening by $\pm\pi i$ and 0 signifies that flattening has been changed by both πi and $-\pi i$ so total change is zero. The log-parameters affected by applying the cycle relation about a single e_i are bold.

Since the log-parameter along the normal curve is a multiple of $2\pi i$ (Corollary 5.4), we can apply a multiple of the above procedure to make sure that desired log-parameter along the curve is zero before we perform the blow-down.

This would complete the proof of Theorem 4.6 except that we have carried out the proof only for ordered 3–cycles whose vertex-orderings come from a global ordering of the 0–simplices of K. We now show that this stronger condition is unnecessary. We will also show, as promised earlier, that K need only be a quasi-simplicial complex (closed simplices do not necessarily embed in $|K|$). An example of such a quasi-simplicial ordered 3–cycle is given by gluing the two 3–simplices in Figure 8 by matching each face of the left 3–simplex with the correspondingly decorated face of the right 3–simplex (the decorations in question are the single and double arrows on the edges that bound the face). The arrows on the edges order the vertices of each 3–simplex as shown, and, by construction, these orderings match on common faces. The resulting complex

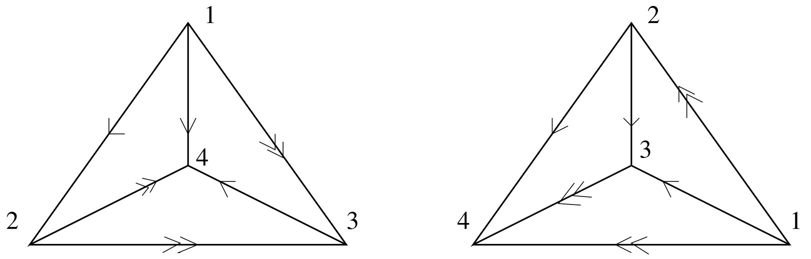

Figure 8: A quasi-simplicial ordered 3–cycle

has one 0–simplex, two 1–simplices, four 2–simplices, and two 3–simplices. (This 3–cycle is simply connected, so it cannot give a non-trivial element of $H_3(G;\mathbb{Z})$ in Theorem 4.6, but we will return to it when discussing invariants of 3–manifolds, since $|K| - K^{(0)}$ is Thurston's ideal triangulation of the figure eight knot complement.)

Proposition 11.2 *Any quasi-simplicial ordered 3–cycle K is related by Pachner moves that respect vertex-orderings of 3–simplices to a simplicial 3–cycle K' whose vertex-orderings are induced by a global ordering of the 0–simplices of K'.*

Proof We will start with an arbitrary ordering of the 0–simplices of K. At first this ordering will not induce the given vertex-orderings of simplices. We will perform various Pachner moves on K. Each time we perform a $1 \to 4$ move we will add the new 0–simplex as a new maximal element both in our ordering of 0–simplices and also in the vertex-orderings of the new 3–simplices created by the move. At each stage in the process, the ordering of the two vertices of a 1–simplex will be induced by the global ordering of the 0–simplices except maybe if both vertices are "old" 0–simplices (ie, they occurred in K). We will therefore be done once no two old 0–simplices are joined by a 1–simplex.

We will assume no 3–simplex of K meets itself across a 2–face (do a $1 \to 4$ Pachner move on any such simplex if it occurs). Suppose K has n 3–simplices. We first do $1 \to 4$ Pachner moves on each of the 3–simplices of K, creating n new 0–simplices. Next, for each old 2–simplex (ie, a 2–simplex that was already in K) we do a $2 \to 3$ Pachner move on the two 3–simplices that meet across it. Each 3–simplex of the resulting complex is in the star of a unique "old" 1–simplex. So if we operate in the star of an old 1–simplex it will not affect the star of any other old simplex.

If there are three or more three 3–simplices in the star of an old 1–simplex, we reduce the number to three by repeatedly doing $2 \to 3$ moves on pairs of adjacent simplices in its star, and then finally do a $3 \to 2$ move on the three 3–simplices in its star to remove the old 1–simplex. If an old 1–simplex has just two 3–simplices in its star we do a $1 \to 4$ move on one of them and then do a $3 \to 2$ move to remove the old 1–simplex.

In this way we alter the complex by Pachner moves until all the old 1–simplices have been removed. Then no two old 0–simplices are still joined by a 1–simplex, so the proof is complete. Since Pachner moves do not affect the represented element in $\widehat{\mathcal{B}}(\mathbb{C})$, this also completes the proof of Theorem 4.6. $\square$

The parity condition in the definition of flattening is probably essential to Theorem 4.6, but it's failure can at most change the resulting element of $\widehat{\mathcal{B}}(\mathbb{C})$ by 2–torsion:

Lemma 11.3 *If we represent an element $\alpha \in H_3(G;\mathbb{Z})$ as in Theorem 4.6 but without requiring the condition on parity then the resulting element differs from $\lambda(\alpha)$ at worst by the element of order 2 in $\widehat{\mathcal{B}}(\mathbb{C})$.*

Proof Suppose the log-parameter condition is satisfied but the parity condition is not satisfied for the flattening of K and let μ be the resulting element in $\widehat{\mathcal{B}}(\mathbb{C})$. Parity along paths determines an element of $H^1(|K| - K^{(0)};\mathbb{Z}/2)$ and hence a 2–fold cover of K (possibly branched at some 0–simplices). The lift of the flattening to this cover will satisfy the parity condition and hence represent $\lambda(2\alpha) = 2\lambda(\alpha)$. But it clearly also represents 2μ, so $2(\mu - \lambda(\alpha)) = 0$. $\square$

12 Cheeger–Chern–Simons class

The following is simply a restatement of Theorem 2.6.

Theorem 12.1 *The homomorphism $\lambda\colon H_3(G;\mathbb{Z}) \to \widehat{\mathcal{B}}(\mathbb{C})$ is an isomorphism and it fits in a commutative square:*

$$\begin{array}{ccc} H_3(G;\mathbb{Z}) & \xrightarrow{\lambda} & \widehat{\mathcal{B}}(\mathbb{C}) \\ {\scriptstyle c}\downarrow & & {\scriptstyle R}\downarrow \\ \mathbb{C}/\pi^2\mathbb{Z} & \longrightarrow & \mathbb{C}/\pi^2\mathbb{Z} \end{array}$$

Here G is, as usual, $\mathrm{PSL}(2,\mathbb{C})$ with discrete topology, c is the Cheeger–Chern–Simons class, and R is the Rogers dilogarithm map of Proposition 2.5.

Proof We first prove commutativity of the diagram. The imaginary part of both c and R gives volume (this was proved for R in slightly different language in [13]; see also [5]) so the only issue is the real part. As pointed out in [5], to prove that the real part of the above diagram commutes, we may replace $\mathrm{PSL}(2,\mathbb{C})$ by $\mathrm{PSL}(2,\mathbb{R})$.

For an element of $H_3(\mathrm{PSL}(2,\mathbb{R}))$, if we represent it by an ordered 3–cycle K labeled in $\mathrm{PSL}(2,\mathbb{R})$ and then choose the point z in Theorem 4.6 in $\partial\mathbb{H}^2 \subset \partial\mathbb{H}^3$ then all the ideal simplices are flat. A flat ideal simplex has a natural flattening in our sense, since two of its dihedral angles are already 0 and the third is $\pm\pi$, so we adjust only the latter angle to zero. If we flatten K this way, we clearly satisfy the logarithmic flattening conditions, but the parity conditions are less clear. By Lemma 11.3 this affects the element of $\widehat{\mathcal{B}}(\mathbb{C})$, and hence also the value of R on it, at most by 2–torsion.

With this flattening, the function R for a simplex with real cross-ratio parameter x becomes the real Rogers dilogarithm of x used by Dupont in [5]. Thus $R \circ \lambda$ is (up to the 2–torsion mentioned above) the cocyle he considers. He shows there that $R\circ\lambda$ equals c if we map to $\mathbb{C}/\frac{\pi^2}{6}\mathbb{Z}$ rather than $\mathbb{C}/\pi^2\mathbb{Z}$ (since $\mathbb{C}/\frac{\pi^2}{6}\mathbb{Z} = (\mathbb{C}/\pi^2\mathbb{Z})/(\mathbb{Z}/6)$, the possible 2–torsion discrepancy maps to zero). Thus $(R\circ\lambda - c)\colon H_3(G;\mathbb{Z}) \to \mathbb{C}/\pi^2 Z$ has image in $\frac{\pi^2}{6}\mathbb{Z}/\pi^2\mathbb{Z} \cong \mathbb{Z}/6$. On the other hand, $H_3(\mathrm{PSL}(2,\mathbb{C});\mathbb{Z})$ is known to be a divisible group by [7], so it has no non-trivial finite quotient, so $(R\circ\lambda - c) = 0$. Thus $R\circ\lambda = c$, as was to be proved.

It remains to show that λ is an isomorphism. If we compose λ with the map to $\mathcal{B}(\mathbb{C})$ then the kernel is the torsion subgroup of $H_3(G;\mathbb{Z})$, so the kernel of λ is a subgroup of the torsion subgroup $\mathbb{Q}/\mathbb{Z}$ of $H_3(G;Z)$. But it was pointed out in [5] and [7] that the Cheeger–Chern–Simons map c is injective on the torsion subgroup of $H_3(G;\mathbb{Z})$, so it follows that λ must also be injective on this subgroup. Thus λ is injective. Surjectivity follows similarly: we need only prove it on the torsion subgroup, and for this it suffices to see that R is an isomorphism on the torsion subgroups, which was shown in the proof of Proposition 7.4. □

One can give an explicit computation of $R\circ\lambda$ on the torsion of $H_3(G;\mathbb{Z})$ (see also [8]). By [7], the torsion subgroup $\mathbb{Q}/\mathbb{Z} \subset H_3(G;\mathbb{Z})$ is given by the image of $\mu/\{\pm1\} \to G$, where μ is the group of roots of unity mapping to diagonal matrices in G. Thus the torsion is generated by the images of generators of $H_3(\mathbb{Z}/n;\mathbb{Z}) \cong \mathbb{Z}/n$ in $H_3(G;\mathbb{Z})$. We can represent a generator of this $\mathbb{Z}/n \subset H_3(G;\mathbb{Z})$ by the lens space $L(n,1)$ with the flat G–bundle given by mapping a generator of $\pi_2(L(n,1)) = \mathbb{Z}/n$ to the $2\pi/n$–rotation in $\mathrm{SO}(2) \subset \mathrm{PSL}(2,\mathbb{R}) \subset G$. We can triangulate $L(n,1)$ by triangulating its standard fundamental domain as in Figure 4. If we order the vertices of each simplex in that figure in the order south, north, west, east, then the resulting representative for this class $\alpha_n \in H_3(G;\mathbb{Z})$ is the sum of homogeneous simplices

$$\sum_0^n \langle h_1, gh_1, g^j h_2, g^{j+1} h_2 \rangle,$$

where g is the $2\pi/n$–rotation and h_1 and h_2 are any elements of $\mathrm{SO}(2)$ chosen to fulfill our requirement that the G–labels of each simplex be distinct. Explicit computation then gives $R(\lambda(\alpha_n)) = \pi^2/n$ modulo π^2 (we omit the details).

13 Unordered simplices

As we have shown in Proposition 11.2, our complexes can be quasi-simplicial (closed simplices not required to embed). The triangulations that arise in practice (eg, ideal triangulations of complete finite volume 3–manifolds, see section 14) are often not simplicial. However, a quasi-simplicial triangulation does not always admit vertex-orderings of its 3–simplices that agree on common faces. Thus, to use such a triangulation one may either have to subdivide, or do without vertex-orderings.

In this section we describe how the theory changes if we discard the orderings of vertices of 3–simplices in the definition of $\widehat{\mathcal{B}}(\mathbb{C})$ in terms of flattened simplices. We show that the result is to quotient $\widehat{\mathcal{B}}(\mathbb{C})$ by its cyclic subgroup of order 6.

Suppose we have a flattened simplex with parameter $(z;p,q)$ with z in the upper half plane. If we retain its geometry but reorder its vertices by an even permutation (which preserves orientation) then the flattening is replaced by one of

$$(1/(1-z); q, -1-p-q) \quad \text{or} \quad (1-1/z; -1-p-q, q),$$

so the element in $\widehat{\mathcal{B}}(\mathbb{C})$ is changed by subtracting one of

$$[z;p,q] - [1/(1-z); q, -1-p-q] \quad \text{or} \quad [z;p,q] - [1-1/z; -1-p-q, p].$$

Similarly, if we reorder its vertices by an odd permutation then the element in $\widehat{\mathcal{B}}(\mathbb{C})$ is changed by subtracting one of

$$[z;p,q] + [1/z; -p, 1+p+q], \quad [z;p,q] + [1-z; -q, -p],$$
$$\text{or} \quad [z;p,q] + [z/(z-1); 1+p+q, -q].$$

Proposition 13.1 *If z is in the upper half plane then, with χ as in Proposition 7.4 and Theorem 7.5*

$$\begin{aligned}
[z;p,q]-[1/(1-z);q,-1-p-q]&=\chi(e^{\pi i/3+q\pi i})\\
[z;p,q]-[1-1/z;-1-p-q,p]&=\chi(e^{-\pi i/3+p\pi i})\\
[z;p,q]+[1/z;-p,1+p+q]&=\chi(e^{p\pi i})\\
[z;p,q]+[1-z;-q,-p]&=\chi(e^{\pi i/3})\\
[z;p,q]+[z/(z-1);1+p+q,-q]&=\chi(e^{2\pi i/3+q\pi i})
\end{aligned}$$

Proof The last three of the above equations correspond to permutations which exchange two vertices. Since two of these involutions, together with the Klein four-group (which leaves flattenings unchanged) generate the full symmetric group, it suffices to prove the third and fourth equations; the others are then easy consequences. We start with the third equation. Recall that

$$\begin{aligned}
&[x;p_0,q_0]-[y;p_1,q_1]+[\tfrac{y}{x};p_1-p_0,q_2]-\\
&-[\tfrac{1-x^{-1}}{1-y^{-1}};p_1-p_0+q_1-q_0,q_2-q_1]+[\tfrac{1-x}{1-y};q_1-q_0,q_2-q_1-p_0]=0
\end{aligned}\tag{14}$$

is an instance of the five-term relation whenever all five of x, y, $\frac{y}{x}$, $\frac{1-x}{1-y}$, and $\frac{1-x^{-1}}{1-y^{-1}}$ are in the upper half plane. Similarly one checks that it is an instance of the five-term relation if the first three are in the lower half plane and the last two in the upper half plane. In particular, if all five are in the upper half plane, then replacing x by x^{-1} and y by y^{-1} we get a relation

$$\begin{aligned}
&[x^{-1};p_0,q_0]-[y^{-1};p_1,q_1]+[\tfrac{x}{y};p_1-p_0,q_2]-\\
&-[\tfrac{1-x}{1-y};p_1-p_0+q_1-q_0,q_2-q_1]+[\tfrac{1-x^{-1}}{1-y^{-1}};q_1-q_0,q_2-q_1-p_0]=0\,.
\end{aligned}\tag{15}$$

Putting $p_0=p_1=q_0=q_1=0$ and $q_2=q$ in equation (14) and $p_0=p_1=0$, $q_0=q_1=1$, $q_2=q+1$ in (15) and then adding, we get:

$$([x;0,0]+[x^{-1};0,1])-([y;0,0]+[y^{-1};0,1])+([\frac{y}{x};0,q]+[\frac{x}{y};0,q+1])=0 \tag{16}$$

This equation holds if x, y and $\frac{y}{x}$ are suitably positioned in the upper half plane. It will continue to hold by analytic continuation if we vary x and y so long as none of x, y, or $\frac{y}{x}$ strays out of the upper half plane. If we vary x and y in the upper half plane so that $\frac{y}{x}$ crosses interval $(0,1)$ and $\frac{x}{y}$ therefore crosses $(1,\infty)$ we get the equation

$$([x;0,0]+[x^{-1};0,1])-([y;0,0]+[y^{-1};0,1])+([\frac{y}{x};0,q]+[\frac{x}{y};0,q-1])=0,$$

and exchanging x and y in this equation gives

$$([y;0,0]+[y^{-1};0,1])-([x;0,0]+[x^{-1};0,1])+([\frac{y}{x};0,q-1]+[\frac{x}{y};0,q])=0, \tag{17}$$

Replacing q by $q+1$ in equation (17) and then adding to equation (16) gives, with $z := y/x$,

$$2\big([z;0,q]+[z^{-1};0,q+1]\big)=0\,.$$

Here z is now arbitrary in the upper half plane. Thus $[z;0,q]+[z^{-1};0,q+1]$ is in the cyclic subgroup of order 2 in $\widehat{\mathcal{B}}(\mathbb{C})$. On the other hand, we know by the proof of Proposition 7.4 that R is an isomorphism on the torsion subgroup of $\widehat{\mathcal{B}}(\mathbb{C})$ and one checks easily that $R([z;0,q]+[z^{-1};0,q+1])=0$. Thus $[z;0,q]+[z^{-1};0,q+1]=0$. As we analytically continue this relation by letting z go repeatedly around the origin crossing the intervals $(-\infty,0)$ and $(0,1)$ (so z^{-1} crosses $(-\infty,0)$ and $(1,\infty)$) we get the relations

$$[z;p,q]+[z^{-1};-p,p+q+1]=0$$

for all even p.

Now, using the notation of the proof of Proposition 7.4 and Lemma 7.1 we find

$$\begin{aligned}[z;-1,q]+[z^{-1};1,q]&=([z;0,q]+[z^{-1};0,q+1])+([z;-1,q]-[z;0,q])+\\&\quad+([z^{-1};1,q]-[z^{-1};0,q])+([z^{-1};0,q]-[z^{-1};0,q+1])\\&=0+\{1-z,0\}-\{1-z^{-1},0\}-\{z,0\}\\&=\chi(1-z)-\chi(1-z^{-1})-\chi(z)=\chi\Big(\frac{1-z}{(1-z^{-1})z}\Big)=\chi(-1)\,.\end{aligned}$$

Analytically continuing this as z circles the origin as before gives us the equations

$$[z;p,q]+[z^{-1};-p,p+q+1]=\chi(-1)$$

for all odd p. Thus the third equation of the proposition is proved.

For the fourth equation of the proposition, we already know by Lemma 7.3 that $[z;p,q]+[1-z;-q,-p]=2[1/2;0,0]$, so we need to show that $2[1/2;0,0]=\chi(e^{\pi i/3})$. Denote $\omega=e^{\pi i/3}$. Then $1-\omega=\omega^{-1}$ so $[\omega;0,0]+[\omega^{-1};0,0]=2[1/2;0,0]$. We have just shown that $[\omega;0,0]+[\omega^{-1};0,1]=0$. And by definition of χ, we have $[\omega^{-1};0,0]-[\omega^{-1};0,1]=-\chi(\omega^{-1})=\chi(\omega)$. Adding the last two equations gives $[\omega;0,0]+[\omega^{-1};0,0]=\chi(\omega)$, whence $2[1/2;0,0]=\chi(\omega)$, as desired (in fact, $[1/2;0,0]=\chi(e^{\pi i/6})$, as can be seen now by applying R). $\square$

Corollary 13.2 *If we define groups analogous to $\widehat{\mathcal{P}}(\mathbb{C})$ and $\widehat{\mathcal{B}}(\mathbb{C})$ using flattened simplices modulo the lifted five term relation, but ignoring orderings of vertices, we obtain the quotients $\widehat{\mathcal{P}}(\mathbb{C})/C_6$ and $\widehat{\mathcal{B}}(\mathbb{C})/C_6$, where C_6 is the unique cyclic subgroup of order 6.*

14 Invariants of 3–manifolds

Suppose we have a compact oriented hyperbolic 3–manifold M, or, more generally, a compact oriented manifold 3–manifold with a flat G–bundle (equivalently, a homomorphism $\pi_1(M) \to G$; as usual G denotes $\mathrm{PSL}(2,\mathbb{C})$). Then Theorems 4.6 and 2.6 give a computation of the corresponding homology class in $H_3(G;\mathbb{Z})$ and its Cheeger–Chern–Simons class from a triangulation of M.

In this section we extend this to the case that M is an oriented hyperbolic manifold which is complete of finite volume but not necessarily compact. We will also extend to the sort of ideal triangulations that the programs Snap [10, 4] and Snappea [22] use for compact hyperbolic manifolds. The underlying complex of such a "Dehn filling triangulation" is homeomorphic not to M itself, but to the result of collapsing to a point a simple closed curve of M, so Theorem 4.6 does not apply directly.

Suppose M is a non-compact oriented complete hyperbolic manifold of finite volume. It is known by Epstein and Penner [9] that M has a triangulation by ideal hyperbolic polytopes, and by subdividing these polytopes we may obtain a triangulation by ideal simplices. As has often been pointed out (eg, [15]), after subdividing the polytopes into ideal simplices, the subdivisions may not match across faces of the polytopes. One can mediate between non-matching subdivisions by including flat ideal simplices in the triangulation. It is still unknown if this is actually necessary in any example. It has been shown by Petronio and Weeks [18] that flat ideal simplices are usually not a serious issue. This is so for our arguments, in fact for us even "folded back tetrahedra" are allowed.

In [13] it is shown that any ideal triangulation of a hyperbolic 3–manifold has a flattening in the sense of Theorem 4.6. If we can appropriately order vertices then the arguments of this paper imply that we get an element of $\widehat{\mathcal{B}}(\mathbb{C})$. In general we cannot do this, so, by the previous section, we only get an element of $\widehat{\mathcal{B}}(\mathbb{C})/C_6$. We can refine our triangulation to make sure that we can order the 0–simplices. The triangulations we must consider for this are hybrids of usual triangulations and ideal ones.

Suppose M has h cusps, and consider the end compactification of M, obtained by adding points p_i, $i = 1,\dots,h$, to compactify each cusp. If K is a complex that triangulates this compactification of M then $p_1,\dots,p_h$ will be 0–simplices of K. We consider $K_0 := K - \{p_1,\dots,p_h\}$ to be a "triangulation" of the non-compact manifold M. Some of the simplices of K have one or more vertices among $\{p_1,\dots,p_h\}$; we call these "ideal vertices."

We wish to assign ideal simplex shapes to all the simplices of K in an appropriate fashion. This is maybe most easily visualized as follows. We position our triangulation of $M = \mathbb{H}^3/\Gamma$ so that it triangulates M by geodesic hyperbolic simplices, with vertices p_i ideal. We then lift to a triangulation of $\tilde{M} = \mathbb{H}^3$. Finally, we move each non-ideal vertex to $\partial\overline{\mathbb{H}}^3$ in a Γ–equivariant fashion, making sure that the resulting ideal simplices are non-degenerate (all four vertices distinct).

Remark 14.1 We can also describe this is terms of a G–valued 1–cocycle "relative to the cusps." We choose a lift $\tilde{p}_i \in \mathbb{H}^3$ for each p_i and denote by P_i the parabolic subgroup of G that fixes $\tilde{p}_i$. The G–valued 1–cocycle relative to the cusps is a 1–cocycle in the usual sense except that its value on an edge that ends (resp. starts) in an ideal vertex p_i is only well defined modulo right (resp. left) multiplication by elements of P_i. This 1–cocycle is well defined up to the usual action of 0–cocycles, but only 0–cocycles that take value $1 \in G$ on ideal vertices are permitted. The 1–cocycle depends on the choice of the lifts $\tilde{p}_i$; a change of lift changes the cocycle on edges to p_i by right multiplication by the element of G that moves the new lift to the old.

There is then an equivalent G–labeling of the 3–simplices as in Section 4.1, except that the label of an ideal vertex is only defined up to right multiplication by elements of the corresponding parabolic subgroup. We then assign ideal simplex shapes to the simplices of K by choosing a point $z \in \partial\overline{\mathbb{H}}^3$ for each 0–simplex of K, as in Remark 10.5, but with the restriction that the point z chosen for an ideal vertex is the fixed point $\tilde{p}_i$ of the corresponding parabolic subgroup. A simplex with G–labels g_1, g_2, g_3, g_4 and whose vertices have been assigned points $z_1, \dots, z_4$ in $\partial\overline{\mathbb{H}}^3$ receives the ideal simplex shape $\langle g_1 z_1, \dots, g_4 z_4 \rangle$.

Suppose we have triangulated M and assigned ideal simplex shapes to the simplices as above. We also assume that we have chosen the resulting complex K so that it admits orderings of its 3–simplices that agree on common faces, and we fix such an ordering. The arguments of sections 9 and 10 then go through to show the following theorem:

Theorem 14.2 *There exists a strong flattening of K as in Theorem 4.5. The resulting element $\sum_i \epsilon_i [x_i; p_i, q_i] \in \widehat{\mathcal{P}}(\mathbb{C})$ is in $\widehat{\mathcal{B}}(\mathbb{C})$ and only depends on the hyperbolic manifold M. We denote it $\hat{\beta}(M)$.* □

Since $\widehat{\mathcal{B}}(\mathbb{C}) \cong H_3(G;\mathbb{Z})$, this gives a "fundamental class" in $H_3(G;\mathbb{Z})$ for any oriented complete finite volume hyperbolic 3–manifold (as usual, $G = \mathrm{PSL}(2,\mathbb{C})$ with discrete topology).

We pointed out in the Introduction that this class is easy to define algebraically if M is compact. I am grateful to the referee for a comment that led me to a similar elementary description of this class if M has cusps. M can be compactified to a manifold $\bar{M}$ with $\partial\bar{M}$ consisting of tori. The fundamental class $[\bar{M},\partial\bar{M}] \in H_3(\bar{M},\partial\bar{M})$ determines a class in $\beta(\bar{M},\partial\bar{M}) \in H_3(BG,BP;\mathbb{Z})$, where $P = \left\{\left(\begin{smallmatrix}1 & *\\ 0 & 1\end{smallmatrix}\right)\right\} \subset \mathrm{PSL}(2,\mathbb{C})$ (as usual, all groups are considered with their discrete topology). The long exact sequence for the pair (BG,BP) simplifies to

$$0 \to H_i(BG;\mathbb{Z}) \to H_i(BG,BP;\mathbb{Z}) \to H_{i-1}(BP;\mathbb{Z}) \to 0\,,$$

for $i > 1$, since the inclusion $P \to G$ factors through the Borel subgroup $B = \left\{\left(\begin{smallmatrix}* & *\\ 0 & *\end{smallmatrix}\right)\right\} \subset \mathrm{PSL}(2,\mathbb{C})$ and $H_i(BP;\mathbb{Z}) \to H_i(BB;\mathbb{Z})$ is zero for $i > 0$.

Proposition 14.3 *There is a natural splitting*

$$\rho\colon\ H_i(BG,BP;\mathbb{Z}) \to H_i(BG;\mathbb{Z})$$

of the above sequence for $i > 1$.

Proof Denote

$$P_2 := \left\{ \begin{pmatrix} 2^k & b \\ 0 & 2^{-k} \end{pmatrix} \mid k \in \mathbb{Z}, b \in \mathbb{C} \right\}.$$

One computes that $H_i(BP_2;\mathbb{Z}) = 0$ for $i > 0$. Thus the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H_i(BG;\mathbb{Z}) & \longrightarrow & H_i(BG,BP;\mathbb{Z}) & \longrightarrow & H_{i-1}(BP;\mathbb{Z}) & \longrightarrow & 0 \\
 & & & & \downarrow & & \| & & \\
 & & 0 & \longrightarrow & H_i(BP_2,BP;\mathbb{Z}) & \xrightarrow{\cong} & H_{i-1}(BP;\mathbb{Z}) & \longrightarrow & 0
\end{array}$$

splits the top sequence. Note that the splitting $H_{i-1}(BP) \to H_i(BG,BP)$ is not canonical, since it depends on the "2" in the definition of P_2, but it is not hard to see that its image does not depend on this choice, so the resulting splitting

$$\rho\colon\ H_i(BG,BP;\mathbb{Z}) \to H_i(BG;\mathbb{Z})$$

is canonical. □

We omit the proof that $\rho(\beta(\bar{M},\partial\bar{M}))$ is indeed the element $\hat{\beta}(M)$ of Theorem 14.2.

Example 14.4 Figure 8 gives a well-known ideal triangulation for the complete hyperbolic structure on the figure eight knot complement (see, eg, [23, 16]).

This manifold is denoted $m004$ in the Callahan-Hildebrand-Weeks cusped census (see [2, 22, 10]). Both simplices are regular ideal simplices but the vertex-order of the second simplex is opposite to its orientation. The cross-ratio parameters are therefore $\omega := e^{\pi i/3}$ and ω^{-1} respectively, and $\hat{\beta}(m004) = [\omega; p, q] - [\omega^{-1}; r, s]$ for suitable p, q, r, s. In the next section we will see that we get a flattening of the complex K if and only if $q = -1-2p$ and $(r, s) = -(p, q)$. Choosing, eg, $p = 0$, we see

$$\hat{\beta}(m004) = [\omega; 0, -1] - [\omega^{-1}; 0, 1].$$

If

$$V_0 = 1.01494160640965362502120255 4\ldots$$

denotes the volume of the regular ideal tetrahedron, then

$$R(\omega; p, q) = \frac{(2p - 2q - 1)\pi^2}{12} + V_0 i\,,$$

$$R(\omega^{-1}; -p, -q) = \frac{(2p - 2q - 1)\pi^2}{12} - V_0 i\,.$$

Thus

$$(\mathrm{vol} + i\,\mathrm{cs})(m004) = 2V_0\,,$$

recovering the known volume and Chern–Simons invariant of the figure eight knot complement.

We now describe what happens when we deform our hyperbolic structure by hyperbolic Dehn surgery to perform a Dehn filling on M (see, eg, [23, 16]). Topologically, the Dehn filled manifold M' differs from M in that some of the cusps have been filled by solid tori, which adds new closed geodesics at these cusps (the cores of the solid tori). Let λ_j be the complex length of the geodesic (length plus i times torsion) added at the j-th cusp. If no geodesic has been added at the j-th cusp we put $\lambda_j = 0$.

The deformation deforms the representation $\Gamma = \pi_1(M) \to G$. If we keep the points $\tilde{p}_i$ at fixed points of the images of the cusp subgroups of Γ then our triangulation deforms to what is called a "degree one ideal triangulation" in [15] (it is not a genuine triangulation because the topology of K has not changed, so its topology is not the topology of M'). By deforming the flattened ideal simplex parameters continuously we obtain flattened ideal simplex parameters $[x'_i; p'_i, q'_i]$ after deformation.

Theorem 14.5 *With the above notation, the element in $\widehat{\mathcal{B}}(\mathbb{C})$ represented by the Dehn filled manifold M' is*

$$\hat{\beta}(M') = -\sum_{j=1}^{h}\chi(e^{\lambda_j}) + \sum_i \epsilon_i[x_i';p_i',q_i']$$

with χ as in Theorem 7.5. Moreover,

$$(\mathrm{vol}+i\,\mathrm{cs})(M') = -\frac{\pi}{2}\sum_{j=1}^{h}\lambda_j - i\sum_i \epsilon_i R(x_i';p_i',q_i')$$

The second formula of this theorem was proved up to a constant (depending on K but independent of the Dehn filling) in [13]. The constant was conjectured to be a multiple of $i\pi^2/6$. Since [13] used unordered simplices, the constant in that version is indeed a multiple of $i\pi^2/6$. Versions prior to 1.10.2 of the program Snap [10, 4] use this formula but ignores the parity condition, giving answers accurate to $i\pi^2/12$. Snappea [22] uses a version, also from [13], that is accurate to an unknown constant. Both programs then bootstrap this to an accurate computation of the Chern–Simons invariant for any manifold that can be related by a sequence of Dehn drillings and fillings to one with known Chern–Simons invariant. The value they print is $\mathrm{cs}(M)/2\pi^2$, hence well defined modulo $1/2$.

Snap now uses the formulae of this paper and can compute Chern–Simons for any manifold. For compact manifolds accessible by the bootstrapping method, Snap also computes the eta-invariant using a related formula of [12]; from this the Riemannian Chern–Simons (defined modulo $2\pi^2$, hence modulo 1 in their normalization) is then computed.

Proof of Theorem 14.5 For simplicity of exposition we assume M has just one cusp represented by the ideal vertex p of K.

The proof depends on the fact that the deformed flattening parameters $[x_i';p_i',q_i']$ still satisfy our flattening conditions around edges, but there is a natural flattening condition at the filled cusps that is not satisfied, which leads to the "correction term" $\sum_{j=1}^{h}\chi(e^{\lambda_j})$ in the theorem.

The link L_p of vertex p of K is a torus. Let C be a simplicial curve in L_p that is a meridian of the solid torus that the cusp has been replaced by. We can obtain a genuine triangulation of M' by the blow-up procedure of Section 11, blowing up using the curve C. As in that section, we want to extend the flattening after blowing up so that matching pairs of added simplices have the same flattening and therefore cancel in the computation of $\hat{\beta}(M')$. To be able to extend the flattening of K after blowing up, we need that the log-parameter along a normal curve parallel to C is zero. As it stands, it is not zero, since it represents the logarithm of the derivative of the holonomy of the meridian curve. It was zero before deformation, but after deformation the meridian curve represents a full rotation about the core curve of the solid torus added by the Dehn filling, so the logarithm of the derivative of its holonomy is $2\pi i$ (see, eg, [23, 16] for more detail). We can use the procedure of Section 11 to correct this: we must modify flattenings on the simplices traversed by a normal curve representing a longitude of the added solid torus as indicated in Figure 7 (the longitude runs horizontally across the center of the figure). The effect on each modified simplex is to change its contribution to $\sum_i \epsilon_i[x_i;p_i,q_i]$ by something of the form $[x;p,q\pm 1]-[x;p,q]$ or the equivalent after a permutation of vertices. By the calculations of Proposition 7.4 this is $\pm\chi(z)$, where z is the cross-ratio parameter corresponding to the edge of the 3–simplex that the curve passes (so z is one of x, $1/(1-x)$, $1/x$, etc.). As we sum these contributions over the affected simplices with appropriate signs, they sum to $-\chi(L)$, where L is product of the corresponding cross-ratio parameters or their inverses for the edges passed by the normal curve. This L is the derivative of holonomy of the longitude and $\log(L)$ is the complex length λ of the core curve of the solid torus added by the Dehn filling (see, eg, [23, 16]). This proves the first equation of the theorem.

For the second equation we use the fact that $R(\hat{\beta}(M')) = i(\mathrm{vol}+i\,\mathrm{cs})(M')$. We know this if M' is compact and we will deduce it below if M' is non-compact, so we assume it for now. $R\circ\chi\colon \mathbb{C}^* \to \mathbb{C}/\pi^2\mathbb{Z}$ is the map $z\mapsto \frac{\pi i}{2}\log z$, so $R\chi(e^\lambda)=\frac{\pi i}{2}\lambda$. Applying $\frac{1}{i}R$ to the first equation of the theorem thus gives the second. □

Meyerhoff's extension of Chern–Simons invariant to cusped hyperbolic manifolds in [11] is given by defining $(\mathrm{vol}+i\,\mathrm{cs})(M)$ to be the limit of $(\mathrm{vol}+i\,\mathrm{cs})(M')$ $+\frac{\pi}{2}\sum_{j=1}^h \lambda_j$ as M' approaches M in hyperbolic Dehn surgery space. The case that M' is compact in the above Theorem thus implies:

Corollary 14.6 *For any oriented complete finite volume hyperbolic 3–manifold M*

$$i(\mathrm{vol}+i\,\mathrm{cs})(M) = R(\hat{\beta}(M)) \in \mathbb{C}/\pi^2\mathbb{Z},$$

where $cs(M)$ is Meyerhoff's extension of the Chern–Simons invariant if M is non-compact. □

The "correction term" $-\sum_{j=1}^{h}\chi(e^{\lambda_j})$ in Theorem 14.5 arises because the sum of flattening parameters corresponding to a meridian of the a solid torus added at a cusp by Dehn filling is $2\pi i$ rather than zero. The proof of the theorem shows that flattening parameters can be chosen so that this sum is zero, and then the correction term is not there. This gives an analog of Theorem 4.6 for these triangulations:

Theorem 14.7 *Consider the degree one triangulation of M' of Theorem 14.5. Then there exists flattenings $[x_i';p_i'',q_i'']$ of the ideal simplices which satisfy the conditions*

- *parity along normal paths is zero;*
- *log-parameter about each edge is zero;*
- *log-parameter along any normal path in the neighborhood of a 0–simplex that represents an unfilled cusp is zero;*
- *log-parameter along a normal path in the neighborhood of a 0–simplex that represents a filled cusp is zero if the path is null-homotopic in the added solid torus.*

For any such choice of flattenings we have

$$\hat\beta(M')=\sum_i[x_i';p_i'',q_i'']\in\widehat{\mathcal{B}}(\mathbb{C})$$

so

$$i(\mathrm{vol}+i\,\mathrm{cs})(M')=\sum_i R(x_i';p_i'',q_i'')\,. \qquad \square$$

15 Example

We return to the figure 8 knot complement $m004$ of Example 14.4 to illustrate the above formulae. Denote the cross-ratio parameters of the two simplices in Figure 8 by z and y, and choose flattenings $[z;p,q]$ and $[y,r,s]$. The consistency condition about each edge can be read off from the figure. The condition for the edge labeled with a single arrow is:

$$2\log z+\log z'-\log y'-2\log y''=2\pi i \tag{18}$$

and the condition for the other edge turns out to be equivalent to this. The log-parameters for the flattenings are

$$(\log z+p\pi i,\ \log z'+q\pi i,\ \log z''+(-1-p-q)\pi i),$$
$$(\log y+r\pi i,\ \log y'+s\pi i,\ \log y''+(1-r-s)\pi i)\,.$$

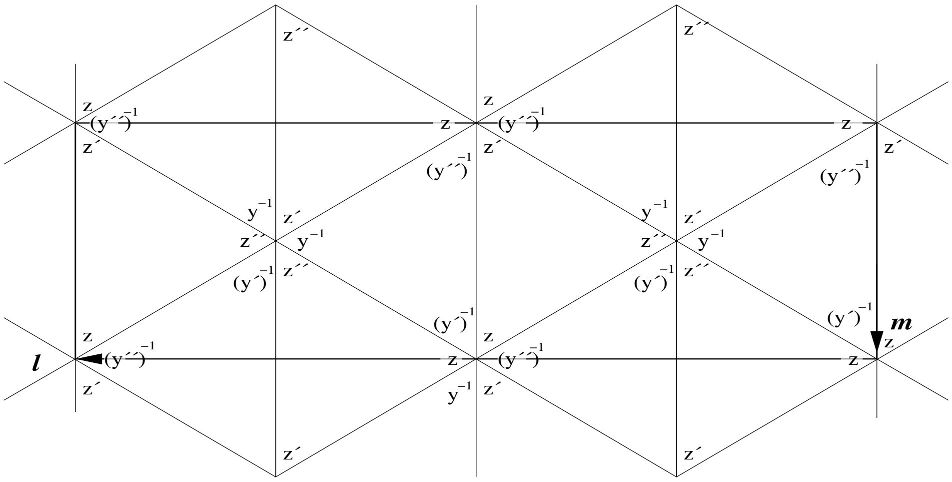

Figure 9: Fundamental domain of the cusp torus, triangulated as the link of the 0–simplex of the ideal triangulation of $m004$.

Thus the flattening condition about the edge is

$$2p + q - s - 2(1 - r - s) = -2\,. \tag{19}$$

The log holonomy for a meridian and longitude can be read off from the fundamental domain of the cusp torus illustrated in Figure 9 (taken from [16] but modified since orientation conventions differ; in particular, w, w', w'' there are our $(y'')^{-1}, (y')^{-1}, y^{-1}$ because of different vertex-ordering for the second simplex). They are:

$$u := \log z'' + \log y'' \qquad v := 2\log z - 2\log z''\,. \tag{20}$$

Their vanishing thus gives flattening conditions

$$\begin{aligned}(1 - r - s) + (-1 - p - q) &= 0\\ 2p - 2(-1 - p - q) &= 0\end{aligned}$$

which simplify to

$$-p - q - r - s = 0 \tag{21}$$

$$2 + 4p + 2q = 0\,. \tag{22}$$

Solving equations (19), (21), (22) gives

$$q = -1 - 2p, \quad (r, s) = -(p, q). \tag{23}$$

Simultaneously solving the consistency condition (18) and the cusp conditions $u = v = 0$ (see (20)) gives $z = \omega$, $y = \omega^{-1}$. Thus, as promised earlier, we see that $\hat\beta(m005) = [\omega; p, q] - [\omega^{-1}; -p, -q]$ with $q = -1 - 2p$. Choosing $p = 0$, we get:

$$\hat\beta(m005) = [\omega; 0, -1] - [\omega^{-1}; 0, 1].$$

Now suppose we deform our parameters to perform hyperbolic (α, β) Dehn filling on this manifold. Let z, y now denote the deformed values of the simplex cross-ratios (they simultaneously satisfy the consistency condition (18) and the Dehn filling condition $\alpha u + \beta v = 2\pi i$). Choose $\gamma, \delta \in \mathbb{Z}$ so

$$\begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} = 1.$$

As described in [16], the complex length of the added geodesic is $\lambda := -(\gamma u + \delta v)$, so Theorem 14.5 gives the formulae

$$\begin{aligned} \hat\beta(m004(\alpha,\beta)) &= -\chi(e^{-\gamma u - \delta v}) + [z; 0, -1] - [y; 0, 1] \\ (\mathrm{vol} + i\,\mathrm{cs})(m004(\alpha,\beta)) &= \frac{\pi}{2}(\gamma u + \delta v) - i(R(z; 0, -1) - R(y; 0, 1)), \end{aligned} \tag{24}$$

with u, v given by (20).

On the other hand, if we use Theorem 14.7, the Dehn filling condition $\alpha u + \beta v = 2\pi i$ gives the flattening condition

$$\alpha(-p - q - r - s) + \beta(2 + 4p + 2q) = -2$$

while the parity condition says $p + q + r + s$ must be even. A simultaneous solution of these and equation (19) is:

$$q = \gamma - 1 - 2p, \quad r = -2\delta - p, \quad s = -\gamma + 4\delta + 1 + 2p.$$

Choosing, eg, $p = 0$, we get

$$\begin{aligned} \hat\beta(m004(\alpha,\beta)) &= [z; 0, \gamma - 1] - [y; -2\delta, -\gamma + 4\delta + 1], \\ (\mathrm{vol} + i\,\mathrm{cs})(m004(\alpha,\beta)) &= -i(R(z; 0, \gamma - 1) - R(y; -2\delta, -\gamma + 4\delta + 1)). \end{aligned} \tag{25}$$

It is not hard to verify directly the equivalence of (24) and (25).

16 Other fields

Although we have worked over $\mathbb{C}$ in this paper, most of what we have done works for a subfield $K \subset \mathbb{C}$ if we replace $G = \mathrm{PSL}(2,\mathbb{C}) = \mathrm{PGL}(2,\mathbb{C})$ by $G = \mathrm{PGL}(2,K)$. In Theorem 7.5 we must replace the group μ of roots of unity by the group μ_K of roots of unity in K. The homomorphism $\lambda\colon H_3(\mathrm{PGL}(2,K)) \to \widehat{\mathcal{B}}(K)$ is still defined. For a hyperbolic 3–manifold $\mathbb{H}^3/\Gamma$ with $\Gamma \subset \mathrm{PGL}(2,K)$

the element $\hat{\beta}(M)$ naturally lies in $\widehat{\mathcal{B}}(K)$, but we have numerical evidence that it lies in $\widehat{\mathcal{B}}(k)$, where k is the invariant trace field of M (k is always contained in K and is generally smaller). The arguments of [15] show that $\hat{\beta}(M) \in \widehat{\mathcal{B}}(k)$ if M is non-compact, while some power of 2 times $\hat{\beta}(M)$ lies in $\widehat{\mathcal{B}}(k)$ in general.

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