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Singular Lefschetz pencils

Denis Auroux
Simon K Donaldson
Ludmil Katzarkov

Department of Mathematics, Massachusetts Institute of Technology
Cambridge, MA 02139, USA
Department of Mathematics, Imperial College
London SW7 2BZ, United Kingdom
Department of Mathematics, University of Miami, Coral Gables, FL 33124, USA
and Department of Mathematics, UC Irvine, Irvine, CA 92612, USA

auroux@math.mit.edu, s.donaldson@imperial.ac.uk, lkatzark@math.uci.edu

Abstract

We consider structures analogous to symplectic Lefschetz pencils in the context of a closed 4–manifold equipped with a "near-symplectic" structure (ie, a closed 2–form which is symplectic outside a union of circles where it vanishes transversely). Our main result asserts that, up to blowups, every near-symplectic 4–manifold (X, ω) can be decomposed into (a) two symplectic Lefschetz fibrations over discs, and (b) a fibre bundle over S^1 which relates the boundaries of the Lefschetz fibrations to each other via a sequence of fibrewise handle additions taking place in a neighbourhood of the zero set of the 2–form. Conversely, from such a decomposition one can recover a near-symplectic structure.

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1 Introduction

The classification of smooth 4–manifolds remains mysterious, but that of *symplectic* 4–manifolds is perhaps a little clearer. The purpose of this article is to extend some of the techniques which have been developed in the symplectic case to more general 4–manifolds.

Let X be a smooth, oriented, 4–manifold and let ω be a closed 2–form on X. Then ω is a symplectic structure, compatible with the given orientation, if and only if $\omega^2 > 0$ everywhere on X. We are interested in relaxing this condition. Any form ω has, at each point of X, a rank which is $0, 2$ or 4. We consider forms with $\omega^2 \geq 0$ and which do not have rank 2 at any point: thus $\omega^2 = 0$ only at the set $\Gamma \subset X$ of points where ω vanishes. The nature of this condition becomes clearer if we recall that the wedge-product defines a quadratic form of signature $(3,3)$ on $\Lambda^2\mathbb{R}^4$. Locally we can regard a 2–form as a map into $\Lambda^2\mathbb{R}^4$ and the condition is that the image of the map only meets the null-cone at the origin. Suppose ω satisfies this condition and let x be a point of the zero-set Γ. Thus there is an intrinsically defined derivative $\nabla\omega_x\colon TX_x \to \Lambda^2 T^*X_x$. The rank of $\nabla\omega_x$ can be at most 3, since the wedge product form is nonnegative on the image.

Definition 1 *A closed 2–form on X is a near-symplectic structure if $\omega^2 \geq 0$, if ω does not have rank 2 at any point and if the rank of $\nabla\omega_x$ is 3 at each point x where ω vanishes.*

It follows from this definition that the zero set Γ of a near-symplectic form is a 1–dimensional submanifold of X. The point of this notion is that, on the one hand, the form defines a *bona fide* symplectic structure outside this "small" set, while on the other hand these near-symplectic structures exist in abundance.

Proposition 1 *Suppose ω is a near-symplectic form on X. Then there is a Riemannian metric g on X such that ω is a self-dual harmonic form with respect to g. Conversely, if X is compact and $b_2^+(X) \geq 1$ then for generic Riemannian metrics on X there is a self-dual harmonic form which defines a near-symplectic structure. Moreover there is a dense subset of metrics on X for which we can choose ω such that the cohomology class $[\omega]$ is the reduction of a rational class.*

This is essentially a standard result, and we give the proof in Section 7. It is also worth mentioning another existence result for near-symplectic forms, recently obtained by Gay and Kirby, in which the 2–form is constructed explicitly from the handlebody decomposition induced by a Morse function on X [7]. In any case, the point we wish to bring out, in formulating things the way we have, is that the near-symplectic condition has a meaning independent of Riemannian geometry. Indeed one can see this as the first case of a hierarchy of conditions, for a closed 2–form on a $2n$–manifold, in which one imposes constraints on the way in which the form meets the different strata, by rank, of $\Lambda^2\mathbb{R}^{2n}$.

Given the abundance of near-symplectic structures, it is natural to try to extend techniques from symplectic geometry to this more general situation. This is, of course, the starting point for Taubes' programme, studying the Seiberg–Witten equations and pseudo-holomorphic curves [13, 14]. This article runs entirely parallel to Taubes' programme, our aim being to extend some of the "approximately holomorphic" techniques developed in [3, 5] to the near-symplectic case. More specifically, recall that any compact symplectic 4–manifold (X,ω) (with rational class $[\omega]$) admits a symplectic Lefschetz pencil. That is, there are disjoint, finite sets $A, B \subset X$ and a map $f\colon X \setminus A \to S^2$ which conforms to the following local models, in suitable oriented (complex) co-ordinates about each point $x \in X$.

- If $x \in A$ the model is $(z_1, z_2) \mapsto z_1/z_2$;
- If $x \in B$ the model is $(z_1, z_2) \mapsto z_1^2 + z_2^2$;
- For all other x the model is $(z_1, z_2) \mapsto z_1$.

Although the map f is not defined at A (the "base points" of the pencil), the fibres $f^{-1}(p)$ can naturally be regarded as closed subsets of X by adjoining the points of A. The connection with the symplectic form ω is that these fibres are symplectic subvarieties, Poincaré dual to $k\omega$, for large k.

Conversely, under mild conditions, a 4–manifold which admits such a Lefschetz pencil is symplectic [8]. The main aim of this paper is to generalise these results to the near-symplectic case. To formulate our result, let Y be any oriented 4–manifold and let $\Delta \subset Y$ be a 1–dimensional submanifold. We say that a map $f\colon Y \to S^2$ has *indefinite quadratic singularities* along Δ if around each point of Δ we can choose local co-ordinates (y_0, y_1, y_2, t) such that Δ is given by $y_i = 0$ and the map f is represented in suitable local co-ordinates on S^2 by

$$(y_0, y_1, y_2, t) \mapsto y_0^2 - \frac{1}{2}(y_1^2 + y_2^2) + it.$$

Definition 2 *A singular Lefschetz pencil on Y, with singular set Δ, is given by a finite set $A \subset Y \setminus \Delta$ and a map $f\colon Y \setminus A \to S^2$ which has indefinite quadratic singularities along Δ and which is a Lefschetz pencil on $Y \setminus \Delta$.*

Given such a singular Lefschetz pencil we define the fibre over a point p in S^2 in the obvious way, adjoining the points of A. Any such fibre is homeomorphic to the space obtained from a disjoint union of compact oriented surfaces by identifying a finite number of disjoint pairs of points. We refer to the image of one of these surfaces under the composite of the homeomorphism and the identification map as a *component* of the fibre. We can now state our main result.

Theorem 1 *Suppose Γ is a 1–dimensional submanifold of a compact oriented 4–manifold X. Then the following two conditions are equivalent.*

- *There is a near-symplectic form ω on X, with zero set Γ,*
- *There is a singular Lefschetz pencil f on X which has quadratic singularities along Γ, with the property that there is a class $h \in H^2(X)$ such that $h(\Sigma) > 0$ for every component Σ of every fibre of f.*

This is a somewhat simplified statement, we actually prove rather more, in both directions. The general drift is, roughly, that there is a correspondence between these two kinds of objects: near-symplectic forms and singular pencils. To state a more precise result, in one direction, we recall a result of Honda [10]. Take $\mathbb{R}^4$ with co-ordinates (x_0, x_1, x_2, t) and consider the 2–form

$$\Omega = dQ \wedge dt + *(dQ \wedge dt),$$

where $Q(x_0, x_1, x_2) = x_0^2 - \frac{1}{2}(x_1^2 + x_2^2)$ and $*$ is the standard Hodge $*$–operator on $\Lambda^2\mathbb{R}^4$. Let $\sigma_-\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the map $\sigma_-(x_0, x_1, x_2) = (-x_0, x_1, -x_2)$. Define $\overline{\sigma}_+\colon \mathbb{R}^4 \to \mathbb{R}^4$ to be the translation

$$\overline{\sigma}_+(\underline{x}, t) = (\underline{x}, t + 2\pi)$$

and let $\overline{\sigma}_-$ be the map

$$\overline{\sigma}_-(\underline{x}, t) = (\sigma_-(\underline{x}), t + 2\pi).$$

The maps $\overline{\sigma}_\pm$ preserve the form Ω so we get induced forms on the quotient spaces. Let $N_\pm$ be the quotients of the tube $B^3 \times \mathbb{R}$ by $\overline{\sigma}_\pm$ with the induced near-symplectic forms. Then, according to Honda, if ω is any near-symplectic form on a 4–manifold X with zero set Γ there is a Lipschitz homeomorphism ϕ of X — equal to the identity on Γ, smooth outside Γ and supported in an arbitrarily small neighbourhood of Γ — such that $\phi^*(\omega)$ agrees with one of the two models $N_\pm$ in suitable trivialisations of tubular neighbourhoods of each component of Γ. Replacing ω by $\phi^*(\omega)$ we may suppose for most purposes that the form agrees with the standard models in these tubular neighbourhoods. Let $f_\pm\colon N_\pm \to \mathbb{R} \times S^1$ be the maps defined by (Q, t) in the obvious way.

Suppose now that ω is a near-symplectic form with $[\omega]$ an integral class in $H^2(X)$. Thus we may choose a complex line bundle $\mathcal{L}$ with connection over X having curvature $-i\omega$. Given the choice of this connection we get, for each component of the singular set, a holonomy in $U(1) \subset \mathbb{C}$. It will be convenient to suppose that all these holonomies are equal to -1. The more precise result we prove in one direction is:

Theorem 2 *Suppose that ω is a near-symplectic form on X equal to one of the standard models in neighbourhoods of the zero set Γ. Suppose that $[\omega] = c_1(\mathcal{L})$ is integral and that $\mathcal{L}$ has holonomy -1 around each component of Γ. Then for all sufficiently large odd integers k there is a singular Lefschetz pencil on X such that*

- *the fibres are symplectic with respect to ω;*
- *the fibres are in the homology class dual to $kc_1(\mathcal{L})$;*
- *in sufficiently small neighbourhoods of the components of the singular set, the map is equal to the composite of one of the maps $f_\pm$ with a diffeomorphism taking $(-\delta,\delta) \times S^1$ to a neighbourhood of the standard equator in S^2.*

In the last part of the statement, the diffeomorphism taking $(-\delta,\delta) \times S^1$ to a neighbourhood of the equator is essentially the same for every component of Γ, as will be clear from the proof. Hence, each component of Γ is mapped bijectively to the equator, and there are well-defined "positive" and "negative" sides of the equator, corresponding to $Q > 0$ and $Q < 0$ in a consistent manner for all components.

It is easy to deduce one half of Theorem 1 from Theorem 2. Given any near-sympletic form we use Honda's result to get a new one compatible with the standard models. Making a small deformation away from Γ we can suppose that $[\omega]$ is a rational class and then multiplying by a suitable integer we obtain an integral class, associated to a line bundle with connection. Making a further small deformation we can suppose that each of the holonomies around the components of Γ is a root of $z^n = -1$, for some large n. Then again replacing the line bundle by its nth power we fit into the hypotheses of Theorem 2.

The more precise result in the converse direction is the following:

Theorem 3 *Let X be a compact oriented 4–manifold, and let $f\colon X \setminus A \to S^2$ be a singular Lefschetz pencil with singular set Γ (ie, a smooth map described by the above local models in oriented local co-ordinates). If there exists a* *cohomology class $h \in H^2(X)$ such that $h(\Sigma) > 0$ for every component Σ of every fibre of f, then X carries a near-symplectic form ω, with zero set Γ, and which makes all the fibres of f symplectic outside of their singular points. Moreover, these properties determine a deformation class of near-symplectic forms canonically associated to f.*

In particular, if every component of every fibre of f contains at least one base point, then the cohomological assumption automatically holds. In that case we can require $[\omega]$ to be Poincaré dual to the homology class of the fibre.

The topology of singular Lefschetz pencils is made quite complicated by the presence of the singular locus Γ. Nonetheless, Theorem 2 leads to an interesting structure result for near-symplectic 4–manifolds. Namely, given a near-symplectic 4–manifold (X, ω) with $\omega^{-1}(0) = \Gamma$ and a singular Lefschetz pencil $f\colon X \setminus A \to S^2$ such that Γ maps to the equator as in Theorem 2, after blowing up the base points we can decompose the manifold into:

- two symplectic Lefschetz fibrations over discs $f_\pm\colon X_\pm \to D^2$, obtained by restricting f to the preimages of two open hemispheres not containing the equator $f(\Gamma)$;
- the preimage W of a neighbourhood of the equator.

The 4–manifold W is a fibre bundle over S^1, whose fibre Y defines a cobordism between the fibres Σ_+ and Σ_- of $f_\pm$ (note that these need not be connected a priori), consisting of a succession of handle additions. Hence the cobordism W relates the boundaries of X_+ and X_- to each other via a sequence of fibrewise handle additions (or "round handle" additions), one for each component of Γ.

The topology of f can be described combinatorially in terms of (a) the monodromies of the Lefschetz fibrations $f_\pm$, which are given by products of positive Dehn twists in the relative mapping class groups of $(\Sigma_\pm, A)$, and (b) gluing data, which can be expressed eg, in terms of a coloured link on the boundary of one of the Lefschetz fibrations (see Section 8). This information determines f completely if the identity components in $\mathrm{Diff}(\Sigma_\pm, A)$ are simply connected (eg, if $\Sigma_\pm$ both have genus at least 2).

The paper is organised in the following way. Sections 2–6 are devoted to the proof of Theorem 2. The proof rests on techniques of approximately holomorphic geometry: roughly speaking, the construction of maps which have the same topological properties as holomorphic maps but in a context where the underlying almost-complex structure is not integrable. In Section 2 we develop the techniques from this theory that we need, encapsulated into a general result (Theorem 4), which may have other applications. (As an aside here, we mention that it would be interesting to compare our results with the methods developed by Presas [12] for symplectic manifolds with contact boundary.) The core of the paper lies in Sections 3–6. Here we show that a 4–manifold with a near symplectic form can be endowed with the geometrical structures required to apply Theorem 4. Almost all of the work is devoted to the geometry in a standard model around the zero set, and we make extensive and explicit calculations here. Again, these geometrical constructions could conceivably be of interest in other contexts. (One can also compare with the detailed study by Taubes of other geometrical phenomena in the same local model [14].) In Section 7 we prove the converse result, Theorem 3, together with Proposition 1 above. Section 8 begins the exploration of the topological aspects of singular Lefschetz pencils and their monodromy data.

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2 Approximately holomorphic theory

In the symplectic case, the construction of Lefschetz pencils relies on the existence of various structures which are the basic building blocks of "approximately holomorphic geometry", which we now review briefly and informally (precise statements are given below in a more general context). Let (X,ω) be a compact symplectic $2n$–manifold, equipped with a compatible almost-complex structure J, and let $\mathcal{L}\to X$ be a hermitian line bundle with a connection having curvature $-i\omega$. Then, for any $\epsilon>0$, the following properties hold up to a suitable rescaling, replacing ω by $k\omega$ and $\mathcal{L}$ by $\mathcal{L}^{\otimes k}$ for some large integer k (such that $k^{-1/2}\ll\epsilon$) [3, 5]:

(1) Near every point of X there exist "approximately holomorphic" co-ordinate charts, ie, local complex co-ordinates in which the almost-complex structure J differs from the standard complex structure by at most a fixed multiple of ϵ, and the symplectic form ω is uniformly bounded.

(2) For every point $p\in X$ there exist $n+1$ "localised" sections $\sigma_0,\dots,\sigma_n$ of $\mathcal{L}\to X$, and a real-valued function F which decreases exponentially fast away from p, such that

- σ_i and its covariant derivatives are bounded by F;
- $\overline{\partial}\sigma_i$ and its covariant derivatives are bounded by ϵF;
- $|\sigma_0|$ is bounded below near p, and the ratios σ_i/σ_0 define local complex co-ordinates near p.

These key ingredients make it possible to construct a pair of sections σ_0, σ_1 of $\mathcal{L}$ such that the map σ_1/σ_0 is a symplectic Lefschetz pencil. More precisely, σ_0 and σ_1 are obtained as linear combinations of the above-mentioned "localised" sections, chosen in a manner which ensures that σ_0, σ_1, and $\partial(\sigma_1/\sigma_0)$ satisfy suitable uniform transversality properties (ie, whenever one of these quantities vanishes, its derivative is surjective and satisfies a uniform *a priori* lower bound) [5].

In the near-symplectic case, we can use the same methods away from the zero set Γ of the near-symplectic form, while over a small neighbourhood of Γ we can construct the pencil explicitly from a local model. The difficulty comes from the intermediate region. To adapt the machinery to this situation, we consider the non-compact symplectic manifold $Z = X \setminus \Gamma$, equipped with a suitably rescaled symplectic form $k\omega$, and two compact subsets $K \subset Z_0 \subset Z$ (the complements of two concentric tubular neighbourhoods of Γ in X). Our goal will be to show that, for a suitable choice of almost-complex structure on Z, the following properties hold (see below for more precise statements) and imply Theorem 2:

(1) near every point of Z_0 there exist local approximately holomorphic co-ordinate charts;

(2) near every point of K there exist localised sections of $\mathcal{L}$, with support contained in Z_0, and with the same properties as in the symplectic case;

(3) there are sections σ_0, σ_1 of $\mathcal{L} \to Z$ such that σ_1/σ_0 is a Lefschetz pencil outside of K, and satisfying appropriate uniform bounds over Z_0 and transversality estimates over the intermediate region $Z_0 \setminus K$.

In the rest of this section, we give precise formulations of these three properties ("hypotheses"), and show how they can be used to construct a Lefschetz pencil. We place ourselves in a more general setting, although the reader may wish to keep in mind the above motivation.

Let (Z, ω) be a symplectic $2n$–manifold, not necessarily compact and let J be a compatible almost-complex structure on Z. Suppose we have a hermitian line bundle $\mathcal{L} \to Z$ with a connection having curvature $-i\omega$. We also suppose that we have given compact subsets Z_0 and K of Z, such that Z_0 contains a neighbourhood of K. We wish to formulate three "hypotheses" bearing on various data in this situation, involving certain numerical parameters. One collection of parameters will be denoted $C_1, C_2, \ldots$ which we abbreviate to a single symbol $\underline{C}$. These give bounds on the geometry of the set-up: the precise number of parameters C_i is unimportant, it would probably be possible to reduce them to a single constant C, but this would mean considerable loss of accuracy if one was actually interested in implementing the proof numerically. The important parameter is a small number ϵ which, roughly, measures the deviation from holomorphic geometry. In the third hypothesis we will introduce three parameters $\kappa_1, \kappa_2, \kappa_3$ which we sometimes denote by $\underline{\kappa}$. These are a measure of transversality of certain data.

Hypothesis 1 *Hypothesis* $H_1(\epsilon, \underline{C})$.

For each point p of Z_0 there is a co-ordinate chart χ_p: $B^{2n} \to Z$ centred on p such that

- *The pull-back $\chi_p^*(J)$ of the almost-complex structure on Z is close to the standard structure I on $B^{2n} \subset \mathbb{C}^n$ in that*

$$\|\chi_p^*(J) - I\|_{C^r} \leq C_1 \epsilon.$$

- *The pull-back of the symplectic form satisfies uniform bounds*

$$\|\chi_p^*(\omega)\|_{C^r} \leq C_2$$

  and $\chi_p^*(\omega)^n \geq C_2^{-1}$.

Here r is a fixed integer, $r = 3$ will do.

We call such a chart an "approximately holomorphic chart", where of course the notion depends on the parameters ϵ, C_i.

Remark In essence, this hypothesis asserts that the manifold has bounded geometry and that the norm of the Nijenhuis tensor is $O(\epsilon)$.

Before stating the next hypothesis we formulate a definition. Let $U \subset V \subset W$ be subsets of Z (with U open) and let F be a positive function on Z.

Definition 3 *An F–localised, ϵ–holomorphic system over U, relative to V and W, consists of $n+1$ sections $\sigma_0, \ldots, \sigma_n$ of $\mathcal{L} \to Z$ such that*

- *The support of any section σ_i is contained in the interior of W;*

- $|\nabla^p \sigma_i| \leq F$ *throughout* Z, *for* $p \leq r$ *and all* i;
- $|\nabla^p \overline{\partial} \sigma_i| \leq \epsilon F$ *in* V *for* $p \leq r$ *and all* i;
- $|\sigma_0| \geq 1$ *in* U *– this means that we can define a map* $f\colon U \to \mathbb{C}^n$ *by the ratios* σ_i/σ_0;
- *The Jacobian of* f *(defined using the volume form* ω^n *on* Z*) is not less than* 1.

Now we can state the following:

Hypothesis 2 *Hypothesis* $H_2(\epsilon, \underline{C})$

There is a finite collection of approximately holomorphic charts χ_i, $i = 1, \dots, M$ *mapping to balls* B_i *contained in* Z_0 *such that*

- *For a fixed* $\lambda = \frac{C_3}{1+C_3}$, *the balls* $\lambda B_i = \chi_i(\lambda B^{2n})$ *cover* K. *We define* K^+ *to be the union of the balls* B_i.
- *There are positive functions* F_i *on* Z *and for each* i *an* F_i*–localised,* ϵ*–holomorphic system over* B_i, *relative to* K^+, Z_0. *For each point* q *in the support of any section making up this system there is an approximately holomorphic chart centred on* q *with image contained in* Z_0.
- *For each point* p *of* Z,
$$\sum_i F_i(p) \leq C_4.$$
- *For all* $D > 1$ *we can divide the set* $\{1, \dots, M\}$ *into* $N = N(D)$ *disjoint subsets* $I_1, \dots, I_N$ *where*
$$N(D) \leq C_5 D^{C_6},$$
and if p *is contained in a ball* B_i *for* $i \in I_\alpha$ *then*
$$\sum_{j \in I_\alpha, j \neq i} F_j(p) \leq C_7 e^{-D}.$$

Remark In essence, this hypothesis states that associated to each point there are approximately holomorphic sections of the line bundle which on the one hand decay rapidly away from the point, and on the other hand give an approximately holomorphic projective embedding of a neighbourhood of the point.

The third hypothesis bears on a pair of sections σ_0, σ_1 which should be thought of as giving a model for a pencil outside Z_0.

Recall that, given a $\mathbb{CP}^1$–valued map F defined over an open subset of Z and a constant $\kappa > 0$, we say that ∂F *is* κ*–transverse to* 0 if at any point where $|\partial F| < \kappa$ the covariant derivative $\nabla \partial F$ is invertible and the inverse has norm less than κ^{-1}.

Hypothesis 3 *Hypothesis* $H_3(\epsilon, \kappa_1, \kappa_2, \kappa_3, \underline{C})$.

There are sections σ_0, σ_1 *of* $\mathcal{L} \to Z$ *such that*

- $F = \sigma_1/\sigma_0$ *is a topological Lefschetz pencil over* $Z \setminus K$*, with symplectic fibres.*
- $|\nabla^p \sigma_i| \leq C_8$ *in* Z_0*, for* $p \leq r$.
- $|\nabla^p \overline{\partial} \sigma_i| \leq C_9 \epsilon$ *in* K^+.
- $|\sigma_0|^2 + |\sigma_1|^2 \geq C_{10}^{-1}$ *in* $Z_0 \setminus K$*; thus* $F = \sigma_1/\sigma_0$ *defines a map from* $Z_0 \setminus K$ *to the Riemann sphere* S^2.
- *The complex-linear component* ∂F *of the derivative of* F *is* κ_1*–transverse to* 0 *throughout* $Z_0 \setminus K$.
- $|\overline{\partial} F| \leq \max(\epsilon\kappa_2, |\partial F| - \kappa_3)$ *throughout* $Z_0 \setminus K$

With all this in place we can state our general theorem.

Theorem 4 *There is a universal function* $\epsilon_0(\underline{\kappa}, \underline{C})$ *with the following property. If we have data satisfying hypotheses* $H_1(\epsilon, \underline{C}), H_2(\epsilon, \underline{C}), H_3(\epsilon, \underline{\kappa}, \underline{C})$ *and if* $\epsilon \leq \epsilon_0(\underline{\kappa}, \underline{C})$ *then there is a topological Lefschetz pencil on* (Z, ω) *with symplectic fibres, equal to* σ_1/σ_0 *outside* Z_0.

We will not say much about the proof of Theorem 4, which would essentially repeat the whole of the paper [5] (see also [3], [1], [2]). While there are no new ideas involved in the proof, the theorem extends the previous results in two different directions. On the one hand the theorem is a "relative" version of the previous results, extending a Lefschetz pencil which is already prescribed over a subset of the manifold. On the other hand, the dependence on parameters is made more explicit: in the earlier results the parameter ϵ is essentially $k^{-1/2}$ where one works with a *fixed* almost complex structure but scales the symplectic form by a factor k. The new result allows us to vary the almost complex structure at the same time as k, which will be one of the main ideas in our construction.

We outline the proof of Theorem 4. Introduce a parameter $c \in (0,1)$ and consider modifying the sections σ_0, σ_1 to

$$\tilde{\sigma}_0 = \sigma_0 + \sum a_j s_j \;, \tilde{\sigma}_1 = \sigma_1 + \sum b_j s_j,$$

where the s_j run over all the sections comprising the systems provided by Hypothesis H_2 and the coefficients a_j, b_j are complex numbers to be chosen, with the constraint that

$$|a_j|, |b_j| \leq c.$$

The arguments of [5] show that for any fixed c and for small enough ϵ we can choose the coefficients such that $\tilde{F} = \tilde{\sigma}_1/\tilde{\sigma}_0$ is close to being a symplectic Lefschetz pencil over K, in that we can find a set of disjoint balls of radii $O(\epsilon)$ and obtain a Lefschetz pencil over K by modifying $\tilde{F}$ inside these balls (in order to obtain the desired local model at the critical points). Since the sections s_j are supported in Z_0 the map $\tilde{F}$ agrees with the model pencil outside Z_0. The new issue has to do with the intermediate region $Z_0 \setminus K$, where we argue as follows.

Suppose that a map $\tilde{F}$ obtained by the procedure above satisfies

- $\partial\tilde{F}$ is $\tilde{\kappa}_1$–transverse to 0,
- $|\overline{\partial}\tilde{F}| \leq \max(\nu, |\partial\tilde{F}| - \tilde{\kappa}_3)$,

for some $\nu, \tilde{\kappa}_1, \tilde{\kappa}_3 > 0$. By construction we will also have bounds

$$|\nabla^p \tilde{F}| \leq C,$$

for $p \leq 3$ and some fixed C. We claim that there is a ν_0 depending only on $C, \tilde{\kappa}_1, \tilde{\kappa}_3$ such that if $\nu \leq \nu_0$ the map $\tilde{F}$ can be modified over a number of small disjoint balls to yield a symplectic Lefschetz fibration.

By construction, the map $\tilde{F}$ agrees with the model F outside the support of the s_j and by Hypothesis H_2 we have a good co-ordinate chart centred on any point q in the union of these supports. If $|\overline{\partial}\tilde{F}| < |\partial\tilde{F}|$ at q then $\tilde{F}$ is a fibration with symplectic fibres near q. If on the other hand $|\overline{\partial}\tilde{F}| \geq |\partial\tilde{F}|$ then we must have $|\partial\tilde{F}| \leq \nu$ at q. It follows from the transversality estimate on $\partial\tilde{F}$ that if ν is sufficiently small compared with $\tilde{\kappa}_1$ then q is close to a zero of $\partial\tilde{F}$: more precisely we can find such a zero p at a distance $O(\nu/\tilde{\kappa}_1)$ from q. Adjusting constants slightly, we can suppose that there is a good co-ordinate chart centred at this point p and contained in Z_0.

Now we clearly have $|\overline{\partial}\tilde{F}| \leq \nu$ at p. We claim that the derivative $|\nabla\overline{\partial}\tilde{F}|$ is $O(\nu^{1/2})$ at p. To see this, suppose that $|\nabla\overline{\partial}\tilde{F}(p)| = A$. Then for any small r, we can find a point p' at distance r from p with $|\overline{\partial}\tilde{F}(p')| \geq Ar - \frac{C}{2}r^2$. If r is small compared with $\tilde{\kappa}_3/C$ we have $|\partial\tilde{F}| < \tilde{\kappa}_3$ at p' so it follows that $|\overline{\partial}\tilde{F}|(p') \leq \nu$. Combining the inequalities gives $A \leq \frac{\nu}{r} + \frac{Cr}{2}$. Taking r of the order of $\nu^{1/2}$ we obtain the desired bound $A = O(\nu^{1/2})$. Now considering the Taylor series of $\tilde{F}$ at p just as in [5], Section 2, we see that $\tilde{F}$ can be modified in a ball of radius ρ to obtain a new map which is a Lefschetz fibration over the ball provided we can find a radius ρ which satisfies

$$\nu^{1/2} \ll \rho \ll \tilde{\kappa}_1/C.$$

This will be possible if ν is small and we see that moreover the original point q will lie inside the ball. So we conclude that, after making these modifications we obtain the desired fibration.

With this discussion in place we now return to complete the proof. Recall that, under our hypotheses, we do not have any ϵ bound on $\overline{\partial} s_j$ outside K^+. What we do have is a bound

$$|\nabla^r(\tilde{F} - F)| \leq Bc$$

for a suitable constant B. It follows that if c is sufficiently small then $\partial\tilde{F}$ is $\kappa_1/2$–transverse to 0. Similarly

$$|\overline{\partial}\tilde{F}| \leq |\overline{\partial}F| + Bc \leq \max(Bc + \epsilon\kappa_2, |\partial\tilde{F}| + 2Bc - \kappa_3).$$

We set $\tilde{\kappa}_1 = \kappa_1/2$ and choose c so small that $2Bc \leq \kappa_3/2$. Then we can take $\tilde{\kappa}_3 = \kappa_3/2$. Thus we have a $\nu_0 = \nu_0(\tilde{\kappa}_1, \tilde{\kappa}_2)$, as above. Now we also choose c so small that $Bc \leq \nu_0/2$. Then if ϵ is so small that $\epsilon\kappa_2 \leq \nu_0/2$ we achieve the desired properties for our function $\tilde{F}$.

3 Definition of the almost-complex structure

3.1 Set-up

In this section we put our problem in the general framework considered in Section 2. To simplify notation we will consider a case where the singular set has just one component and the model is N_+. (At the end of the proof, in Section 6.3 below, we discuss the easy extensions to the general case.) Thus we suppose that X is a compact Riemannian 4–manifold containing an isometrically embedded copy $N \subset X$ of the standard model N_+ and that ω is a closed self-dual 2–form on X which is equal to the standard form Ω in N_+ and which does not vanish outside N_+. We suppose that there is a unitary line bundle with connection $\mathcal{L} \to X$ having holonomy -1 around the zero set and with curvature $-i\omega$. For large odd integers k we consider the line bundle $\mathcal{L}^{\otimes k}$ with curvature $-ik\omega$. Clearly the standard form Ω on $\mathbb{R}^4$ scales with weight 3. Thus we can identify the pair $(N, k\omega)$ with the form induced by Ω on the quotient of $B^3(k^{1/3}) \times \mathbb{R}$ under the translations $t \mapsto t + 2\pi\mathbb{Z}k^{1/3}$, where $B^3(k^{1/3})$ is the ball in $\mathbb{R}^3$ of radius $k^{1/3}$. We will denote this form again by Ω. It is convenient to put $\epsilon = k^{-1/3}$; this is the essential parameter in the construction which will eventually be made very small. Throughout the proof our attention will be focussed on this region N on which we take our standard co-ordinates (x_0, x_1, x_2, t) (so $|\underline{x}| \leq \epsilon^{-1}$). We recall that Ω is given by

$$\Omega = (2x_0 dx_0 - x_1 dx_1 - x_2 dx_2) \wedge dt + 2x_0 dx_1 \wedge dx_2 - x_1 dx_2 \wedge dx_0 - x_2 dx_0 \wedge dx_1. \quad (1)$$

So

$$\Omega^2 = (4x_0^2 + x_1^2 + x_2^2)\, dx_0 \wedge dx_1 \wedge dx_2 \wedge dt.$$

It will be convenient to write

$$p = (4x_0^2 + x_1^2 + x_2^2)^{1/4}, \quad (2)$$

so Ω^2 is p^4 times the standard volume form.

To match up with the set-up in Section 2, we let $K \subset X \setminus \Gamma$ be the subset corresponding to $|\underline{x}| \geq 10$ and let X_0 be the subset corresponding to $|\underline{x}| \geq 1$.

The great benefit for us given by Honda's result [10], reducing to this standard model, is that there are two obvious symmetries: translation in the t–direction and rotation in the (x_1, x_2) plane. We use the standard polar co-ordinates (r, θ) in the (x_1, x_2) plane and we define

$$H = x_0 r^2. \quad (3)$$

Then one readily checks that H is the Hamiltonian for the rotation action and that

$$\Omega = dQ \wedge dt + dH \wedge d\theta.$$

Recall here that Q is the quadratic form

$$Q(\underline{x}) = x_0^2 - \frac{1}{2}(x_1^2 + x_2^2). \quad (4)$$

In these (Q, t, H, θ) co-ordinates the Euclidean co-ordinate x_0 is defined implicitly as the root of the cubic equation

$$x_0^3 - Q x_0 - \frac{H}{2} = 0, \quad (5)$$

having the same sign as H.

We want to define a suitable almost-complex structure J on $X \setminus \Gamma$. This structure will depend on the parameter ϵ. It is a standard fact that the compatible almost-complex structures on an oriented Riemannian 4–manifold are parametrised by the unit self-dual 2–forms, so we have one structure J_0 corresponding to the form $\frac{\omega}{|\omega|}$, which is smooth away from Γ. In our co-ordinates on

N this structure J_0 can be described as follows. We let $\underline{n}$ be the unit vector field on $\mathbb{R}^3$

$$\underline{n} = p^{-2}(2x_0, -x_1, -x_2).$$

Then J_0 is characterised by the conditions that

$$J_0(\underline{n}) = \frac{\partial}{\partial t} \quad , \quad J_0(\frac{\partial}{\partial t}) = -\underline{n},$$

while on the orthogonal plane $\underline{n}^{\perp}$ in $\mathbb{R}^3$, J_0 is given by the standard rotation by $\pi/2$ (with orientation fixed by that of $\underline{n}$). Notice that $\underline{n}$ is the normalised gradient vector field of the quadratic function Q on $\mathbb{R}^3$, so the planes $\underline{n}^{\perp}$ are tangent to the family of real quadric surfaces qiven by the level sets $\{Q(\underline{x}) = q\}$ of Q. Thus these quadric surfaces are *complex curves* for the almost-complex structure J_0. More precisely, we have a 2–parameter family $\Sigma_{q,t}$ of Riemann surfaces in N.

The almost-complex structure J we want to use is a modification of J_0. We set

$$J(\underline{n}) = p^2\psi^{-2}\frac{\partial}{\partial t} \quad , \quad J(\frac{\partial}{\partial t}) = -p^{-2}\psi^2\underline{n}; \tag{6}$$

where $\psi = \psi_\epsilon(\underline{x})$ is a function which we will specify shortly. On the orthogonal plane $\underline{n}^{\perp}$ we define J to be the same as J_0, thus the $\Sigma_{q,t}$ are still complex curves for the almost-complex structure J. We require that the function ψ be equal to p once $|\underline{x}| \geq \epsilon^{-1} = k^{1/3}$ so we can extend J over the whole of X by the standard structure J_0. The form $k\omega$ and the almost-complex structure J define a Riemannian metric $g = g_\epsilon$ on $X \setminus \Gamma$ in the standard way: outside N this is just the original metric scaled by a factor $k\frac{|\omega|}{\sqrt{2}}$.

In terms of the (Q, t, H, θ) co-ordinates, the almost complex structure J in the $(\frac{\partial}{\partial Q}, \frac{\partial}{\partial t})$ plane is given by

$$J(\frac{\partial}{\partial Q}) = \psi^{-2}\frac{\partial}{\partial t} \ , \ J(\frac{\partial}{\partial t}) = -\psi^2\frac{\partial}{\partial Q}. \tag{7}$$

Writing the almost-complex structure in the $\frac{\partial}{\partial H}, \frac{\partial}{\partial \theta}$–plane explicitly is equivalent to finding the conformal structure induced on the quadric surfaces – which is just the structure induced from the embedding in $\mathbb{R}^3$. A short calculation, which we leave as an exercise for the reader, shows that the metric g is given in these co-ordinates by

$$g = \psi^{-2}dQ^2 + \psi^2 dt^2 + p^{-2}r^{-2}dH^2 + p^2r^2d\theta^2. \tag{8}$$

Thus

$$J(\frac{\partial}{\partial H}) = p^{-2}r^{-2}\frac{\partial}{\partial \theta} \quad , \quad J(\frac{\partial}{\partial \theta}) = -p^2r^2\frac{\partial}{\partial H}. \tag{9}$$

We will now specify the function ψ and hence the almost complex structure. It is convenient to make ψ a function of p, depending also on the parameter ϵ. Notice that p is essentially equivalent to the square root of the Euclidean norm:

$$|\underline{x}| \leq p^2 \leq 2|\underline{x}|.$$

Lemma 1 *There are constants c_r such that for all sufficiently small ϵ we can find a smooth, positive, non-decreasing, function $\psi(p)$ on the interval $[1, \epsilon^{-1/2}]$ with following properties:*

- $\psi(p) = \epsilon$ *if* $p \leq \frac{1}{2}\epsilon^{-1/2}$;
- $\psi(p) = p$ *if* $p \geq \frac{9}{10}\epsilon^{-1/2}$;
- $\psi(p) \leq c_0 p$;
- $\psi(p) \leq c_0 \epsilon p^4$;
- $|\frac{\psi^{(r)}}{\psi}| \leq c_r \epsilon p^{2r}$ *(where $\psi^{(r)}$ denotes the r-th derivative of ψ.)*

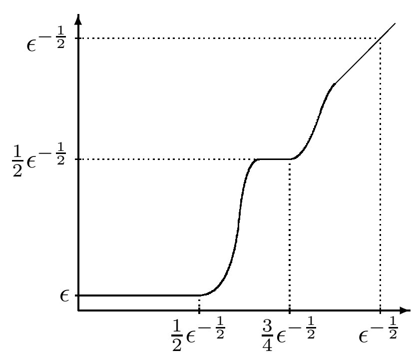

Figure 1: The function $\psi(p)$

To prove the Lemma we give an explicit construction. Choose a smooth function on $[0,1]$ equal to 0 for small values and to 1 for values near 1. Using this in an obvious way, we define for any $T > 1$ a function α_T, equal to 1 on the interval $[1, T]$ and supported in $(0, T+1)$. Likewise we choose a smooth function $g(t)$, equal to t for $t \geq \frac{9}{10}$ and to $\frac{1}{2}$ for $t \leq \frac{3}{4}$. For fixed T, let f be the solution of the differential equation

$$\frac{df}{dt} = \alpha_T f$$

with $f(t) = 1$ for $t \leq 0$. Thus f takes a constant value $L(T)$ say for large values of t (that is, for $t \geq T+1$). Clearly L is approximately e^T for large values of T. Given a small ϵ we choose T so that $L = \frac{1}{2}\epsilon^{-3/2}$. Thus this

$T = T(\epsilon)$ is much less than $\epsilon^{-1/2}$ for small ϵ: we can assume that $T < \frac{1}{4}\epsilon^{-1/2}$. Now define

$$\psi_0(p) = \epsilon f(p - \frac{1}{2}\epsilon^{-1/2}).$$

Thus $\psi_0(p) = \epsilon$ for $p \leq \frac{1}{2}\epsilon^{-1/2}$ and $\psi_0(p) = \frac{1}{2}\epsilon^{-1/2}$ for $p \geq \frac{3}{4}\epsilon^{-1/2}$. Next define

$$\psi_1(p) = \epsilon^{-1/2} g(\epsilon^{1/2} p).$$

Thus $\psi_1(p) = p$ for $p \geq \frac{9}{10}\epsilon^{-1/2}$ and $\psi_1(p)$ takes the same constant value $\frac{1}{2}\epsilon^{-1/2}$ as does ψ_0 for p near $p_0 = \frac{3}{4}\epsilon^{-1/2}$. So finally we define ψ to be equal to ψ_0 for $p \leq p_0$ and to ψ_1 for $p \geq p_0$ (see Figure 1).

It is straightforward to check that this function satisfies the requirements of the Lemma.

We now fix the almost complex structure to be the one defined by any function ψ which satisfies the requirements of Lemma 1, for example the function constructed above.

Proposition 2 *There are constants C_1, C_2 such that the symplectic manifold $(X \setminus \Gamma, k\omega)$ with the prescribed subsets $K \subset\subset X_0$ and almost complex structure J depending on $\epsilon = k^{-1/3}$ satisfies Hypothesis $H_1(\epsilon, \underline{C})$ for all large enough k.*

This is proved in Subsection 3.3 below. The essential idea of the proof is the following. Away from Γ what we have is just the familiar "flattening" of the manifold by rescaling. The region N is foliated by the Riemann surfaces $\Sigma_{q,t}$ and the almost-complex structure gives vector fields $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial Q}$ transverse to these. If the flow defined by these vector fields preserved the conformal structure of the Riemann surfaces we would have an integrable structure and we could introduce genuine local holomorphic co-ordinates. The flow by $\frac{\partial}{\partial t}$ obviously preserves the conformal structure, so the whole difficulty comes from the distortion in the conformal structure appearing in the flow of $\frac{\partial}{\partial Q}$. However, the almost-complex structure and resulting metric g have been arranged so that the small parameter ϵ makes the length of $\frac{\partial}{\partial Q}$ very large so, measured with respect to this metric, the conformal distortion is very small and we can find approximately holomorphic co-ordinates.

3.2 Holomorphic co-ordinates

While it is not really essential for the proof of Proposition 2, we will now find explicit holomorphic co-ordinates – ie, holomorphic functions – on the Riemann surfaces $\Sigma_{q,t}$. These functions will also be crucial to the work in the later parts of the proof. The existence of the circle symmetry means that we are able to construct these by elementary methods.

Consider first the surface in $\mathbb{R}^3$ defined by the equation $Q(\underline{x}) = -1$, and take x_0 and θ as co-ordinates. We seek a holomorhic function f on this surface of the form $f = u(x_0)e^{i\theta}$. If we differentiate Equation (5) we find that, with Q fixed,

$$\frac{\partial x_0}{\partial H} = p^{-4}. \tag{10}$$

By Equation (9), the Cauchy-Riemann equations for f on the surface are

$$\frac{\partial f}{\partial H} + i\, p^{-2} r^{-2} \frac{\partial f}{\partial \theta} = 0,$$

so we see that $u(x_0)$ must satisfy the equation

$$\frac{du}{dx_0} = \frac{p^2}{r^2} u = \frac{\sqrt{3x_0^2+1}}{\sqrt{2}(x_0^2+1)} u. \tag{11}$$

We choose u to be the solution of this equation with $u(0) = 1$. Thus

$$u(x_0) = \exp\left(\int_0^{x_0} \frac{\sqrt{3x^2+1}}{\sqrt{2}(x^2+1)}\, dx \right). \tag{12}$$

We can evaluate this integral explicitly in terms of elementary functions, but the formula that results is too cumbersome to be much use to us. Notice that $u(-x_0) = u(x_0)^{-1}$. Clearly u has the asymptotic behaviour

$$u(x_0) \sim A x_0^{\nu} \tag{13}$$

as $x_0 \to +\infty$, where $\nu = \sqrt{3/2}$ and

$$A = \exp\left(\int_0^{\infty} \frac{\sqrt{3x^2+1} - \sqrt{3}\, x}{\sqrt{2}(x^2+1)}\, dx \right) = (2\sqrt{3})^{\sqrt{3/2}} (\sqrt{3} - \sqrt{2}). \tag{14}$$

We now define the function F^+ on the set $\{\underline{x} : Q(\underline{x}) < 0\}$ by

$$F^+(\underline{x}) = a^{\nu} u(\frac{x_0}{a}) e^{i\theta}, \tag{15}$$

where $a = \sqrt{-Q}$. The function F^+ is holomorphic on each quadric surface $Q(\underline{x}) = -a^2$ for $a > 0$, since scaling by a^{-1} maps these conformally to the quadric $Q(\underline{x}) = -1$. The asymptotic behaviour (13) implies that as $\underline{x}$ tends to the null cone with x_0 fixed and positive $F^+(\underline{x})$ tends to $A x_0^{\nu} e^{i\theta}$, while if x_0 is fixed and negative $F^+(\underline{x})$ tends to zero on the null cone. We take these limiting values as the definition of F^+ on the null-cone. Symmetrically, we define a function F^- on $\{\underline{x} : Q(\underline{x}) < 0\}$ by

$$F^-(\underline{x}) = a^\nu u(-\frac{x_0}{a})e^{-i\theta},$$

so F^- is also holomorphic on each surface, and $F^+F^- = a^{2\nu} = (-Q(\underline{x}))^\nu$. The function F^- now tends to zero on the part of the null cone where $x_0 > 0$.

We follow a similar procedure on the set where $Q(\underline{x}) > 0$. On the sheet of the surface $\{Q(\underline{x}) = 1\}$ on which x_0 is positive we have a holomorphic function of the form $v(x_0)e^{i\theta}$ where, for $x_0 > 1$, the function v satisfies

$$\frac{dv}{dx_0} = \frac{\sqrt{3x_0^2 - 1}}{\sqrt{2}(x_0^2 - 1)} v.$$

This defines v (with $v(1) = 0$) up to a multiplicative constant, and we fix the constant by requiring that $v(x_0) \sim Ax_0^\nu$, where A is given by Equation (14) above. Then we define F^+ on $\{\underline{x} : x_0 > 0,\ Q(\underline{x}) > 0\}$ by

$$F^+(\underline{x}) = b^\nu v(\frac{x_0}{b})e^{i\theta},$$

where $b = \sqrt{Q}$. Symmetrically, we define $F^-(\underline{x})$ to be $\overline{F^+(-\underline{x})}$ on the set $\{\underline{x} : x_0 < 0,\ Q(x_0) > 0\}$.

To summarise, define open sets

$$G^+ = \{\underline{x} \in \mathbb{R}^3 : x_0 > 0 \text{ if } Q(\underline{x}) \geq 0\},$$
$$G^- = \{\underline{x} \in \mathbb{R}^3 : x_0 < 0 \text{ if } Q(\underline{x}) \geq 0\}.$$

Then we have:

Proposition 3 *The functions $F^\pm$ are smooth on $G^\pm$ and holomorphic on each connected component of the quadric surfaces $Q(\underline{x}) = q$ in $G^\pm$.*

The proof of this is a straightforward calculus argument involving the analytic continuation of the function $u(x_0)$ to imaginary values of x_0.

3.3 Proof of Proposition 2

Let $\underline{r}$ be a point in $\mathbb{R}^3$ with $|\underline{r}| = 1$ and let Σ be the quadric surface passing through $\underline{r}$. We choose a map

$$L: D \times (-\tfrac{1}{4}, \tfrac{1}{4}) \to \mathbb{R}^3,$$

where D is the unit disc in $\mathbb{C}$, with the following properties.

- $L(0,0) = \underline{r}$ and $z \mapsto L(z,0)$ gives a conformal parametrisation of a neighbourhood of $\underline{r}$ in Σ.
- $H(L(z,q))$ and $\theta(L(z,q))$ are independent of q
- $Q(L(z,q)) = Q(\underline{r}) + q$.

To construct this map we first choose a conformal parametrisation $L(z,0)$ and then extend by integrating the vector field $\frac{\partial}{\partial Q}$. This can all be done explicitly, using the conformal parametrisation by F^+ above, but we do not need the detailed formulae; the crucial point for the proof of Proposition 2 is the behaviour of the data under scaling. The complex structure on the quadric surfaces pulls back to a leaf-wise structure on $D \times (-\frac{1}{4}, \frac{1}{4})$ which is described by a matrix-valued function $J(z,q)$. By construction $J(z,0)$ is the standard matrix J_0 so

$$J(z,q) = J_0 + qK(z,q)$$

say, with K smooth. The pull-back by L of the 2–form $*(dQ \wedge dt)$ can be written as

$$A(z,q)\, i\, dz \wedge d\overline{z},$$

for some positive function A, with $A(z,q) \geq A_0 > 0$. As $\underline{r}$ varies in the unit sphere we get a family of such maps and it is clear that, by compactness of the sphere, we can choose these so that K and A satisfy uniform C^∞–estimates on their derivatives, and A_0 is fixed independent of $\underline{r}$. Having said this we will not complicate our notation by keeping the $\underline{r}$–dependence explicitly.

Now consider the point $\underline{R} = \lambda \underline{r}$ for some $\lambda \geq 1$. Let ψ_0 be the value of the function ψ at this point. We define a map $M(z,q,\tau)$ into $\mathbb{R}^4$

$$M(z,q,\tau) = \left(\lambda L\left(\frac{z}{\lambda^{3/2}}, \frac{\psi_0}{\lambda^2} q\right), \psi_0^{-1}\tau\right).$$

The fourth condition of Lemma 1 implies that $\psi_0/\lambda^2 = O(\epsilon)$, so we can suppose that M is defined on $D \times I \times I$ for some fixed interval I. Then

$$M^*(\Omega) = dq \wedge d\tau + A\left(\frac{z}{\lambda^{3/2}}, \frac{\psi_0}{\lambda^2} q\right) i\, dz \wedge d\overline{z}.$$

Clearly, then, $M^*(\Omega)$ satisfies uniform C^∞ bounds and with volume form bounded below by A_0 as the point $\underline{R} = \lambda \underline{r}$ ranges over the set $\{|\underline{R}| \geq 1\}$. To prove Proposition 2 we need to show that the almost-complex structure differs from the standard one in these co-ordinates by $O(\epsilon)$, with all derivatives. This almost-complex structure is given by a matrix valued function which is the direct sum

$$J\left(\frac{z}{\lambda^{3/2}}\right) \oplus \begin{pmatrix} 0 & -\Psi^2/\psi_0^2 \\ \psi_0^2/\Psi^2 & 0 \end{pmatrix} \tag{16}$$

where Ψ is the composite $\psi \circ p \circ M$.

Now the first term is

$$J(\frac{z}{\lambda^{3/2}}) = J_0 + \frac{\psi_0 q}{\lambda^2} K(\frac{z}{\lambda^{3/2}}, \frac{\psi_0 q}{\lambda^2}).$$

This satisfies the required bound since $\psi_0 \lambda^{-2} = O(\epsilon)$. Thus the real work involves the second term: we want to show that all derivatives of $1 - \Psi/\psi_0$ are $O(\epsilon)$.

Return again to the function $L(z,q)$. Write

$$p(L(z,q)) = G(z,q).$$

Using homogeneity, our function Ψ is given in the co-ordinates $M(z,q,\tau)$ by

$$\Psi(z,q,\tau) = \psi(\lambda^{1/2} G(\frac{z}{\lambda^{3/2}}, \frac{\psi_0 q}{\lambda^2})).$$

We are left then with the elementary task of showing that the hypotheses in Lemma 1 bound the derivatives of this composite function. For simplicity we will just work at the origin of the co-ordinates. We claim that

$$\lambda^{1/2} G(\frac{z}{\lambda^{3/2}}, \frac{\psi_0}{\lambda^2} q) = \lambda^{1/2} G(0,0) + \lambda^{-1} B(z,q),$$

where B is a smooth function, depending on the parameters λ, ψ_0 but all of whose derivatives are bounded. For if we write the Taylor series of G in the schematic form $G(z,q) = \sum a_{IJ} z^I q^J$, then

$$B(z,q) = \sum_{(I,J)\neq(0,0)} a_{IJ} \lambda^{3/2} \lambda^{-3I/2} \lambda^{-2J} \psi_0^J \, z^I q^J.$$

Now the assertion follows from the fact that $\psi_0 \leq C\lambda^{1/2}$. Thus our function is

$$1 - \Psi/\psi_0 = 1 - \psi(p_0)^{-1} \psi(p_0 + \lambda^{-1} B(z,w)),$$

The fact that all derivatives of this are $O(\epsilon)$ follows from the condition

$$\psi^{(r)} \leq \epsilon c_r p^{2r} \psi,$$

in Lemma 1.

It is now straightforward to complete the proof of Proposition 2. We use the maps M as above, together with their obvious translates in the t variable, to get co-ordinate charts over a neighboorhood of $N \cap X_0$. Over the remainder of X_0 we can use the familiar rescaled osculating co-ordinates, just as in the case of compact symplectic manifolds.

4 Construction of approximately holomorphic sections

We now start to work towards the verification of Hypothesis 2, involving sections of the line bundle $\mathcal{L}^{\otimes k}$ over X. The crucial constructions and arguments will take up this Section 4 and the following Section 5. As one would expect, the essential issues involve the local model around the zero set. Thus in Sections 4 and 5 we will work with a line bundle $\mathcal{L}$ over $\mathbb{R}^4$ with a connection of curvature $-i\Omega$. We use the almost-complex structure J, defined in the previous section, over the complement in $\mathbb{R}^4$ of the t–axis. In Section 6 we will adapt our constructions to the 4–manifold X. We write the line bundle $\mathcal{L}$ over $\mathbb{R}^4$ as the tensor product

$$\mathcal{L} = \mathcal{L}_1 \otimes \mathcal{L}_2$$

where $\mathcal{L}_1$ has curvature $-i\, dH \wedge d\theta$ and $\mathcal{L}_2$ has curvature $-i\, dQ \wedge dt$.

We will omit some of the steps required to give a complete verification of Hypothesis 2. The proofs that we do give seem to us quite long enough, having in mind that the whole discussion is largely a matter of elementary calculus and geometry in $\mathbb{R}^3$, and the techniques we develop can easily be extended to cover the parts we do not go through in detail.

4.1 Holomorphic sections over the quadric surfaces

In this section we will work with the Hermitian line bundle $\mathcal{L}_1$. We can ignore the t–variable and consider $\mathcal{L}_1$ as a line bundle over $\mathbb{R}^3$. Our goal is to find sections of $\mathcal{L}_1$ over suitable open sets in $\mathbb{R}^3$ which are *holomorphic* along the quadric surfaces and with appropriate localisation and smoothness properties. Exploiting the fact that the rotations in the x_1, x_2 plane act as symmetries of the whole set-up, we can find the desired sections by elementary methods.

Fix a trivialisation of $\mathcal{L}_1$ in which the connection form is $-iH d\theta$. We define the section σ of $\mathcal{L}_1$, in this trivialisation, to be

$$\sigma = \exp(-\frac{p^6}{18}) \tag{17}$$

Lemma 2 *The section σ is holomorphic along each of the quadric surfaces in $\mathbb{R}^3 \setminus \{0\}$.*

With the connection form $-iHd\theta$, the Cauchy-Riemann equation for a holomorphic section σ of $\mathcal{L}_2$ is:

$$p^2r^2\frac{\partial\sigma}{\partial H}+i\left(\frac{\partial\sigma}{\partial\theta}-iH\sigma\right)=0.$$

We have

$$p^4=6x_0^2-2Q \tag{18}$$

so, on a surface with $Q(\underline{x})$ constant,

$$4p^3\frac{\partial p}{\partial H}=12x_0\frac{\partial x_0}{\partial H}=\frac{12x_0}{p^4},$$

using Equation (10). Thus

$$\frac{\partial p}{\partial H}=\frac{3x_0}{p^7}, \tag{19}$$

and the Cauchy-Riemann equation for a section with no θ dependence is

$$\frac{3x_0r^2}{p^5}\frac{\partial\sigma}{\partial p}=-H\sigma.$$

But, since $H=x_0r^2$, this is just

$$\frac{\partial\sigma}{\partial p}=-\frac{p^5}{3}\sigma,$$

with solution $\sigma=\exp(-p^6/18)$.

The section σ can obviously be regarded as being localised at the origin in $\mathbb{R}^3$, with exponential decay as we move away from the origin. We obtain more sections – holomorphic along the quadric surfaces – by multiplying σ by suitable functions. The basic model to have in mind here is that in ordinary flat space, say $\mathbb{C}$. The Gaussian $\exp(-\frac{|z|^2}{4})$ represents a holomorphic section s_0 of the Hermitian line bundle with curvature $-idx\wedge dy$ in a trivialisation in which the connection matrix is $-\frac{i}{2}(xdy-ydx)$. Given a point $a\in\mathbb{C}$ let f_a be the holomorphic function

$$f_a(z)=\exp(\frac{\overline{a}z}{2}-\frac{|a|^2}{4}).$$

Then $f_a s_0$ is a holomorphic section with norm $\exp\left(-\frac{|z-a|^2}{4}\right)$, concentrated around the point a in $\mathbb{C}$.

To implement this idea in our setting, consider a section $\hat{\tau}=\exp(f)\,\sigma$ on one of the quadric surfaces, where $f=\mu+i\nu$ is a holomorphic function on the surface. We want to locate the points where $|\hat{\tau}|$ is stationary. In our trivialisation, these are points where the H and θ derivatives of $\mu + \log|\sigma|$ vanish. Since σ is independent of θ and $\frac{\partial\sigma}{\partial H} = -p^{-2}r^{-2}H\sigma$, the conditions are:

$$\frac{\partial\mu}{\partial H} = p^{-2}r^{-2}H, \quad \frac{\partial\mu}{\partial\theta} = 0.$$

But the Cauchy-Riemann equations are

$$\frac{\partial\mu}{\partial H} = p^{-2}r^{-2}\frac{\partial\nu}{\partial\theta}, \quad \frac{\partial\nu}{\partial H} = -p^2r^2\frac{\partial\mu}{\partial\theta},$$

so the conditions just become:

$$\frac{\partial f}{\partial\theta} = iH \tag{20}$$

Now, given fixed H_0, θ_0 we want to construct a section $\tau = \tau_{H_0,\theta_0}$ of the line bundle $\mathcal{L}_1$ over a suitable open set in $\mathbb{R}^3$ which, on each quadric surface $Q(\underline{x}) = q$, is holomorphic and which can be regarded as concentrated at the point in the surface with co-ordinates $H = H_0$, $\theta = \theta_0$. For simplicity we suppose $H_0 \neq 0$. The construction is simpler in the region where $Q(\underline{x}) > 0$, and we begin with that case. We first assume $H_0 > 0$, in which case we consider the component where $x_0 > 0$. Here we define

$$\hat{\tau} = \hat{\tau}_{H_0,\theta_0} = \exp\left(\frac{H_0}{F^+(H_0,\theta_0,Q)}\, F^+\right)\sigma. \tag{21}$$

That is, we take the function f above to be AF^+ where A is, on each surface $Q(\underline{x}) = q$, the constant $H_0/F^+(H_0,\theta_0,q)$. Now $\frac{\partial F^+}{\partial\theta} = iF^+$ so $\frac{\partial f}{\partial\theta} = iAF^+$ which, by construction, is equal to iH when $H = H_0$, $\theta = \theta_0$. So the modulus of this section $\hat{\tau}$ has a critical point at (H_0,θ_0), which we will see is a maximum (cf Section 5.1). Now we normalise by defining

$$\tau_{H_0,\theta_0} = \lambda\hat{\tau}_{H_0,\theta_0},$$

where $\lambda = |\hat{\tau}(H_0,\theta_0)|^{-1}$. Thus the value of $|\tau|$ at the point with co-ordinates (H_0,θ_0) on each quadric surface is 1.

If $H_0 < 0$, we work symmetrically on the region where $Q(\underline{x}) > 0$ but $x_0 < 0$ with the function F^-, setting

$$\hat{\tau} = \exp\left(-\frac{H_0}{F^-(H_0,\theta_0,Q)}F^-\right)\sigma.$$

The complication comes from the region $Q(\underline{x}) < 0$ where we need to use a combination of the functions $F^{\pm}$, smoothly interpolating between the two cases already defined.

Consider the quadric surface $\{Q(\underline{x}) = -1\}$ on which we have functions H, p, and $u = u(x_0)$ (defined by Equation (12)). Any of u, H, x_0 can (along with θ) be used as a co-ordinate on the surface. For example we can regard x_0 as a function $x_0(H)$. Equation (13) implies that the positive function on this quadric

$$D = (p^2 - x_0)r^2 u$$

tends to infinity as $x_0 \to \pm\infty$. Thus D has a strictly positive minimum value, η say. (The significance of this number will appear in the proof of Lemma 6 in Section 5.1.) Now given small $\delta > 0$ choose an even function g on $\mathbb{R}$ with

- $g'(h) \geq 0$ for $h \geq 0$
- $g(h) \geq |h|$
- $g(h) = \delta/2$ for $|h| \leq \delta/4$ and $g(h) = |h|$ for $|h| \geq \delta$.

It is clear that if δ is sufficiently small we will have

$$g(h) - h \leq \frac{\eta}{U(h)} \tag{22}$$

for all $h > 0$, where $U(h) = u(x_0(h))$. We fix such a δ and hence, once and for all, a function g. Define $\varphi(h) = \frac{1}{2}(h + g(h))$ so

$$\varphi(h) - \varphi(-h) = h$$

$$\varphi(h) + \varphi(-h) = g(h),$$

and $\varphi(h)$ vanishes if $h < -\delta$. Now, on the set where $Q(\underline{x}) < 0$ write $Q(\underline{x}) = -a^2$ and define a section $\hat{\tau}_{H_0,\theta_0}$ of $\mathcal{L}_1$ by

$$\hat{\tau}_{H_0,\theta_0} = \exp\left(\frac{\alpha}{F^+(H_0,\theta_0,-a^2)}F^+ + \frac{\beta}{F^-(H_0,\theta_0,-a^2)}F^-\right)\sigma, \tag{23}$$

where

$$\alpha = a^3\varphi\left(\frac{H_0}{a^3}\right),$$

$$\beta = a^3\varphi\left(-\frac{H_0}{a^3}\right).$$

Thus

$$\alpha - \beta = H_0, \quad \alpha + \beta = a^3 g\left(\frac{H_0}{a^3}\right).$$

On each quadric surface $Q(\underline{x}) = -a^2$ the section $\hat{\tau} = \hat{\tau}_{H_0,\theta_0}$ is holomorphic, since α, β and $F^{\pm}(H_0, \theta_0, -a^2)$ are all constant on the surface. We claim that, on each surface, $|\hat{\tau}|$ is stationary at the point where $H = H_0$ and $\theta = \theta_0$.

Indeed $\hat{\tau} = e^f \sigma$ where $f = AF^+ + BF^-$ and A, B are constants on the surface. So

$$\frac{\partial f}{\partial \theta} = iAF^+ - iBF^-$$

which is equal to $i(\alpha - \beta)$ at the given point. Then the claim follows from the fact that $\alpha - \beta = H_0$. Once again, we define τ_{H_0,θ_0} by normalising so that the modulus is 1 at the critical point.

To sum up, if $H_0 > 0$ we have defined sections τ_{H_0,θ_0} separately over the two regions $\{Q(\underline{x}) < 0\}$ and $\{Q(\underline{x}) > 0,\ x_0 > 0\}$. However it follows from the construction that these sections have the same limit over the positive part of the null cone, and define a smooth section over the region $G^+ \subset \mathbb{R}^3$. This is because the coefficient B of F^- vanishes near the positive part of the null cone. Likewise if $H_0 < 0$ we get a section τ_{H_0,θ_0} defined over G^-. We obtain the following:

Proposition 4 *For any $H_0 \neq 0$, θ_0 the section τ_{H_0,θ_0} defined above is a smooth section of $\mathcal{L}_1$ over G^+ or G^-. The section is holomorphic along each connected component of the quadric surfaces in its domain of definition and has modulus 1 at the points with co-ordinates (H_0, θ_0).*

Note that some of the steps in the construction work equally well when $H_0 = 0$ but there are some difficulties. From one point of view this is because we are really attempting to define a family of sections indexed by the set of integral curves of the vector field $\frac{\partial}{\partial Q}$ on $\mathbb{R}^3 \setminus \{0\}$ and this set, in its natural topology, is not Hausdorff. To avoid these essentially irrelevant complications we do not define sections τ_{H_0,θ_0} when $H_0 = 0$.

4.2 Sections of $\mathcal{L}_2$ and cut-off functions

In this subsection we first define suitable sections of the line bundle $\mathcal{L}_2$ over $\mathbb{R}^4$. Recall that this has curvature $-idQ \wedge dt$. let $(\underline{x}', t')$ be a point of $\mathbb{R}^4$ where $\underline{x}'$ has (Q, H, θ) co-ordinates (Q_0, H_0, θ_0). Let ψ_0 be the value of the function ψ at $\underline{x}'$. We can choose a trivialisation of the bundle such that the connection form is

$$-\frac{i}{2}((Q - Q_0)dt - (t - t')dQ).$$

In this trivialisation, we define a section by

$$\hat{\rho}_{\underline{x}',t'} = \exp\left(-\frac{\psi_0^2 (t - t')^2 + \psi_0^{-2} (Q - Q_0)^2}{4}\right). \tag{24}$$

(The trivialisation is ambiguous up to an overall phase, so this definition is not strictly precise, but we can ignore this here.) Notice that in a region where ψ is constant the section will be a holomorphic section of $\mathcal{L}_2$; we postpone until Section 5 the estimates for $\overline{\partial}\hat{\rho}$ in general. Obviously $|\hat{\rho}|$ achieves its maximum value 1 at points where $Q = Q_0$, $t = t'$.

The section $\hat{\rho}_{\underline{x}',t'}$ decays rapidly away from the surface $Q(\underline{x}) = Q_0$. We will now introduce a cut-off function to construct a section which vanishes outside a neighbourhood of this surface. Let $\chi(q)$ be a fixed, standard, cut-off function equal to 1 for $|q| \leq 1$ and vanishing when $|q| \geq 2$. Let b_1 be a small positive constant, to be fixed later, and define a function χ_{Q_0} on $\mathbb{R}^3$ by

$$\chi_{Q_0} = \chi\left(\frac{\epsilon}{b_1}\frac{Q - Q_0}{\psi_0}\right). \tag{25}$$

Then set

$$\rho_{\underline{x}',t'} = \chi_{Q_0}\hat{\rho}_{\underline{x}',t'} \tag{26}$$

We now return to the sections τ_{H_0,θ_0} defined in the previous section. We want to modify these by suitable cut-off functions to overcome the difficulties with their domains of definition. This cut-off construction will depend on another small positive parameter b_2. Let c_0 be the constant from Lemma 1, so $\psi(p) \leq c_0\epsilon p^4$ for $|p| \geq 1$. Recall that p^4 is the quadratic form $4x_0^2 + r^2$ on $\mathbb{R}^3$. We choose the constant b_2 so that $b_2c_0 < \frac{1}{10}$, say. Then the quadratic form $Q + b_2c_0p^4$ is indefinite. Define $G_0^{\pm} \subset \mathbb{R}^3$ by

$$G_0^{\pm} = \{\underline{x} : Q(\underline{x}) + b_2c_0p^4 < 0 \ \text{ or } \ Q(\underline{x}) + b_2c_0p^4 \geq 0 \ \text{ and } \ \pm x_0 > 0\}.$$

Thus $G_0^{\pm} \subset G^{\pm}$ and $G_0^{+} \cup G_0^{-} = \mathbb{R}^3 \setminus \{0\}$. Let N be the 1–neighbourhood, in the metric g, of the plane-minus-disc $\{(0, x_1, x_2) \in \mathbb{R}^3 : x_1^2 + x_2^2 \geq 4\}$. It is easy to check (using the fact that $Q = -\frac{1}{2}p^4$ for $x_0 = 0$, the formula $\|\frac{\partial}{\partial Q}\|_g = \psi^{-1}$

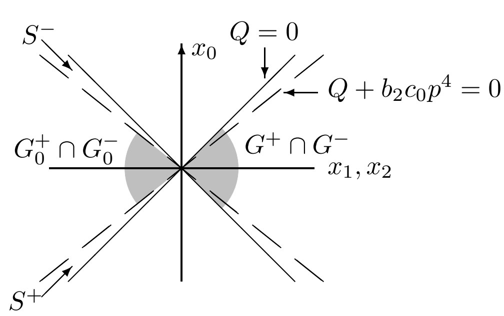

Figure 2: The sets $G^{\pm}$ and $G_0^{\pm}$

and the estimates on ψ) that we can choose b_2 small enough (depending on the constants in Lemma 1) so that

$$G_0^+ \cap G_0^- \supset N. \tag{27}$$

We now fix a value of b_2 such that (27) holds. Let λ be a standard cut-off function with $\lambda(q) = 1$ for $q \leq -1$ and $\lambda(q) = 0$ for $q > -1/2$. Define a function L on the set $\{|\underline{x}| \geq 1\}$ in $\mathbb{R}^3$ by

$$L = \lambda(\frac{\epsilon}{b_2}\frac{Q}{\psi}).$$

Suppose a point $\underline{x}$ lies in the support of ∇L. Then we must have

$$-\frac{b_2}{\epsilon}\psi < Q < 0.$$

Thus $-b_2 c_0 p^4 < Q < 0$. So the support of ∇L is contained in the set

$$\{\underline{x} : |\underline{x}| > 1,\ Q < 0,\ Q + b_2 c_0 p^4 > 0\}$$

which is the disjoint union of two components,

$$S^{\pm} = (G^{\pm} \setminus G_0^{\pm}) \cap \{|\underline{x}| > 1\}.$$

It follows that there are smooth functions $\hat{L}^+, \hat{L}^-$ on $\{|\underline{x}| > 1\}$, supported in G^+, G^- respectively and equal to 1 on G_0^+, G_0^- respectively, such that $\hat{L}^{\pm}$ and L have the same restriction to $S^{\pm}$. Finally, define

$$L^{\pm} = \chi(\frac{2}{|\underline{x}|})\ \hat{L}^{\pm}.$$

Now suppose that the point $\underline{x}' \in \mathbb{R}^3$ with co-ordinates Q_0, H_0, θ_0 has $|\underline{x}'| > 3$. Suppose that $H_0 \neq 0$. We define a section $\tau^*_{\underline{x}'}$ of $\mathcal{L}_1$ as follows. If $H_0 > 0$ the section τ_{H_0,θ_0} is smooth on G^+ and we set

$$\tau^*_{\underline{x}'} = L^+ \tau_{H_0,\theta_0},$$

extending in the obvious way by zero outside the support of L^+. Thus $\tau^*_{\underline{x}'}$ is equal to the section τ_{H_0,θ_0} – holomorphic along the quadric surfaces – near $\underline{x}'$, and the modulus of $\tau^*_{\underline{x}'}$ at the point $\underline{x}'$ is 1. In fact, because of (27), the 1–ball $B_{\underline{x}'}$ centred at $\underline{x}'$ in the metric g is contained in G_0^+, and by estimating the norm of $d(|\underline{x}|^2)$ one can verify that $B_{\underline{x}'} \subset \{|\underline{x}| > 2\}$. Hence $\tau^*_{\underline{x}'}$ is equal to τ_{H_0,θ_0} on the unit ball $B_{\underline{x}'}$.

We proceed similarly if $H_0 < 0$ (using L^- instead of L^+). Finally, we combine this with the other construction. For $(\underline{x}', t')$ as above, we set

$$s_{\underline{x}',t'} = \tau^*_{\underline{x}'} \otimes \rho_{\underline{x}',t'}. \tag{28}$$

What we have now achieved is a collection of sections of the line bundle $\mathcal{L}$ and in the next section we will derive the estimates which will ultimately allow us to verify Hypothesis 2. That hypothesis requires rather more input data. Associated to each point $(\underline{x}', t')$ we need not just one section $s_{\underline{x}',t'}$ of $\mathcal{L}$ but a triple of sections (s, s', s'') say, so that s'/s and s''/s give local approximately holomorphic co-ordinates. We will not go through this part of the construction in detail, since it would not contain any new ideas. For example, one approach is to define s', s'' by differentiating the section $s_{\underline{x}',t'}$ with respect to the parameters $\underline{x}', t'$.

Remark To illustrate this possible approach to the construction of s' and s'', consider the much simpler example of a line bundle with curvature $-i\,dx \wedge dy$ over $\mathbb{C}$. Given any $z' = x' + iy' \in \mathbb{C}$, this line bundle admits a holomorphic section $s_{x',y'}$ with $|s_{x',y'}(z)| = \exp(-\frac{1}{4}|z - z'|^2)$; in a trivialisation where the connection is $\nabla = d + \frac{i}{2}(y\,dx - x\,dy)$, such a section is given eg, by $s_{x',y'}(z) = \exp(-\frac{1}{4}|z|^2 + \frac{1}{2}\overline{z'}\,z - \frac{1}{4}|z'|^2)$. Differentiating with respect to x' and y', one easily checks that $(\frac{\partial}{\partial x'} - \frac{i}{2}y')\,s_{x',y'} = \frac{1}{2}(z - z')\,s_{x',y'}$ and $(\frac{\partial}{\partial y'} + \frac{i}{2}x')\,s_{x',y'} = -\frac{i}{2}(z - z')\,s_{x',y'}$. The ratio of either one of these sections to $s_{x',y'}$ defines a local holomorphic co-ordinate near z'. In our case the sections $s_{\underline{x}',t'}$ depend on four real parameters; as in the example, the sections obtained by differentiating in directions which belong to a same complex line are essentially redundant, and we are left with two sections s' and s'' whose ratio to $s_{\underline{x}',t'}$ gives local approximately holomorphic co-ordinates near $(\underline{x}', t')$.

5 Estimates for approximately holomorphic sections

5.1 Estimates for τ

In this subsection (5.1) and the following (5.2) we develop estimates for the sections constructed in Section 4.2. Fix H_0 and θ_0 and suppose that $H_0 > 0$ (of course there will be symmetrical statements for the case $H_0 < 0$). Then we have defined a section $\tau = \tau_{H_0,\theta_0}$ of $\mathcal{L}_1$ over the open set $G^+ \subset \mathbb{R}^3$. We introduce some notation. Let $\underline{x}$ be a point in G^+, with co-ordinates (Q, H, θ). Let $\underline{x}'$ be the point in G^+ with co-ordinates (Q, H_0, θ_0). Let $\underline{x}''$ be the point with co-ordinates (Q, H_0, θ) if $\underline{x}$ does not lie on the positive x_0–axis, and otherwise set $\underline{x}'' = \underline{x}'$. We define two functions $S = S_{H_0,\theta_0}$ and $L = L_{H_0,\theta_0}$ on G^+. The value $S(\underline{x})$ is the distance in the metric g from $\underline{x}$ to $\underline{x}''$, measured along the quadric surface through $\underline{x}$. The value $L(\underline{x})$ is $1/2\pi$ times the length, in the metric g, of the orbit of $\underline{x}'$ under the rotation action. Now for $\alpha > 0$ we define a function $E_\alpha = E_{\alpha,H_0,\theta_0}$ on G^+ by

$$E_\alpha(\underline{x}) = \exp(-\alpha\left(S(\underline{x})^2 + (\theta-\theta_0)^2 L(\underline{x})^2\right)) \tag{29}$$

(Here we interpret $(\theta-\theta_0)$ as taking values in $(-\pi,\pi]$; thus $L(\underline{x})(\theta-\theta_0)$ is the distance in the metric g from $\underline{x}'$ to $\underline{x}''$, measured along the circle orbit.)

Now given $c > 0$ let Ω_c^+ be the set

$$\Omega_c^+ = \{\underline{x} : \underline{x} \in G^+,\ |\underline{x}| > 1,\ Q(\underline{x}) < -c \ \text{ if } \ x_0 < 0\}. \tag{30}$$

The result we will prove in this section is the following:

Proposition 5 *For any c there are C, α (independent of H_0, θ_0) such that in Ω_c^+,*

$$|\tau| \leq C E_\alpha.$$

Recall that, given $\underline{x}$ and H_0, θ_0, we write $\underline{x}'$ for the point with co-ordinates H_0, θ_0 on the quadric surface through $\underline{x}$. In Section 5.2 below we will prove:

Proposition 6 *For any c, r there are C, α such that at points $\underline{x} \in \Omega_c^+$ for which $|\underline{x}'| \geq 1$ and for all $p \leq r$:*

- $|\nabla^p \tau| \leq C E_\alpha$,
- $|\nabla^p \overline{\partial}\tau| \leq \epsilon C E_\alpha$ *at points where* $|\underline{x}| \geq 3$.

Here, more precisely, $\overline{\partial}\tau$ is defined by extending the section τ to $G^+ \times \mathbb{R}$ but since there is no t dependence we can formulate the result entirely within $\mathbb{R}^3$.

We begin the proof of Proposition 5 by considering the restriction of τ to the sheet $\{x_0 > 0\}$ of the quadric $Q(\underline{x}) = 1$. We may obviously suppose that $\theta_0 = 0$ and to begin with we consider the restriction to $\theta = 0$. Thus we are considering the section τ over a single arc, homeomorphic to $[0, \infty)$. In our analysis we will use two convenient co-ordinates on this arc. One co-ordinate is the function v, the modulus of the holomorphic function F^+. The other co-ordinate is the arc length s, measured from the intersection with the x_0–axis, in the metric g. We write v_0, s_0 for the co-ordinate values corresponding to $H = H_0$; ie, corresponding to the point $\underline{x}'$. The co-ordinates v and s both run from 0 to ∞ and the asymptotic relation between them is

$$s \sim C v^{\sqrt{3/2}},$$

as $s, v \to \infty$. In fact, in terms of the radial co-ordinate r, we have

$$s \sim C' r^{3/2} \ , \ v \sim C'' r^{\sqrt{3/2}}.$$

The corresponding asymptotic relations hold for the mutual derivatives of these different co-ordinate functions.

Recall that our basic section is $\sigma = \exp(-p^6/18)$. We can write $p^6/18$ as a function of v, $f(v)$ say, on this arc. Thus f is an increasing function of v, asymptotic to a multiple of v^λ, where $\lambda = \sqrt{6} > 2$. We introduce a piece of notation. For a function g of a real variable $v \in [0, \infty)$ we write

$$\Delta_g(v, v_0) = g(v) - g(v_0) - (v - v_0) g'(v_0). \tag{31}$$

The relevance of this, working with the co-ordinate v over the arc, is that our definition of the section τ is just

$$\tau(v) = \exp\left(-\Delta_f(v, v_0)\right).$$

To see this, note that $\frac{dp}{dx_0} = 3x_0 p^{-3}$ (by differentiating $p^4 = 6x_0^2 - 2Q$ with Q fixed), and $\frac{dv}{dx_0} = p^2 r^{-2}\, v$ (by definition of v). Therefore $f'(v) = H/v$, and $\Delta_f(v, v_0) = \frac{1}{18}(p^6 - p_0^6) - \frac{H_0}{v_0}(v - v_0) = -\log|\tau|$. The first point to note is:

Lemma 3 *The function f is a convex function of v, its second derivative is strictly positive.*

To see this, recall that $f'(v) = H/v$, so we have to show that H/v is an increasing function of v, or equivalently of the variable x_0. Now, with Q fixed,

$$\frac{dH}{dx_0} = 6x_0^2 - 2 = p^4 \ , \ \frac{dv}{dx_0} = \frac{p^2 x_0}{H} v.$$

Thus

$$\frac{d}{dx_0}(H/v) = \frac{p^4}{v} - \frac{H}{v^2}\frac{p^2 x_0}{H} v = \frac{p^2}{v}(p^2 - x_0).$$

This is positive since $p^2 = \sqrt{4x_0^2 + r^2} > x_0$.

This Lemma shows that the modulus of the section τ does indeed attain a unique maximum at the point $v = v_0$. Next we need:

Lemma 4 *Suppose f is a function of $v \in [0, \infty)$ and $f''(v) \geq k(1 + v^{\lambda - 2})$ for some $k > 0$, $\lambda > 2$. Then there is a constant c such that*

$$\Delta_f(v, v_0) \geq c\left((1 + v)^{\lambda/2} - (1 + v_0)^{\lambda/2}\right)^2.$$

To prove this Lemma note that we can write

$$\Delta_f(v_1, v_0) = \int_{v_0}^{v_1} f''(v)(v_1 - v)\, dv. \tag{32}$$

Thus the hypothesis implies that $\Delta_f(v, v_0) \geq \Delta_g(v, v_0)$ where $g(v) = k(\frac{v^2}{2} + \frac{v^\lambda}{\lambda(\lambda-1)})$. So, for a suitable constant c,

$$\Delta_f(v, v_0) \geq c\left((v^\lambda - v_0^\lambda) - \lambda(v - v_0)v_0^{\lambda-1} + (v - v_0)^2\right).$$

The convexity of the function v^λ implies that the expression

$$v^\lambda - v_0^\lambda - \lambda(v - v_0)v_0^{\lambda-1}$$

is non-negative. By considering the scaling behaviour under simultaneous scaling of v and v_0 (or by using the Taylor formula), one sees that it is bounded below by a positive multiple of $(v - v_0)^2(v^{\lambda-2} + v_0^{\lambda-2})$. Thus

$$\Delta_f(v, v_0) \geq c(v - v_0)^2(1 + v^{\lambda-2} + v_0^{\lambda-2}).$$

On the other hand it is clear that

$$\begin{aligned}\left((1 + v_0)^{\lambda/2} - (1 + v)^{\lambda/2}\right)^2 &\leq c'(v - v_0)^2\left((1 + v)^{\lambda/2-1} + (1 + v_0)^{\lambda/2-1}\right)^2 \\ &\leq c''(v - v_0)^2\left(1 + v^{\lambda-2} + v_0^{\lambda-2}\right)\end{aligned}$$

which implies the desired result.

In the case of $f(v) = \frac{1}{18}p^6$ the quantity $f''(v) = \frac{d}{dv}(\frac{H}{v}) = \frac{r^2}{v^2}(p^2 - x_0)$ is bounded below by a positive multiple of $(1 + v^{\lambda-2})$, so by Lemma 4 we obtain a bound on $|\tau|$. Now the functions s and $\tilde{s} = (1 + v)^{\lambda/2}$ have the same asymptotic behaviour, so the derivative $\frac{ds}{d\tilde{s}}$ is bounded above and below by positive constants. Thus we see that on this arc

$$|\tau(s)| \leq \exp(-c(s - s_0)^2),$$

which is precisely the statement of Proposition 5 (on the arc).

Still working on the surface $Q(\underline{x}) = 1$, $x_0 > 0$, we now consider the dependence on the angular variable θ. (Recall that we are assuming $\theta_0 = 0$.) We have

$$\log|\tau(s, \theta)| = \log|\tau(s, 0)| - \frac{v}{v_0}H_0(1 - \cos\theta).$$

Now H_0 is bounded below by a multiple of s_0^2. For $\theta \in [-\pi, \pi]$, the function $(1 - \cos\theta)$ is bounded below by a multiple of θ^2. Thus we have

$$|\tau(s, \theta)| \leq \exp(-c\left((s - s_0)^2 + \frac{v}{v_0}s_0^2\theta^2\right)). \tag{33}$$

Recall that we defined $L = L(\underline{x}')$ to be $1/2\pi$ times the length of the circle orbit through $\underline{x}'$. This is $p\,r$, evaluated at $\underline{x}'$, which is bounded above and below by multiples of s_0. Thus, on this quadric surface, we have

$$E_\alpha(s,\theta) = \exp(-\alpha\left((s-s_0)^2 + Cs_0^2\theta^2\right)).$$

The difficulty comes from the term $\frac{v}{v_0}$ in Equation (33). For this we use:

Lemma 5 *There is a constant $C > 0$ such that*

$$(s-s_0)^2 + s_0^2\theta^2 \le C((s-s_0)^2 + \frac{v}{v_0}s_0^2\theta^2)$$

for all $s, s_0 \ge 0$ and $\theta \in [-\pi,\pi]$.

We consider the function v/s as a function of s. This tends to a positive limit as $s \to 0$ and tends to zero as $s \to \infty$. Thus there is a constant b, independent of s and s_0, such that

$$\frac{v_0}{s_0} \le b\frac{v}{s}$$

whenever $v < v_0$. This means that whenever $v/v_0 \le 1/2b$ we have $s \le s_0/2$. So either $v/v_0 > 1/2b$ in which case the desired inequality holds with $C = 2b$, or $s_0^2 \le 4(s-s_0)^2$ in which case the inequality holds with $C = 4\pi^2 + 1$.

Lemma 5 implies that τ is bounded by a suitable function E_α on the quadric surface $Q(\underline{x}) = 1,\ x_0 > 0$. We can extend this bound to the entire cone $Q(\underline{x}) > 0,\ x_0 > 0$ in a very simple way, by homogeneity. We will use this principle repeatedly below, so we will spell it out clearly now. If we write, in the fixed trivialisation of the line bundle $\mathcal{L}_1$

$$\tau = \tau_{H_0,\theta_0} = \exp(-A(\underline{x}; H_0, \theta_0)) \tag{34}$$

then the function A satisfies

$$A(\lambda\underline{x}; \lambda^3 H_0, \theta_0) = \lambda^3 A(\underline{x}; H_0, \theta_0). \tag{35}$$

The functions $\log E_\alpha$ satisfy exactly the same scaling behaviour

$$\log E_{\alpha,\lambda^3 H_0,\theta_0}(\lambda\underline{x}) = \lambda^3 \log E_{\alpha,H_0,\theta_0}(\underline{x}).$$

Thus the bound $|\tau| \le E_\alpha$ on the quadric surface, for all choices of the parameter H_0, immediately gives the same bound over the whole cone.

This scaling behaviour may be clearer if we change notation and regard A and E_α as functions of pairs of points $\underline{x}, \underline{x}'$ on the same quadric surface. Then the scaling reads

$$A(\lambda\underline{x}, \lambda\underline{x}') = \lambda^3 A(\underline{x}, \underline{x}')\ ,\ \log E_\alpha(\lambda\underline{x}, \lambda\underline{x}') = \lambda^3 \log E_\alpha(\underline{x}, \underline{x}').$$

We now follow a similar argument for the region where $Q(\underline{x}) < 0$. By the same scaling argument it suffices to work on the quadric $Q(\underline{x}) = -1$. Again, we begin with arc on this quadric where $\theta = 0$. We have two different co-ordinates on this arc. One is the arc length s in the metric g which now runs from $-\infty$ to ∞. The other is the function u, the modulus of F^+, which runs over $(0, \infty)$. The function u is asymptotic to $|s|^{\pm\sqrt{2/3}}$ as $s \to \pm\infty$. The choice of the parameter $H_0 > 0$ defines corresponding values $u_0 > 1$, $s_0 > 0$. Recall that over this arc our section τ is given by

$$\tau = \exp(-f(u) + \alpha \frac{u}{u_0} + \beta \frac{u_0}{u} + c(u_0))$$

where now f is $p^6/18$, expressed as a function of u on the quadric, $c(u_0)$ is a normalisation constant ensuring that $|\tau(u_0)| = 1$, and α and β are defined by u_0 as in Section 4.1.

Lemma 6 $\log|\tau(u)|$ *has just one critical point, when* $u = u_0$ *and this point is a global maximum.*

To see this, we have

$$\frac{d}{du}(-\log|\tau|) = \frac{H}{u} - \frac{\alpha}{u_0} + \beta\frac{u_0}{u^2}.$$

We want to see that this vanishes only when $u = u_0$, where it vanishes by construction (since $\alpha - \beta = H_0$). Thus it suffices to show that the function $\frac{H}{u} + \beta\frac{u_0}{u^2}$ is an increasing function of u, or equivalently of H. Now

$$\frac{d}{dH}\left(\frac{H}{u} + \beta\frac{u_0}{u^2}\right) = \frac{1}{u} - \frac{H}{p^2r^2u} - 2\beta\frac{u_0}{p^2r^2u^2},$$

using the fact that $\frac{du}{dH} = \frac{u}{p^2r^2}$. Rearranging terms, we need

$$2\beta u_0 < (p^2 - x_0)r^2u.$$

But this is precisely the condition we required in the choice of α, β defining τ (see Equation (22), and recall that $\beta = \varphi(-H_0) = \frac{1}{2}(g(H_0) - H_0) \leq \frac{\eta}{2u_0}$, where $\eta = \min\{(p^2 - x_0)r^2u\}$), so the assertion follows. This discussion also shows that $\log|\tau|$ is a concave function of u along the arc $\theta = 0$. Thus u_0 is a maximum along this arc. On the other hand the θ–dependence is again proportional to $\cos\theta$ so clearly the maximum on each circle $u =$ constant is attained when $\theta = 0$.

We claim that, on the arc $\theta = 0$,

$$\tau \leq \exp(-\alpha(s - s_0)^2).$$

The proof follows the same pattern as in the positive case above. Recall that $\beta = 0$ once u_0 is bigger than some $K > 1$ say. When $u, u_0 > K$ the argument is identical. There are then various other cases to check, a task which we will largely leave to the reader. We just discuss two representative sample cases. First, if $u_0 = 1$ then we have to show that

$$f(u) - f(1) - c(u + u^{-1} - 2) \geq cs^2.$$

This holds when u is close to 1 by the critical point analysis above. When $u \to \infty$ the left hand side grows like $f(u) \sim u^{\sqrt{6}}$ since $\sqrt{6} > 1$, and this is the same growth as s^2. Similarly when $u \to 0$. For the second case, consider $u \to 0$ and $u_0 \to \infty$. Then we have to show that

$$f(u) - f(u_0) - (u - u_0)f'(u_0) \geq c(s_0^2 + s^2).$$

Now $f(u_0)+(u-u_0)f'(u_0)$ grows like $(1-\sqrt{6})u_0^{\sqrt{6}}$, which is large and negative, while $f(u)$ grows like $u^{-\sqrt{6}}$, which is large and positive. Thus the left hand side is bounded below by a multiple of $u_0^{\sqrt{6}} + u^{-\sqrt{6}}$, or equivalently $s_0^2 + s^2$.

5.2 Estimates for derivatives of τ

In this section we obtain estimates for the derivatives of a section $\tau = \tau_{H_0,\theta_0}$. We suppose that $H_0 > 0$, so τ is defined over G^+. Recall that given $\underline{x} \in G^+$ we write $\underline{x}'$ for the point with co-ordinates (Q, H_0, θ_0) where $Q = Q(\underline{x})$. In the fixed trivialisation of $\mathcal{L}_1$ we write $\tau = \exp(-A)$ as above. We fix a positive integer r and $c > 0$. The result we prove is:

Proposition 7 *For any $\tilde{\alpha} > 0$ there is a constant C such that*

$$|\underline{x}|^{2r}\left|\left(\frac{\partial}{\partial Q}\right)^r A\right| \leq C E_{\tilde{\alpha}}^{-1},$$

in the set where $|\underline{x}|, |\underline{x}'| \geq 1$ and $Q(\underline{x}) < -c$ if $x_0 < 0$.

It is not hard to deduce Proposition 6 from this. The simplest case is the estimate on $|\overline{\partial}\tau|$. Since τ is holomorphic along the quadric surfaces we have

$$|\overline{\partial}\tau| = \psi\left|\frac{\partial\tau}{\partial Q}\right| = \psi\left|\frac{\partial A}{\partial Q}e^{-A}\right| \leq C\psi\left|\frac{\partial A}{\partial Q}\right| E_{\alpha},$$

using Proposition 5. Now $\psi \leq C\epsilon p^4 \leq C\epsilon|\underline{x}|^2$ so Proposition 7 yields

$$|\overline{\partial}\tau| \leq \epsilon\, C E_{\tilde{\alpha}}$$

(over the given set) for some $\tilde{\alpha}$ smaller than α. The other estimates in Proposition 6 are obtained similarly. Using the fact that τ is holomorphic along the surfaces we can bound the partial derivatives in the (H,θ) directions in terms of $|\tau|$ (via either elliptic theory or the Cauchy integral formula). Thus we can estimate any partial derivative of τ by the derivatives in the Q variable. We leave the details to the reader.

We follow the same pattern as in the previous subsection, proving Proposition 7 first on the cone $\{Q(\underline{x})>0, x_0>0\}$. Again we can exploit homogeneity under scaling $\underline{x}\mapsto\lambda\underline{x}$, $H_0\mapsto\lambda^3 H_0$. Thus we begin by considering the restriction of $|\underline{x}|^{2r}(\frac{\partial}{\partial Q})^r A$ to the surface $Q(\underline{x})=1$, $x_0>0$. The θ–variable will play essentially no role, so we suppose $\theta_0=0$ and restrict to the arc Γ where $\theta=0$. We recall from the previous subsection that we have two useful co-ordinates along this arc, one the function v and the other the arc length s. In what follows we will also have to bring in a third co-ordinate, the restriction of the function H. Recall also that the fixed parameter H_0 corresponds to values v_0, s_0 – ie, the co-ordinates of the point $\underline{x}'$ in the different parametrisations of the arc. For a suitable fixed N we write

$$R(s,s_0)=\left(\frac{1+s}{1+s_0}\right)^N+\left(\frac{1+s_0}{1+s}\right)^N.$$

With all these preliminaries out of the way, what we actually prove is:

Proposition 8 *For any r there are N, C such that on the arc Γ*

$$|\underline{x}|^{2r}\left|\left(\frac{\partial}{\partial Q}\right)^r A\right|\leq CR(s,s_0)(s-s_0)^2.$$

To see that this implies Proposition 7 in the positive cone, we argue as follows. For any point $\underline{x}$ in this cone we define s to be the length of the obvious arc in the quadric surface through $\underline{x}$, running from the x_0 axis to $\underline{x}$. Similarly we define s_0 to be the length of the arc in the same quadric surface to the point $\underline{x}'$. Thus $S(\underline{x})=s-s_0$. The function $|\underline{x}|^{2r}|(\frac{\partial}{\partial Q})^r A|$ is homogeneous of degree 3 under rescaling, while s, s_0 are homogeneous of degree 3/2. Thus the estimate in Proposition 8 scales to the general estimate

$$|\underline{x}|^{2r}\left|\left(\frac{\partial}{\partial Q}\right)^r A\right|\leq CR(Q^{-3/2}s, Q^{-3/2}s_0)\,(s-s_0)^2. \tag{36}$$

We use the following:

Lemma 7 *For any b, $\beta>0$ there is a C such that*

$$R(Q^{-3/2}s, Q^{-3/2}s_0)\,(s-s_0)^2\, e^{-\beta(s-s_0)^2}\leq C$$

provided that $Q\geq b$ or s, $s_0\geq b$.

The proof is elementary and left to the reader. We can obviously choose b so that $|\underline{x}| \geq 1$, $|\underline{x}'| \geq 1$ implies that $Q \geq b$ or $s, s_0 \geq b$. Hence the Lemma and Equation (36) imply Proposition 7 in the positive cone.

We now turn to the heart of the matter: the proof of Proposition 8. The complication here is the interaction between the three co-ordinates H, v, s on Γ. For a function f on Γ and two points $\underline{x}, \underline{x}'$ on Γ we write

$$\Delta(f; \underline{x}, \underline{x}') = \Delta_f(v, v_0)$$

where on the right hand side we understand that we use the co-ordinate v to parametrise Γ and v, v_0 are the co-ordinates of $\underline{x}, \underline{x}'$.

Lemma 8 *Suppose f is a smooth function on Γ and $f \sim H^\mu$, $\frac{df}{dH} \sim \mu H^{\mu-1}$, $\frac{d^2 f}{dH^2} \sim \mu(\mu-1)H^{\mu-2}$. Then for a suitable N depending on μ we have:*

$$|\Delta(f; \underline{x}, \underline{x}')| \leq C R(s, s_0)(s - s_0)^2 (1 + H)^{\mu - 1},$$

where H corresponds to the point $\underline{x}$.

To see this we express f as a function of v, $f \sim C v^{\sqrt{6}\mu}$. We have

$$\frac{d^2 f}{dv^2} \sim C v^{\sqrt{6}\mu - 2}$$

The integral formula Equation (32) gives

$$|\Delta(f; \underline{x}, \underline{x}')| \leq C(v - v_0)^2 (1 + v_*)^{\sqrt{6}\mu - 2},$$

where v_* is one of v, v_0 (which one depending on the sign of $\sqrt{6}\mu - 2$ and which of v, v_0 is the larger). The function s is asymptotic to a multiple of $v^{\sqrt{3/2}}$, hence

$$|s - s_0| \geq C|v - v_0|(1 + v_{**})^{\sqrt{3/2} - 1},$$

where v_{**} is the smaller of v, v_0. Then the result follows by elementary arguments. (The point is that introducing the function R allows us to essentially interchange v, v_0 in our estimates.)

Now consider the function $A = A(Q, H, H_0)$. By construction this satisfies

$$A(Q, H_0, H_0) = 0; \quad \left.\frac{\partial A}{\partial H}\right|_{H = H_0} = 0.$$

In other words, A vanishes to second order along the "diagonal" $H = H_0$. Differentiating r times with respect to Q, we see that $(\frac{\partial}{\partial Q})^r A$ also vanishes to second order along the diagonal. This means that on the arc Γ it is equal to

$$\Delta(\Big(\frac{\partial}{\partial Q}\Big)^r A; \underline{x}, \underline{x}').$$

Thus we see that, on Γ,

$$\Big(\frac{\partial}{\partial Q}\Big)^r A = B_1 - B_2,$$

where

$$B_1 = \Delta(\Big(\frac{\partial}{\partial Q}\Big)^r \frac{p^6}{18}; \underline{x}, \underline{x}')$$

$$B_2 = \Delta(\Big(\frac{\partial}{\partial Q}\Big)^r \frac{H_0 F^+(H,Q)}{F^+(H_0,Q)}; \underline{x}, \underline{x}').$$

Now $f = (\frac{\partial}{\partial Q})^r p^6$ is a homogeneous function of degree $3-2r$ on $\mathbb{R}^3$. It follows that $f \sim CH^\lambda$ on Γ, where $\lambda = 1 - \frac{2r}{3}$ (since H is homogeneous of degree 3); similarly for the derivatives of f. Applying Lemma 8 we see that

$$|B_1| \leq CR(s,s_0)(s-s_0)^2(1+H)^{-2r/3}$$

Now on Γ, $|\underline{x}|^2 \leq C(1+H)^{2/3}$ so we obtain

$$|\underline{x}|^{2r}|B_1| \leq CR(s,s_0)(s-s_0)^2,$$

which is just the form of estimate we need.

The term B_2 is more complicated. Regard v as a function of H – taking $Q=1$. Then we can write

$$\frac{H_0 F^+(H,Q)}{F^+(H_0,Q)} = Q^{\sqrt{3/8}} v(H/Q^{3/2}) \frac{H_0}{Q^{\sqrt{3/8}} v(H_0/Q^{3/2})}.$$

Set

$$f_p = \Big(\frac{\partial}{\partial Q}\Big)^p Q^{\sqrt{3/8}} v(H/Q^{3/2}),$$

$$g_q = \Big(\frac{\partial}{\partial Q}\Big)^q \frac{H_0}{Q^{\sqrt{3/8}} v(H_0/Q^{3/2})}.$$

Then f_p, g_q are smooth functions on Γ (ie, we set $Q=1$ after performing the differentiation). We have

$$B_2 = \sum_{p+q=r} g_q(H_0)\Delta(f_p; \underline{x}, \underline{x}').$$

Now, regarded as a function of x_0, it is easy to see that v has a series expansion for x_0 large:

$$v = x_0^{\sqrt{3/2}}(a_0 + a_1 x_0^{-2} + \ldots).$$

This means that $v(H)$ has an expansion

$$v(H) = H^{1/\sqrt{6}}(b_0 + b_1 H^{-2/3} + \ldots).$$

Hence $$Q^{\sqrt{3/8}}v(H/Q^{3/2}) = H^{1/\sqrt{6}}(b_0 + b_1 QH^{-2/3} + \ldots).$$

So we see that $f_p \sim b_p H^{1/\sqrt{6}-2p/3}$. Applying Lemma 8 we get

$$|\Delta(f_p; \underline{x}, \underline{x}')| \leq CR(s, s_0)(s - s_0)^2 (1 + H)^{1/\sqrt{6}-1-2p/3}.$$

Similarly $$|g_q| \leq C(1 + H_0)^{1-1/\sqrt{6}-2q/3}.$$

So $$(1 + H)^{2r/3} |g_q \Delta(f_p, \underline{x}, \underline{x}')| \leq CR(s, s_0)(s - s_0)^2 \left(\frac{1 + H}{1 + H_0}\right)^{\frac{1}{\sqrt{6}}-1+\frac{2q}{3}}.$$

Now changing the value of N suitably, the power of $(1 + H)/(1 + H_0)$ can be absorbed into $R(s, s_0)$ and we get

$$|\underline{x}|^{2r} |g_q \Delta(f_p, \underline{x}, \underline{x}')| \leq CR(s, s_0)(s - s_0)^2,$$

Hence $|\underline{x}|^{2r} B_2$ is bounded by a multiple of $R(s, s_0)(s-s_0)^2$ and we have finished the proof of Proposition 8 over the positive cone.

We omit the details of the extension of this argument to the region $Q(\underline{x}) < 0$. Let us just explain where the condition $Q(\underline{x}) < -c$ enters, if $x_0 \leq 0$. Using homogeneity we can throw the calculations onto the quadric $Q(\underline{x}) = -1$. We consider the arc $\theta = 0$ in this quadric on which we have arc length co-ordinates s for the point $\underline{x}$ and s_0 for the point $\underline{x}'$. Alternatively, we can use the co-ordinates H, H_0. Then H_0 and s_0 are positive by hypothesis. The problem comes when H and s are large and *negative*. The function $u(H)$ for large positive H has a series expansion

$$u(H) = H^{1/\sqrt{6}}(b_0 + b_1 H^{-2/3} + \ldots),$$

just as before. For large negative H on the other hand the series is

$$u(H) = u(-H)^{-1} = (-H)^{-1/\sqrt{6}}(b_0^{-1} + \ldots).$$

This means that the ratio $H_0 \frac{F^+(H,Q)}{F^+(H_0,Q)}$, for $H_0 \gg 0$ and $H \ll 0$, is

$$\frac{H_0^{1-1/\sqrt{6}}(-H)^{-1/\sqrt{6}}(-Q)^{\sqrt{3/2}}}{(b_0 - b_1 Q H_0^{-2/3} + \ldots)(b_0 - b_1 Q(-H)^{-2/3} + \ldots)}.$$

The presence of the term $(-Q)^{\sqrt{3/2}}$ makes for the difference with the previous case. When we differentiate r times this term contributes so we only get the bound:

$$\left(\frac{\partial}{\partial Q}\right)^r H_0 \frac{F^+(H, Q)}{F^+(H_0, Q)} \leq C H_0^{1-1/\sqrt{6}}(-H)^{-1/\sqrt{6}}.$$

This means that we get

$$|\underline{x}|^{2r}|B_2| \leq C(-s)^{4r/3-\sqrt{2/3}}\, s_0^{2-\sqrt{2/3}}.$$

Now scaling back and using homogeneity the derivative bound becomes

$$(-Q)^{\sqrt{3/2}-r}\,(-s)^{4r/3-\sqrt{2/3}}\, s_0^{2-\sqrt{2/3}}.$$

If $r \geq 2$ this blows up as $Q \to 0$ for fixed $s < 0$, $s_0 > 0$. (As we know it must since the functions are only Hölder continuous along the null cone.) On the other hand if $Q < -c$ then we can proceed to obtain a subexponential bound much as before. We leave it to the reader to check that the additional subtleties induced by the presence of F^- in the definition of τ for $Q < 0$ (Equation (23)) do not affect things in any significant manner.

5.3 Estimates for s

Given a point $(\underline{x}', t')$ in $\mathbb{R}^4$ with $|\underline{x}'| > 3$ we have defined a section $s = s_{\underline{x}',t'}$ of $\mathcal{L}$. For $\alpha > 0$ define a function

$$F_\alpha = \exp(-\alpha \left(\psi_0^{-2}(Q-Q_0)^2 + \psi_0^2(t-t')^2\right)) \tag{37}$$

Also define
$$\Psi(\underline{x}, \underline{x}') = \frac{\psi}{\psi_0} + \frac{\psi_0}{\psi},$$
and
$$\delta(\underline{x}, \underline{x}') = \left|\frac{\psi}{\psi_0} - \frac{\psi_0}{\psi}\right|.$$

The goal of this subsection is to prove the following:

Proposition 9 *For any r, c there are α, C such that for $p \leq r$*

- $|\nabla^p s| \leq C\Psi(\underline{x}, \underline{x}')^p E_\alpha F_\alpha$ *everywhere,*
- $|\nabla^p(\overline{\partial} s)| \leq C(\epsilon + \delta(\underline{x}, \underline{x}'))\Psi(\underline{x}, \underline{x}')^{p+1} E_\alpha F_\alpha$ *throughout* $\{(\underline{x}, t) : |\underline{x}| \geq 3\}$.

The proof of this will require a number of steps. For simplicity we will just prove the estimate on $|\overline{\partial} s|$ – the extension to higher derivatives is straightforward (using the appropriate results from Section 5.2). Since $s = \tau^* \otimes \rho$ we have

$$|\overline{\partial} s| \leq |\overline{\partial}\tau^*||\rho| + |\tau^*||\overline{\partial}\rho|. \tag{38}$$

Throughout this subsection and the next we will make frequent use of the bounds on the derivative of the function ψ. Note that we have

$$\frac{\partial p}{\partial Q} = \frac{3x_0^2 + Q}{p^7} = O(p^{-3}) \tag{39}$$

(by differentiating (5) and (18) with H fixed), while Equation (19) gives

$$\frac{\partial p}{\partial H} = \frac{3x_0}{p^7} = O(p^{-5}). \tag{40}$$

Thus Lemma 1 implies that

$$\psi^{-1}\Big|\frac{\partial \psi}{\partial Q}\Big| \le C\epsilon p^{-1}, \quad \psi^{-1}\Big|\frac{\partial \psi}{\partial H}\Big| \le C\epsilon p^{-3}. \tag{41}$$

Lemma 9 *For suitable C, α we have*

$$|\overline{\partial}\tau^*| \le C\epsilon E_\alpha$$

when $|\underline{x}| \ge 3$.

Recall that when $|\underline{x}| > 3$ the section τ^* is equal to $\hat{L}^+\tau$, so

$$|\overline{\partial}(\tau^*)| \le |\hat{L}^+\overline{\partial}\tau| + |\nabla\hat{L}^+||\tau|. \tag{42}$$

There are two issues here. The first issue is that the estimates of Proposition 6 for $\overline{\partial}\tau$ only hold in a region Ω_c^+. However $\hat{L}^+$ vanishes at points where $x_0 < 0$ and $Q(\underline{x}) > -\frac{b_2\psi}{2\epsilon}$. Since $\psi \ge \epsilon$ we see that the support of $\hat{L}^+$ lies in Ω_c^+ with $c = b_2/2$ and the estimates of Proposition 6 deal with the first term in Equation (42). The second issue concerns the term involving $\nabla\hat{L}^+$. Thus it suffices to show that

$$|\nabla(\lambda(\frac{\epsilon}{b_2}\frac{Q}{\psi}))| \le C\epsilon.$$

The derivative of $\lambda(\frac{\epsilon}{b_2}\frac{Q}{\psi})$ vanishes if $|Q| > \psi b_2/\epsilon$. Since the function λ has bounded derivative it suffices to show that

$$|\nabla\Big(\frac{\epsilon}{b_2}\frac{Q}{\psi}\Big)| \le C\epsilon,$$

when $|Q| \le \psi b_2/\epsilon$. The relevant components of ∇ with respect to the standard orthonormal basis of tangent vectors for our metric g are $\psi\frac{\partial}{\partial Q}$ and $pr\frac{\partial}{\partial H}$. Consider first the Q derivative. We have

$$\Big|\psi\frac{\partial}{\partial Q}\Big(\frac{\epsilon}{b_2}\frac{Q}{\psi}\Big)\Big| = \frac{\epsilon}{b_2}\Big|1 - Q\psi^{-1}\frac{\partial\psi}{\partial Q}\Big| \le \frac{\epsilon}{b_2} + \Big|\frac{\partial\psi}{\partial Q}\Big| \le \frac{\epsilon}{b_2} + C\epsilon p^{-1}\psi.$$

Now $\psi \le Cp$ by the third item of Lemma 1, so we are done.

For the H derivative we have similarly:

$$\Big|pr\frac{\partial}{\partial H}\Big(\frac{\epsilon}{b_2}\frac{Q}{\psi}\Big)\Big| = \Big|pr\frac{\epsilon}{b_2}\frac{Q}{\psi^2}\frac{\partial\psi}{\partial H}\Big| \le pr\psi^{-1}\Big|\frac{\partial\psi}{\partial H}\Big| \le C\epsilon,$$

which completes the proof of Lemma 9.

We now turn attention to the section ρ. We begin with $\hat{\rho}$.

Lemma 10 *For any $\alpha < 1$ there is a constant C such that*

$$|\overline{\partial}\hat{\rho}| \leq C\delta(\underline{x}, \underline{x}')F_\alpha.$$

In our standard orthonormal frame, and the given trivialisation of $\mathcal{L}_2$,

$$\overline{\partial}\hat{\rho} = \left(\psi\left(\frac{\partial}{\partial Q} + \frac{i}{2}(t-t')\right) + i\psi^{-1}\left(\frac{\partial}{\partial t} - \frac{i}{2}(Q-Q_0)\right)\right)\hat{\rho}$$

where $$\hat{\rho} = \exp\left(-\frac{\psi_0^2(t-t')^2 + \psi_0^{-2}(Q-Q_0)^2}{4}\right).$$

This is $$\frac{1}{2}\left(\frac{\psi_0}{\psi} - \frac{\psi}{\psi_0}\right)\left(\frac{Q-Q_0}{\psi_0} - i\psi_0(t-t')\right)\hat{\rho}.$$

The Lemma follows from the fact that for any $\alpha < 1$ there is a C such that

$$Ae^{-A^2} \leq Ce^{-\alpha A^2}.$$

Next we have the following:

Lemma 11 *For any $\alpha < 1$ there is a constant C, depending on b_1, such that*

$$|\overline{\partial}\rho| \leq C(\epsilon + \delta(\underline{x}, \underline{x}'))\Psi(\underline{x}, \underline{x}')F_\alpha,$$

in the set where $|\underline{x}| > 3$.

Given the preceding lemma, we just have to estimate the derivative of the cut-off function $\chi(\frac{\epsilon}{b_1}\frac{Q-Q_0}{\psi_0})$. This is bounded in modulus by $C\frac{\epsilon}{b_1}\frac{\psi}{\psi_0}$, which gives the desired result.

The main result (Proposition 9) in the case of $|\overline{\partial}s|$ follows from Equation (38) and Lemmas 9 and 11, since we clearly have

$$|\tau^*| \leq |\tau| \leq CE_\alpha, \quad |\rho| \leq |\hat{\rho}| = F_1.$$

Notice that if we estimate $\overline{\partial}\tau^*$ over the region $|\underline{x}| \leq 2$ we get a new term involving the cut-off function $\chi(2/|\underline{x}|)$ and our estimate is only as good as that on the full covariant derivative $\nabla\tau^*$. This is why we only consider the case $|\underline{x}| \geq 3$ in the second half of Proposition 9.

5.4 Estimates on sums

For each point $(\underline{x}',t')$ with $|\underline{x}'| \geq 3$ we have now got a section $s_{\underline{x}',t'}$ obeying estimates expressed in terms of functions $E_\alpha, F_\alpha, \Psi(\underline{x},\underline{x}'), \delta(\underline{x},\underline{x}')$. Moreover, $s_{\underline{x}',t'}$ is supported in a set $S(\underline{x}') \times \mathbb{R}$ where $S(\underline{x}')$ is the set of points $\underline{x}$ in $\mathbb{R}^3$ which satisfy the conditions

- $|Q(\underline{x}) - Q(\underline{x}')| \leq \frac{2b_1}{\epsilon}\psi(\underline{x}')$,
- $Q(\underline{x}) \leq -\frac{b_2}{2\epsilon}\psi(\underline{x})$ if x_0 and x_0' have different signs,
- $|\underline{x}| \geq 1$.

Given $\underline{x}$ with $|\underline{x}| \geq 1$ let $N(\underline{x}) \subset \mathbb{R}^3$ be the set

$$N(\underline{x}) = \{\underline{x}' : |\underline{x}'| \geq 3,\ \underline{x} \in S(\underline{x}')\}.$$

The modulus of the section $s_{\underline{x}',t'}$ at the point $(\underline{x}',t')$ is 1 and it is quite clear from the constructions that the section is not small on a ball (in the metric g) of uniform size. Let us say $|s_{\underline{x}',t'}| \geq C^{-1}$ on the ball of radius $1/10$ centred at $(\underline{x}',t')$. Our goal in this subsection is to prove:

Proposition 10 *For any α we can find a countable collection of points $(\underline{x}_i', t_i')_{i\in I}$ with $|\underline{x}_i'| \geq 3$ having the following properties:*

- *The balls B_i of radius $1/10$ centred at the $(\underline{x}_i', t_i')$ cover $\{(\underline{x},t) : |\underline{x}| \geq 4\}$.*
- *Let $E_{\alpha,i}, F_{\alpha,i}$ denote the functions associated with these points and write $\delta_i = \delta(\,\cdot\,,\underline{x}_i')$ and $\Psi_i = \Psi(\,\cdot\,,\underline{x}_i')$. Then for any p there is a C such that*

$$\sum_{i,\ \underline{x}_i'\in N(\underline{x})} \delta_i E_{\alpha,i} F_{\alpha,i} \Psi_i^p \leq C\epsilon,$$

  and

$$\sum_{i,\ \underline{x}_i'\in N(\underline{x})} E_{\alpha,i} F_{\alpha,i} \Psi_i^p \leq C.$$

- *There is a constant K such that for all $D > 1$ we can divide the index set I into at most KD^4 disjoint subsets I_μ, such that if $(\underline{x},t)$ is contained in a ball B_i for $i \in I_\mu$ then for any p there is a C such that*

$$\sum_{j\in I_\mu,\ j\neq i,\ \underline{x}_j'\in N(\underline{x})} \delta_j E_{\alpha,j}(\underline{x},t) F_{\alpha,j}(\underline{x},t) \Psi_j^p \leq C\epsilon\, e^{-D},$$

$$\sum_{j\in I_\mu,\ j\neq i,\ \underline{x}_j'\in N(\underline{x})} E_{\alpha,j}(\underline{x},t) F_{\alpha,j}(\underline{x},t) \Psi_j^p \leq C\, e^{-D}.$$

Notice that this Proposition does not involve the sections we have constructed, only the geometry of the metric g and the functions $F_\alpha, E_\alpha, \Psi, \delta$.

To begin the proof of Proposition 10 we consider the restriction of the metric g to the (x_0, x_1)–plane. We first choose a sequence of points on the x_0–axis such that the $\frac{1}{20}$–balls about these points cover the portion $|x_0| > 3$ of this axis. It is easy to check then that the corresponding $\frac{1}{10}$–balls cover the neighbourhood $T = \{pr < \delta\}$ for some small δ. We then choose a collection of points in the half-plane $x_1 > 0$ and outside T such that the $\frac{1}{10}$–discs (in the metric g) about these points cover the complement U of T and the Euclidean ball $\{x_0^2 + x_1^2 \leq 9\}$ in the half-plane. We denote the centres obtained in this way by P'_j and the $\frac{1}{10}$–discs by D_j. It is fairly clear that we can do this in such a way that any intersection of more than n discs D_j is empty, for some fixed n.

We now move to 3–space. We use the balls centred on the axis to cover the relevant portion of the x_0–axis in 3–space in the obvious way. Recall that the length of the circle orbit under rotations is $2\pi pr$. It is straightforward to check that there is a constant R such that for each point P'_j which is not on the axis

$$\frac{\max_{D_j \cap U}(pr)}{\min_{D_j \cap U}(pr)} \leq R.$$

As a consequence of this we can, for each such P'_j, choose an integer m_j which is comparable to pr for all points in $D_j \cap U$. Then we get a cover of $\mathbb{R}^3$, minus the Euclidean ball of radius 3, in the following way. We take the images of these points P'_j under rotations through multiples of $2\pi/Mm_j$ for suitable fixed M, and the balls of radius $\frac{1}{10}$ centred on these points. In this way we get a collection of $\frac{1}{10}$–balls B_k with centres $\underline{x}'(k)$ in $\mathbb{R}^3$ such that

- The balls B_k cover $\{|\underline{x}| > 3\}$,
- The centre of any ball B_k either lies on the x_0–axis or is contained in the orbit of a P'_j under a cyclic subgroup of the rotation group, where the order of the cyclic group is bounded by a fixed multiple of pr, evaluated at the centre.

Next we move to 4–space. Equation (41) above shows that

$$\psi^{-1}|\nabla\psi| \leq C\epsilon. \tag{43}$$

This means that, once ϵ is sufficiently small, we can suppose that

$$\frac{\max_{B_k}(\psi)}{\min_{B_k}(\psi)} \leq \frac{11}{10},$$

say. We fix a constant M' and for each centre $\underline{x}'(k)$ we take a countable collection of points

$$\Big(\underline{x}'(k), \frac{\nu}{M'\,\psi(\underline{x}'(k))}\Big), \quad \nu \in \mathbb{Z}.$$

This finally gives us our collection of centres $(\underline{x}'_i, t'_i)$ in $\mathbb{R}^4$. For a suitable choice of the constants M and M' we can arrange that the $\frac{1}{10}$–balls about the $(\underline{x}'_i, t'_i)$ cover $\{(\underline{x}, t) : |\underline{x}| > 3\}$.

Now let $(\underline{x}, t)$ be a point with $|\underline{x}| \geq 3$. We want to study the sum

$$B(\underline{x}, t) = \sum_i E_{i,\alpha}(\underline{x}, t) F_{i,\alpha}(\underline{x}, t) \Psi(\underline{x}, \underline{x}'_i)^p \delta(\underline{x}, \underline{x}'_i) \tag{44}$$

with the set of centres $(\underline{x}'_i, t'_i)$ obtained above. The manner in which these centres were chosen allows us to easily sum over the θ and t–variables.

Lemma 12 *Let u_i be an arithmetic progression $u_i = Ai + C$, $A > 0$, labelled by $i \in \mathbb{Z}$. Then there are universal constants k_0, k_1 such that for all $B > 0$*

$$\sum_{i \in \mathbb{Z}} \exp(-\Big(\frac{u_i}{B}\Big)^2) \leq k_0 + k_1 \frac{B}{A}.$$

This is standard and elementary. When we consider the contribution to the sum in Equation (44) from the centres which lie in the same orbit under the translation action we get terms precisely of the form considered in the Lemma (with $A = \frac{1}{M'\psi_0}$ and $B = (\alpha\psi_0^2)^{-1/2}$, where $\psi_0 = \psi(\underline{x}'_i)$). Thus we can reduce to a 3-dimensional problem by summing over translation orbits (which yields at most a uniform constant factor).

The rotation action can also be factored out in a similar way, but requires a more careful treatment. Let

$$\lambda = pr = (4H^2 + r^6)^{1/4}.$$

The centres in a same rotation orbit yield (finitely many) terms of the form considered in Lemma 12, but now $A = \frac{2\pi}{Mm_j} \sim \lambda(\underline{x}'_i)^{-1}$, while $B = (\alpha L(\underline{x})^2)^{-1/2} \sim \lambda(\underline{x}')^{-1}$, where $\underline{x}'$ is the point introduced in Section 5.1, lying on the same quadric as $\underline{x}$ but with $H(\underline{x}') = H(\underline{x}'_i)$. Hence, denoting by (Q, H) and (Q_0, H_0) the co-ordinates of $\underline{x}$ and $\underline{x}'_i$ respectively, the factor Σ resulting from summation over a rotation orbit satisfies

$$|\Sigma| \leq \min\Big(C + C\frac{\lambda(Q_0, H_0)}{\lambda(Q, H_0)},\ C'\lambda(Q_0, H_0)\Big) \tag{45}$$

(using Lemma 12 and the fact that the number of centres in the orbit is of the order of $\lambda(Q_0, H_0)$). We now use:

Lemma 13 *There is a constant C such that*

$$|\Sigma| \leq C + C\psi_0^{-1}|Q - Q_0|,$$

where $\psi_0 = \psi(\underline{x}_i')$.

There are several cases to consider. First assume that $Q_0 \geq -|H_0|^{2/3}$. Then the co-ordinates of $\underline{x}_i'$ satisfy $|x_0| \geq cr$ for some $c \in (0, \frac{1}{2})$ (the positive root of the equation $c^2 + c^{2/3} = \frac{1}{2}$). Hence $r \leq C|H_0|^{1/3}$, and $\lambda(Q_0, H_0) = (4H_0^2 + r^6)^{1/4} \leq C|H_0|^{1/2}$. On the other hand $\lambda(Q, H_0) \geq |2H_0|^{1/2}$, so we get a constant bound on $|\Sigma|$ using Equation (45). In the other case $Q_0 \leq -|H_0|^{2/3}$, the co-ordinates of $\underline{x}_i'$ satisfy $|x_0| \leq cr$, so $r \sim |Q_0|^{1/2}$ and $p \sim |Q_0|^{1/4}$, so $\lambda \sim |Q_0|^{3/4}$. If $Q \leq \frac{1}{2}Q_0$ then

$$|\Sigma| \leq C + C(Q_0/Q)^{3/4}$$

is bounded by a uniform constant. Otherwise, we have $|Q - Q_0| \geq \frac{1}{2}|Q_0|$, so

$$|\Sigma| \leq C'\lambda(Q_0, H_0) = C'pr \leq C\psi_0^{-1}p^2 r \leq C\psi_0^{-1}|Q_0| \leq C\psi_0^{-1}|Q - Q_0|.$$

This completes the proof of the Lemma. Since the factor $\psi_0^{-1}|Q - Q_0|$ can be absorbed into F_α up to an arbitrarily small modification of the constant α, Lemma 13 allows us to sum over rotation orbits.

Thus we can reduce to a 2–dimensional problem. For this we adapt our notation slightly. We regard Q and H as functions on $\mathbb{R}^2$ in the obvious way and for P in the half-space $x_1 \geq 0$ in $\mathbb{R}^2$ let $\underline{\Sigma}(P)$ be the part of the corresponding quadric through P which lies in the half-space. Thus $\underline{\Sigma}(P)$ can be identified with the quotient of one of our quadrics in $\mathbb{R}^3$ under the rotation action. We write $N(P)$ for the quotient of the corresponding set N defined above. For each of the centres P_j' we have chosen above we write $E_j(P), F_j(P), \delta(P, P_j'), \Psi(P, P_j')$ for the corresponding functions on the half-plane.

To prove the second item of Proposition 10 it suffices to prove:

Proposition 11 *Let $\{P_j'\} \in \mathbb{R}^2$ be the set of centres constructed above. Then there is a C such that for any $P \in \mathbb{R}^2$*

$$\sum_{j : P_j' \in N(P)} \delta(P, P_j')\, E_j(P)\, F_j(P)\, \Psi(P, P_j')^p \leq C\epsilon$$

$$\sum_{j : P_j' \in N(P)} E_j(P)\, F_j(P)\, \Psi(P, P_j')^p \leq C$$

The essential thing now is to understand the set $N(P)$. Notice first that if $P' \in N(P)$ and if the x_0 co-ordinates P_0 and P'_0 have different signs then we have $Q(P) \leq -\frac{b_2}{2}\frac{\psi(P)}{\epsilon}$ which implies that $Q(P) \leq -\frac{1}{2}b_2$. Thus we have the following "quarter-space property": if $Q(P) > -\frac{1}{2}b_2$ the sign of the co-ordinate x_0 on the whole of $N(P)$ is the same as that at P (see Figure 3).

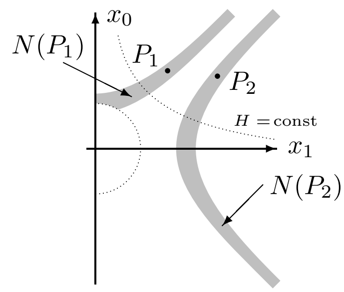

Figure 3: The set $N(P)$ (case 1: $Q(P) > -\frac{b_2}{2}$; case 2: $Q(P) < -\frac{b_2}{2}$)

Now, given P, let $\underline{\Sigma} = \underline{\Sigma}(P)$ be the the quotient of the quadric through P as above. We claim that $N(P)$ is contained in a "thin neighbourhood" of $\underline{\Sigma}$. To state what we need precisely:

Lemma 14 *If b_1 is sufficiently small then for any point P' in $N(P)$ the corresponding level set $H^{-1}(H(P'))$ of H meets $\underline{\Sigma}(P)$ in exactly one point P'', and moreover if $\Gamma(P')$ is the connected arc of the level set joining P' to P'' then*

$$\frac{\max_{P^* \in \Gamma(P')} |P^*|}{\min_{P^* \in \Gamma(P')} |P^*|} \leq 11/10.$$

This is fairly clear from a picture (see Figure 3), and can be verified by routine calculations. Next we have the following:

Lemma 15 *If b_1 is sufficiently small then for any P' in $N(P)$ we have*

$$\frac{\max_{P^* \in \Gamma(P')} \psi(P^*)}{\min_{P^* \in \Gamma(P')} \psi(P^*)} \leq 11/10.$$

To prove this recall that by Equation (41) and Lemma 1 we have

$$\left|\frac{\partial \psi}{\partial Q}\right| \leq C\epsilon.$$

The variation of Q over the connected arc $\Gamma(P')$ is at most $\frac{2b_1}{\epsilon}\psi(P')$. Integrating over the arc we find that for any point P^* on $\Gamma(P')$,

$$|\psi(P^*) - \psi(P')| \leq Cb_1\psi(P').$$

Now we choose b_1 so small that $Cb_1 \leq 1/50$ (say).

We now define a map M from $N(P)$ to $\underline{\Sigma}(P) \times \mathbb{R}$ by

$$M(P') = (P'', Q(P')).$$

We define a metric g_0 on $\underline{\Sigma}(P) \times \mathbb{R}$ as follows. In the $\underline{\Sigma}$ factor we take the metric induced by g, and in the $\mathbb{R}$ factor, with co-ordinate Q, we take

$$\psi(P'')^{-2} dQ^2.$$

In other words, if we take Q and H as co-ordinates we obtain the metric by "freezing" the coefficients of dQ^2 and dH^2 at their values on $Q = Q(P)$. Now by the quarter-space property above, the image of M lies in a connected subset $\Sigma_0 \times \mathbb{R}$ of $\underline{\Sigma} \times \mathbb{R}$, where Σ_0 lies in $\{P : |P| \geq c\}$ for some fixed $c > 0$ depending on b_2.

Lemma 16 *If b_1 is sufficiently small then M is an $11/10$ quasi-isometry from the metric g restricted to $N(P)$ to an open subset in $\Sigma_0 \times \mathbb{R}$ with metric g_0.*

For the Q variable this follows from Lemma 15. For the H variable we have to check that the variation of $\log pr$ along the arc Γ is small, which follows from calculations similar to those above.

Now choose the arc length s along $\underline{\Sigma}(P)$ as co-ordinate, taking the point P as the origin $s = 0$. Thus we can regard the restriction of ψ to $\underline{\Sigma}$ as a function $\psi(s)$. (This notation is not really consistent with that used in Section 3, but we hope this will not cause confusion). On Σ_0 we have

$$|\frac{d\psi}{ds}| \leq C\epsilon\psi$$

so if s_1, s_2 are the arc-length co-ordinates of two points in Σ_0

$$\frac{\psi(s_1)}{\psi(s_2)} \leq e^{C\epsilon|s_1 - s_2|}. \tag{46}$$

We can now prove Proposition 11. We just consider the first inequality, the second being similar. The points P'_j which contribute to the sum lie in $N(P)$ and we can map these by M to get points (s'_j, Q'_j) in $\Sigma_0 \times \mathbb{R}$. We use three facts:

- The quasi-isometry property implies that $E(P, P'_j) \leq \exp(-\alpha(s'_j - s)^2)$, for some α.
- The function $\log \psi$ varies little over the arcs $\Gamma(P')$, so we can replace $\psi(P')$ by $\psi(P'')$ in estimating the sum.

- The terms $\Psi(P, P'_j)$ and $\psi(P)/\psi(P'_j)$ appearing in the sum can be replaced by the exponential bound Equation (46) above.

Putting all of this together, it suffices to bound the sum

$$\sum \exp\left(-\alpha((s'_j)^2 + \psi(s'_j)^{-2}(Q'_j - Q)^2)\right) \exp(C\epsilon|s'_j|)(\exp(C\epsilon|s'_j|) - 1) \quad (47)$$

Now it is easy to check that for any $\alpha' < \alpha$ we have an inequality

$$(e^{\epsilon A} - 1)e^{-\alpha A^2} \leq C\epsilon e^{-\alpha' A^2}.$$

This means that, changing the value of α slightly, it suffices to bound the sum

$$\sum \exp(-\alpha((s'_j)^2 + \Big(\frac{Q'_j - Q}{\psi(s'_j)}\Big)^2)) \quad (48)$$

To do this we compare with the corresponding integral. We consider the image $M(D_j)$ of the 1/10–disc centred on P'_j under the map M and let

$$I_j(\beta) = \int_{M(D_j)} e^{-\beta f} \frac{dQ'}{\psi(s')} ds',$$

where

$$f(s', Q') = (s')^2 + \Big(\frac{Q' - Q}{\psi(s')}\Big)^2.$$

Now over $M(D_j)$ the function $\psi(s')$ is essentially constant and the variations in $Q'/\psi(s')$ and s' are $O(1)$. It follows then that there are constants A, B such that

$$\sup_{M(D_j)} f \leq Af(s'_j, Q'_j) + B.$$

This implies that

$$e^{\beta B} I_j(\beta) \geq e^{-\beta A f(s'_j, Q'_j)} \int_{M(D_j)} \frac{dQ'}{\psi(s')} ds'.$$

We take $\beta = \alpha/A$. Clearly

$$\int_{M(D_j)} \frac{dQ'}{\psi(s')} ds' \geq c$$

for some fixed $c > 0$. We see then that the sum in Equation (48) is bounded by a multiple of

$$\sum_j I_j(\beta).$$

By construction of our open sets D_j, no more than n of the $M(D_j)$ intersect, so

$$\sum_j I_j(\beta) \leq n \int_{\mathbb{R}^2} e^{-\beta f} \frac{dQ'}{\psi(s')} ds'.$$

But this last integral can be evaluated explicitly

$$\int_{\mathbb{R}^2} e^{-\beta(s'^2 + \left(\frac{Q-Q'}{\psi(s')}\right)^2)} \frac{dQ'}{\psi(s')} ds' = \frac{\pi}{\beta}.$$

This completes the verification of the first two items of Proposition 10. We omit the verification of the third item which follows similar lines.

6 Completion of proof

6.1 Verification of Hypothesis 2

In this subsection we will bring together the different strands of the analysis in Sections 4 and 5 to complete the verification of Hypothesis 2. The main issue we have to deal with is the fact that the model for our neighbourhood N of the zero set Γ is a quotient of a tube in $\mathbb{R}^4$ under translations $t \mapsto t + 2\pi\mathbb{Z}\epsilon^{-1}$ whereas in Sections 4 and 5 we have worked in $\mathbb{R}^4$. To deal with this we go back to examine the definition of the section $\hat{\rho}_{\underline{x}',t'}$ in Section 4.2. To construct the line bundle corresponding to $\mathcal{L}_2$ on the quotient space we proceed as follows. On $\mathbb{R}^4$ we take a trivialisation of $\mathcal{L}_2$ in which the connection form is $-i(Q + \frac{\epsilon}{2})dt$. This 1–form is preserved by the translations so we get a line bundle with connection over the quotient space in the obvious way. The factor $\frac{\epsilon}{2}$ means that the holonomy is -1 around the zero set, as required. Now given Q_0, H_0, t', the section $\hat{\rho}_{\underline{x}',t'}$ we defined in Section 4.2 is given, in this trivialisation, by

$$\exp(-\frac{1}{4}\left(\psi_0^2(t-t')^2 + \psi_0^{-2}(Q-Q_0)^2\right))\exp(iU)$$

where

$$U = \frac{1}{2}(Q + Q_0 + \epsilon)(t - t').$$

We now replace t' by $t'_\nu = t' + 2\pi\nu\epsilon^{-1}$ and form the sum

$$\Theta_{\underline{x}',t'} = \sum_{\nu \in \mathbb{Z}} \hat{\rho}_{\underline{x}',t'_\nu}, \tag{49}$$

working always in the fixed trivialisation of $\mathcal{L}_2$. Then $\Theta_{\underline{x}',t'}$ is a $2\pi\epsilon^{-1}$–periodic section. Essentially these are the standard θ–functions.

The modulus of $\Theta_{\underline{x}',t'}$ at the point $(\underline{x}', t')$ is no longer 1. However it is very close to 1, the difference is bounded by the sum

$$2\sum_{\nu\geq 1} e^{-\pi^2\nu^2},$$

which is very small. More generally, the section $\Theta_{\underline{x},t'}$ is very close to $\hat{\rho}_{\underline{x}',t'}$ over a ball (in the metric g) of radius $1/10$ centred on $(\underline{x}', t')$. This means that these sections have essentialy the same local behaviour as those considered before.

The sections $\Theta_{\underline{x}',t'}$ define sections of the corresponding line bundle over the quotient space N and we can repeat all the constructions of Sections 4 and 5 using these in place of the $\hat{\rho}$. However it easier to keep working in $\mathbb{R}^4$. We can the reduce all the estimates for this modified construction to those established before by the following simple device. Recall that for any point $\underline{x}'$, we have $\psi_0 = \psi(\underline{x}') \geq \epsilon$. We can choose an integer q such that

$$q \leq \frac{\psi_0}{\epsilon} \leq 2q.$$

Now we modify the construction in Section 5.4, when we go from a covering in 3-space to a covering in 4–space, slightly. We have centres $\underline{x}'(k)$ in $\mathbb{R}^3$ as before and we take the sequence of centres

$$(\underline{x}'(k), \frac{\nu}{Nq\epsilon}) \quad \nu \in \mathbb{Z},$$

where N is some suitable fixed integer (independent of $\underline{x}'(k)$, while q depends on $\psi(\underline{x}'(k))$). The separation between these centres, in the metric g, is $\frac{\psi}{Nq\epsilon}$ which lies between N^{-1} and $2N^{-1}$: bounded above and below independently of $\underline{x}'(k)$. When we estimate the sum over these centres and combine with the sum involved in the definition of $\Theta_{\underline{x}',t'}$ we get exactly the same form of sum considered in Lemma 12. (Since we estimate via the sum of moduli, the phase factors are irrelevant.)

The verification of Hypothesis 2 should now be clear.

- For fixed k, and hence ϵ, we choose a covering of an appropriate annular region around Γ from the covering in $\mathbb{R}^4$ constructed in Section 5.4, adapted to the quotient as above. Along with this covering we get a collection of approximately holomorphic sections, multiplying the sections of Section 4 by cut-off functions to extend over the 4–manifold. There is just one very small point to mention. In the covering constructed in Section 5.4 some of the centres are taken to lie on the x_0–axis, where the co-ordinate H vanishes. On the other hand, when we defined the sections

$s_{\underline{x}',t'}$ we ruled out this case. However this is a completely artificial problem and we merely need to take sections associated to points arbitrarily close to the axis.

- We extend this covering to the remainder of the 4–manifold using the familiar approximately holomorphic co-ordinates. Likewise for each ball in the covering we have approximately holomorphic sections, defined just as in the theory for compact symplectic manifolds.
- The localisation properties of the sections, expressed through the convergence of the sums in the last two items of Hypothesis 2, follow from the estimates in Section 5.

6.2 The local model, verification of Hypothesis 3

In this subsection we will construct sections σ_0, σ_1 satisfying Hypothesis 3. The construction is completely explicit but is reasonably complicated so we will perform it in four stages.

Stage I

Consider the Riemann surface $\mathbb{C}/2\pi i\mathbb{Z}$ with the symplectic form $dx \wedge dy$, where $z = x + iy$ is the standard co-ordinate on $\mathbb{C}$. Let $\mathbb{L}$ be the Hermitian holomorphic line bundle over $\mathbb{C}/2\pi i\mathbb{Z}$ with a connection having curvature $-i\,dx \wedge dy$ and with holonomy -1 around the circle C corresponding to the imaginary axis.

Lemma 17 *There are holomorphic sections θ_0, θ_1 of $\mathbb{L}$ such that*

- *The θ_i are bounded.*
- *The sections θ_0, θ_1 have no common zeros and the map*
  $$f^I = \theta_1/\theta_0 \colon \mathbb{C}/2\pi i\mathbb{Z} \to \mathbb{CP}^1$$
  maps the circle C bijectively to the circle $i\mathbb{R} \cup \{\infty\}$ in $\mathbb{CP}^1$.
- *The derivative ∂f^I is λ–transverse to 0 for some $\lambda > 0$.*

These sections can be constructed as follows. Recall that the Weierstrass $\wp$–function of the rectangular lattice $\Lambda = 2\mathbb{Z} \oplus 2\pi i\mathbb{Z}$ is an even meromorphic function on the elliptic curve $\mathbb{C}/\Lambda$ with a double pole at the origin, representing it as a double cover of $\mathbb{CP}^1$ ramified at $p_0 = 0$, $p_1 = 1$, $p_2 = i\pi$ and $p_3 = 1 + i\pi$. The meromorphic function $\wp$ is the quotient of two holomorphic sections of the line bundle $\mathcal{O}(2p_0)$ over $\mathbb{C}/\Lambda$. Since $\wp(z)$ and $\wp(1-z)$ have the same ramification points, they must differ by an automorphism of $\mathbb{CP}^1$ (this also follows from the fact that $\mathcal{O}(2p_0)$ and $\mathcal{O}(2p_1)$ are isomorphic). Setting $a=\wp(1)$ and $b=\wp(\frac{1}{2})^2-2\wp(\frac{1}{2})\wp(1)$, we have

$$\wp(1-z)=\frac{a\wp(z)+b}{\wp(z)-a}.$$

The line $\mathrm{Re}(z)=\frac{1}{2}$ is one of the two components of the fixed point locus of the antiholomorphic involution $z\mapsto 1-\overline{z}$ of $\mathbb{C}/\Lambda$, and is mapped bijectively by $\wp$ to the fixed point locus Θ of the involution

$$w\mapsto \frac{a\overline{w}+b}{\overline{w}-a}.$$

Choose a fractional linear transformation $\varphi\in\mathrm{Aut}(\mathbb{CP}^1)$ mapping the circle Θ to the imaginary axis $i\mathbb{R}\cup\{\infty\}$, and let

$$f^I(z)=\varphi(\wp(z+\frac{1}{2})).$$

Then f^I is a doubly-periodic meromorphic function which maps the imaginary axis to itself, without ramification. We can write f^I as the quotient $f^I=\theta_1/\theta_0$ of two holomorphic sections of the line bundle $\mathcal{O}(2p')$ over $\mathbb{C}/\Lambda$, where $p'=-\frac{1}{2}$. This degree 2 line bundle can easily be seen to admit a holomorphic connection with curvature $-i\,dx\wedge dy$ and holonomy -1 around the circle corresponding to the imaginary axis.

Now recall that in our standard model around a component of Γ we write our line bundle $\mathcal{L}$ as $\mathcal{L}_1\otimes\mathcal{L}_2$, where $\mathcal{L}_1$ has curvature $-idH\wedge d\theta$ and $\mathcal{L}_2$ has curvature $-idQ\wedge dt$. Writing $z=\epsilon^{-1}Q+i\epsilon t$, the pullback of $\mathcal{L}_2$ descends to the quotient $\mathbb{C}/\Lambda$, where it can be identified with $\mathbb{L}$. Here we use the condition that the holonomy around each component of Γ is -1. Thus we can regard θ_0 and θ_1 as sections of $\mathcal{L}_2$. Then define

$$\sigma_0^I=\theta_0\otimes\sigma\ ,\ \sigma_1^I=\theta_1\otimes\sigma,$$

where σ is the section of $\mathcal{L}_1$ constructed in Section 4 above.

These sections σ_0^I,σ_1^I have some of the properties required by Hypothesis 3. Let $z_r\in\mathbb{C}$ be the branch points of f^I. We can choose disjoint discs in $\mathbb{C}$ of a fixed radius δ centred on the z_r. We also suppose that δ is chosen small enough that $|\mathrm{Re}(z_r)|>2\delta$ for all r. Let N_r be the tubular region in $\mathbb{R}^4$ defined by the condition $|z-z_r|\leq\delta$. Then the sections have all the desired properties outside the region

$$(\bigcup N_r)\cap(X\setminus K),$$

where we recall that K is the set defined by $|\underline{x}| \geq 10$. In the following stages we will modify the sections to achieve all the required properties. (In fact, except for the very last step, the modifications will only involve the "numerator" σ_1^I.)

Stage II

In the second stage we improve the sections over the intersection of the tubular regions N_r with the annulus $\Omega = \{2 < |\underline{x}| < 5\}$. We take a standard cut-off function β supported in $[0, \delta)$ and equal to 1 on $[0, \delta/2]$. Then define $\beta_r = \beta(|z - z_r|)$. Thus β_r is supported in the tube N_r and equal to 1 on a half-sized tube. Recall that we have functions F_+, F_- which are holomorphic along the quadric surfaces $z = \text{const}$. In Section 3 these were only defined over the subsets $G^\pm$, but we now extend them by zero over the complement of $G^\pm$. For a small parameter α, to be chosen later, we set:

$$\sigma_0^{II} = \sigma_0^I, \qquad \sigma_1^{II} = \sigma_1^I + \alpha \sum_r \beta_r (F_+ + F_-) \sigma_0^I. \tag{50}$$

Thus $f^{II} = \sigma_1^{II} / \sigma_0^{II}$ is

$$f^{II} = f^I + \alpha \sum_r \beta_r (F_+ + F_-).$$

Lemma 18 *For sufficiently small α, ϵ there are $\kappa_1, \kappa_2, \kappa_3 > 0$ such that, over Ω,*

- *∂f^{II} is κ_1–transverse to 0;*
- *$|\overline{\partial} f^{II}| \leq \max(\epsilon \kappa_2, |\partial f^{II}| - \kappa_3)$.*

There are positive constants, independent of ϵ, so that over Ω

- $|\nabla \beta_r| \leq k_1$
- $|F_+ + F_-| \leq k_2$
- $|\nabla (F_+ + F_-)| \leq k_3$
- $|\nabla_z (F_+ + F_-)| \leq k_4 \epsilon$.

Here we write ∇_z for the component of the derivative in the z direction. The existence of these bounds is fairly clear, there is just one point we want to spell out here. The function F_+ is not smooth along the part of the null cone where $x_0 < 0$, but behaves like $(-Q)^\nu$ when $Q < 0$ and vanishes when $Q \geq 0$, where ν is $\sqrt{3/2}$. Since $\nu > 1$ we have a uniform bound on the first derivative, but one might worry about the higher derivatives. In terms of $x = \mathrm{Re}(z) = Q/\epsilon$, F_+ behaves like $\epsilon^\nu (-x)^\nu$; so on the set where $x < -\delta$ all derivatives with respect to z are bounded by multiples of ϵ^ν. Since our formulae only involve F_+ over the tubes N_r, on which $|x| > \delta$, we do not encounter any problems from the singularities of $F_\pm$.

Let $M_r \subset N_r$ be the interior tube on which $\beta_r = 1$. Then there is a $K_1 > 0$ such that $|\partial f^I| \geq K_1$ outside the M_r (but inside Ω). So on this set

$$|\partial f^{II}| \geq |\partial f^I| - \alpha \left|\sum \nabla\beta_r(F_+ + F_-) + \beta_r\nabla(F_+ + F_-)\right|.$$

At any given point there is at most one term contributing to the sum (since the N_r are disjoint) so we have

$$|\partial f^{II}| \geq K_1 - \alpha(k_1 k_2 + k_3).$$

Thus if we choose $\alpha < K_1/(10(k_1k_2 + k_3))$ we have $|\partial f^{II}| \geq 9K_1/10$ outside the M_r. On the other hand, outside the M_r, we have

$$|\overline{\partial} f^{II}| \leq \alpha(k_1k_2 + k_3) \leq K_1/10.$$

Now consider the situation inside a tube M_r where

$$f^{II} = f^I + \alpha(F_+ + F_-).$$

Then

$$|\overline{\partial} f^{II}| = \alpha|\overline{\partial}(F_+ + F_-)| \leq \alpha k_4 \epsilon,$$

since $F_+ + F_-$ is holomorphic along the quadric surfaces and only the z derivative contributes. Now on each quadric surface the holomorphic function $F_+ + F_-$ is either unramified (for $Q > 0$) or has two ramification points (where $x_0 = 0$ and $\theta \in \{0, \pi\}$, for $Q < 0$). Let $p_r^\pm$ be the ramification points on the surface corresponding to z_r and $B_r^\pm$ be the δ–balls about $p_r^\pm$. It is clear then that there is a $K_2 > 0$ such that in the intersection of Ω and $M_r \setminus (B_r^+ \cup B_r^-)$, and once ϵ is sufficiently small, we have

$$|\partial_w(F_+ + F_-)| \geq K_2,$$

where ∂_w denotes the derivative along the quadric surfaces. Thus, on this set,

$$|\partial f^{II}| \geq \alpha K_2.$$

On the other hand it is also clear that if $B_r^\pm$ meets the annulus Ω we have a bound on the inverse of the Hessian of f^{II} over $B_r^\pm$:

$$|(\nabla\partial f^{II})^{-1}| \leq K_3 \alpha^{-1}.$$

In sum then, ∂f^{II} is κ_1–transverse to 0 over Ω with

$$\kappa_1 = \min(\tfrac{9}{10}K_1,\, \alpha K_2,\, \alpha K_3^{-1}),$$

while

$$|\overline{\partial} f^{II}| \leq \max(\kappa_2 \epsilon,\, |\partial f^{II}| - \kappa_3)$$

with $\kappa_2 = \alpha k_4$, $\kappa_3 = 8K_1/10$.

Stage III

The formulae (50) define σ_i^{II} over all of $\mathbb{R}^4$ but they do not satisfy the requirements of Hypothesis 3. One problem is that the section σ_1^{II} is not $\epsilon-$holomorphic over $\{|x| \geq 10\}$ because when we differentiate we pick up a term from $\nabla\beta_r$ which is multiplied by the small parameter α but is not controlled by ϵ. We now get over this problem.

First we address the fact that the functions F_+, F_- are not smooth along the null cone. This is similar to the construction in Section 4.2. We define a function γ_+ in the region $|\underline{x}| > 0.5$ in the following way. We let $\gamma_+(\underline{x}) = 1$ if $x_0 > 0$ and $\gamma_+(\underline{x}) = \gamma_\epsilon(Q(\underline{x}))$ if $x_0 \leq 0$, where γ_ϵ is a standard cut-off function, with $\gamma_\epsilon(t) = 1$ if $t \leq -\delta\epsilon$ and $\gamma_\epsilon(t) = 0$ if $t \geq -\frac{1}{2}\delta\epsilon$. Once ϵ is sufficiently small, the function γ_+ is smooth in $\{|\underline{x}| > 0.5\}$. Now we put $\tilde{F}_+ = \gamma_+ F_+$. Then $\tilde{F}_+$ is a smooth function over $\{|\underline{x}| > 0.5\}$, holomorphic along the quadric surfaces. We define $\tilde{F}_-$ in a symmetrical fashion. Notice that $\tilde{F}_\pm = F_\pm$ over $\bigcup N_r$.

Let $\chi = \chi(|\underline{x}|)$ be a standard cut-off function, equal to 1 when $|\underline{x}| \leq 5$ and zero when $|\underline{x}| \geq 10$. Now we set $\sigma_0^{III} = \sigma_0^{II} = \sigma_0^I$ and

$$\sigma_1^{III} = \chi\sigma_1^{II} + (1-\chi)(\sigma_1^I + \alpha(\tilde{F}_+ + \tilde{F}_-)\sigma_0^I).$$

These sections are well-defined everywhere, even though the $\tilde{F}_\pm$ are not, because the factor $(1-\chi)$ vanishes when $|\underline{x}| \leq 0.5$.

Lemma 19 *There are constants $C, \kappa_1, \kappa_2, \kappa_3$ such that for small enough α and ϵ we have*

- $|\overline{\partial}\sigma_i^{III}| \leq C\epsilon$ *in* $\{|\underline{x}| \geq 10\}$
- *if* $f^{III} = \sigma_1^{III}/\sigma_0^{III}$ *then over* $\{2 \leq |\underline{x}| \leq 10\}$, ∂f^{III} *is* κ_1*–transverse to* 0 *and* $|\overline{\partial} f^{III}| \leq \max(\epsilon\kappa_2, |\partial f^{III}| - \kappa_3)$

Consider the second item of the Lemma. The proof of the previous Lemma applies equally well to any fixed annulus, with suitable adjustment of constants. Thus here we have to deal with extra terms introduced by, on the one hand, the passage from $F_\pm$ to $\tilde{F}_\pm$ and on the other hand the introduction of the cut-off function χ. The first issue is essentially covered by the discussion at the beginning of the proof of Lemma 18, which applies equally well to $\tilde{F}_\pm$. So we will simply ignore the distinction between $\tilde{F}_\pm$ and $F_\pm$, and consider the function

$$f^I + \alpha\,(\chi \textstyle\sum \beta_r + (1-\chi))(F_+ + F_-).$$

When we differentiate this we get a new term

$$\alpha\,\nabla\chi\,(\textstyle\sum \beta_r - 1)(F_+ + F_-)$$

which is supported outside the M_r. The size of $\nabla\chi$ is bounded (independently of ϵ): $|\nabla\chi| \leq k_5$ say. Then the size of the new term is bounded by $\alpha k_2 k_5$. Thus the estimates inside M_r are completely unchanged and outside M_r we have

$$|\partial f^{III}| \geq \frac{9}{10}K_1 - \alpha k_2 k_5, \quad |\overline{\partial} f^{III}| \leq \frac{1}{10}K_1 + \alpha k_2 k_5.$$

This establishes the second item of the lemma, once α is sufficiently small and the constants κ_i are adjusted suitably.

The first item of the lemma follows from the fact that on $\{|\underline{x}| \geq 10\}$ we have simply

$$f^{III} = f^I + \alpha(\tilde{F}_+ + \tilde{F}_-)$$

and we can apply the bounds on the derivatives of $\tilde{F}_\pm$, together with the rapid exponential decay of σ.

Stage IV

In this final stage, we modify the construction to ensure that we get a topological Lefschetz fibration over the inner region. For each point z in one of the discs $|z - z_r| < \delta$ we have a corresponding quadric surface $\Sigma(z)$, say. We can use our standard co-ordinates H, θ to identify these surfaces for different values of z, so we have diffeomorphisms $\tau_z : \Sigma(z) \to \Sigma(z_r)$. Let ρ be a standard cut-off function with $\rho(\underline{x}) = 0$ if $|\underline{x}| \leq 1$ and $\rho(\underline{x}) = 1$ if $|\underline{x}| \geq 2$. On the surface $\Sigma(z)$ we define

$$F_{\pm,r} = \rho F_\pm + (1-\rho) F_\pm \circ \tau_z.$$

This defines new functions $F_{\pm,r}$ on the tube N_r which are equal to $F_\pm$ when $|\underline{x}| \geq 2$. Now define σ_1^{IV} to be equal to σ_1^{III} in $|\underline{x}| \geq 2$ and to be given by the modified formula

$$\sigma_1^{IV} = \sigma_1^I + \alpha \sum \beta_r (F_{+,r} + F_{-,r}) \sigma_0^I$$

in the inner region $|\underline{x}| \leq 2$. Again, we keep the same "denominator" $\sigma_0^{IV} = \sigma_0^{III}$.

Lemma 20 *When α is sufficiently small the ratio $f^{IV} = \sigma_1^{IV}/\sigma_0^{IV}$ is a topological Lefschetz fibration over $\{|\underline{x}| \leq 1\}$, with symplectic fibres.*

Notice that the statement of this lemma does not involve any almost complex structure or quantative estimates. Clearly the only issue involves the behaviour over the tubes N_r and to prove the Lemma we consider an auxiliary almost-complex structure on the tubes – just the integrable product structure given by the identification with $D_r \times \Sigma(z_r)$. Thus taking w as a complex co-ordinate on $\Sigma(z_r)$ our function has the simple form on N_r,

$$f^{IV}(z,w) = f^I(z) + \alpha\, \beta_r(z)\, g(w),$$

where g is the holomorphic function $F^+ + F^-$ on $\Sigma(z_r)$. This function f^{IV} is holomorphic on the interior tube M_r with nondegenerate critical points. So it suffices to show that

$$|\overline{\partial} f^{IV}| < |\partial f^{IV}|$$

on $N_r \setminus M_r$, where now $\overline{\partial}, \partial$ refer to the product complex structure. Then on this region we still have

$$|\partial f^I| \geq K_1, \quad |\nabla \beta_r| \leq k_1, \quad |F_+ + F_-| \leq k_1.$$

Now

$$|\overline{\partial} f^{IV}| = \alpha |\nabla \beta_r|\, |g| \leq \alpha k_1 k_2,$$

while

$$|\partial f^{IV}| \geq |\partial_z f^{IV}| = |\partial_z f^I + \alpha\, \nabla\beta_r\, g| \geq K_1 - \alpha k_1 k_2.$$

Thus the result follows once $\alpha < K_1/2k_1k_2$. (The point of this proof is that we do not need to control the derivatives of $F_\pm$ in the inner region where $|\underline{x}| < 1$.)

This essentially completes our construction. There is just one last issue; that we want to have sections defined over the whole manifold X while up to now we have been working in the local model. So we define

$$\sigma_i = \phi\, \sigma_i^{IV}$$

where ϕ is a cut-off function equal to 1 for $|\underline{x}| \leq c\,\epsilon^{-1}$ and to zero when $|\underline{x}| \geq 2c\,\epsilon^{-1}$ (for some fixed $c > 0$). Thus these sections σ_i can be extended by 0 over the whole of X. Our final result is:

Proposition 12 *There are constants $\kappa_1, \kappa_2, \kappa_3, \underline{C}$ such that for a suitable value of α, and once ϵ is sufficiently small, the sections σ_0, σ_1 satisfy Hypothesis $H_3(\epsilon, \kappa_1, \kappa_2, \kappa_3, \underline{C})$.*

The proof of this proposition has been largely covered in the preceding lemmas. There is one point left over from Stage IV: we need to check that the map f^{IV} satisfies the required transversality estimates over the annulus $\{1 \leq |\underline{x}| \leq 2\}$. Here the discussion follows the same lines as in Stage II, except that we replace the functions $F_\pm$ by the linear combinations

$$F_{\pm,r} = \rho F_\pm + (1-\rho) F_\pm \circ \tau_z = F_\pm + (1-\rho)(F_\pm \circ \tau_z - F_\pm).$$

But $F_{\pm,r} - F_{\pm}$ is $O(\epsilon)$ (along with its derivatives). So the extra term introduced here causes no problem.

Finally, we check that the terms introduced by the cut-off function ϕ are much smaller than ϵ due to the rapid decay of σ away from the origin; this completes the argument.

6.3 The odd case

In all our discussion so far we have focussed on the case when the zero set Γ has just one component and the local model is the "even" version N_+. We now consider the modifications required for the general case. It is quite obvious that the existence of several components makes no difference to the argument, all we have to discuss is the case of the "odd" model N_-. In this case the map σ_- interchanges the two components of the positive cone and maps (H, θ) to $(-H, -\theta)$ but preserves the co-ordinate Q.

We begin with the last part of the construction, the local model in Section 6.2 above. Since the sections σ_0^I, σ_1^I only depend on the Q, t variables the first step goes through unchanged. In the later stages we use the fact that the involution interchanges the functions F_+ and F_-, and so preserves their sum. The upshot is that the whole construction in Section 6.2 goes over immediately to the odd case.

The slightly more substantial discussion involves the construction of the localised sections in the odd case. Working in $\mathbb{R}^4$, in Section 6.1 we have defined $2\pi\epsilon^{-1}$–periodic sections $\Theta_{\underline{x}',t'}$. We write these as

$$\Theta_{\underline{x}',t'} = \Theta^+_{\underline{x}',t'} + \Theta^-_{\underline{x}',t'},$$

taking the even and odd terms respectively in the sum (49). Thus $\Theta^{\pm}_{\underline{x}',t'}$ are $4\pi\epsilon^{-1}$–periodic and the translation $t \mapsto t + 2\pi\epsilon^{-1}$ interchanges the two sections. Now we define

$$s_{\underline{x}',t'} = \Theta^+_{\underline{x}',t'} \otimes \tau^*_{\underline{x}'} + \Theta^-_{\underline{x}',t'} \otimes \tau^*_{\sigma_-(\underline{x}')}.$$

These sections are invariant under the map $\overline{\sigma}_-$ on $\mathbb{R}^4$ so descend to N_-.

7 The converse result

7.1 Proof of Theorem 3

The proof of Theorem 3 is very similar to that of Gompf's result for symplectic Lefschetz fibrations and pencils [8], which in turn relies on a classical argument of Thurston [15].

Let X be a compact oriented 4–manifold, and let $f\colon X \setminus A \to S^2$ be a singular Lefschetz pencil with singular set Γ. Let B be the finite set of isolated critical points of f in $X \setminus \Gamma$, near which f is modelled on $(z_1, z_2) \mapsto z_1^2 + z_2^2$. We assume that there exists a cohomology class $h \in H^2(X)$ such that $h(\Sigma) > 0$ for every component Σ of a fibre of f (if every component Σ contains a base point of the pencil, then we can choose h to be Poincaré dual to the homology class of the fibre).

Step 1 We start by constructing a closed 2–form ω_0 over a regular neighbourhood U of $A \cup B \cup \Gamma$, non-degenerate outside of Γ and positive on the fibres of f, in the following manner. Near $A \cup B$, we take ω_0 to be the standard Kähler form of $\mathbb{C}^2$ in some local oriented co-ordinates in which f is given by the standard models $(z_1, z_2) \mapsto z_1/z_2$ and $(z_1, z_2) \mapsto z_1^2 + z_2^2$. Near a point $p \in \Gamma$, we have oriented local co-ordinates in which f is modelled on $(x_0, x_1, x_2, t) \mapsto x_0^2 - \frac{1}{2}(x_1^2 + x_2^2) + it$. Then we let

$$\omega_p = d\Big(\chi(|t|)\, x_0(x_1\, dx_2 - x_2\, dx_1)\Big),$$

where χ is a suitable smooth cut-off function, and we extend ω_p into a closed 2–form defined over a tubular neighbourhood of the component of Γ containing p, supported near p. The 2–form ω_p vanishes on Γ, and its restriction to the fibres of f is non-negative, and positive near p (outside of Γ). By choosing a suitable finite subset $\{p_i\}$ of Γ and setting $\omega_0 = \sum_i \omega_{p_i} + f^*(\omega_{S^2})$, we obtain a closed 2–form defined over a neighbourhood of Γ, positive on the fibres, vanishing on Γ and non-degenerate outside of Γ.

Step 2 Our next task is to construct local closed 2–forms over neighbourhoods of the fibres of f, compatible with our local model ω_0 near $A \cup B \cup \Gamma$, and restricting positively to the vertical tangent spaces; we will then glue these into a globally defined 2–form. For this purpose, we choose a closed 2–form $\eta \in \Omega^2(X)$, with $[\eta] = h$. Since U retracts onto a union of points and circles, $H^2(U) = 0$, and there is a 1–form β such that $\omega_0 - \eta = d\beta$ over U. Extending β to an arbitrary 1–form on M with support in a neighbourhood of U, and replacing η by $\eta + d\beta$, we can assume that $\eta_{|U} = \omega_0$.

Given any point $q \in S^2$, we can find a regular neighbourhood V_q of the fibre $F_q = f^{-1}(q) \cup A$, and neighbourhoods $U'' \subset U' \subset U$ of $A \cup B \cup \Gamma$, with the following properties:

- $V_q \cap U'$ retracts onto $F_q \cap (A \cup B \cup \Gamma)$;

- $V_q \setminus (V_q \cap U'')$ is diffeomorphic to a product $D^2 \times (F_q \setminus (F_q \cap U''))$;
- there exists a smooth map $\pi\colon V_q \to V_q$ with image in $F_q \cup (V_q \cap U')$, equal to identity over $F_q \cup (V_q \cap U'')$.

The first and second properties can easily be ensured by shrinking V_q so that all critical points of f over V_q lie close to the singular locus of F_q; the map π can then be built by interpolating between the identity map over $V_q \cap U'$ and the projection map from $V_q \setminus (V_q \cap U'')$ to $F_q \setminus (F_q \cap U'')$ given by the product structure.

Since by assumption $[\eta] = h$ evaluates positively over each component of F_q, shrinking U' if necessary we can equip F_q with a (near) symplectic form σ_q which coincides with η over $F_q \cap U'$, is symplectic over the smooth part of F_q, and such that $[\sigma_q - \eta_{|F_q}] = 0$ in $H^2(F_q, F_q \cap U')$ (ie, $\int_\Sigma \sigma_q = h(\Sigma)$ for every component Σ of F_q). Using the projection π to pull back the 2–forms η on $V_q \cap U'$ and σ_q on F_q, we obtain a 2–form $\tilde\eta_q$ on V_q with the following properties:

- $\tilde\eta_q$ is closed, and $[\tilde\eta_q] = h_{|V_q}$;
- $\tilde\eta_q$ coincides with η over $V_q \cap U''$;
- $[\tilde\eta_q - \eta] = 0$ in $H^2(V_q, V_q \cap U'') \simeq H^2(F_q, F_q \cap U'')$;
- (shrinking V_q if necessary) the restriction of $\tilde\eta_q$ to $\mathrm{Ker}(df)$ is positive at every regular point of f in V_q.

By the third property, there is a 1–form β_q on V_q, vanishing identically over $V_q \cap U''$, such that $\tilde\eta_q = \eta + d\beta_q$.

Step 3 For each $q \in S^2$, the above construction yields a 2–form $\tilde\eta_q$ defined over a neighbourhood V_q of the fibre F_q. By compactness, each V_q contains the preimage of a neighbourhood D_q of q in S^2, and there is a finite set $Q \subset S^2$ such that the open subsets $(D_q)_{q\in Q}$ cover S^2. Consider a smooth partition of unity $\sum_{q\in Q} \rho_q = 1$ with ρ_q supported inside D_q, and define

$$\tilde\eta = \eta + d\Big(\sum_{q\in Q} (\rho_q \circ f)\,\beta_q\Big). \tag{51}$$

The closed 2–form $\tilde\eta$ coincides with η over the intersection $\tilde U$ of the neighbourhoods U'' considered above for all $q \in Q$, and hence is well-defined over all X even though (51) only makes sense outside of A. Moreover, the restriction of $\tilde\eta$ to a fibre F_p of f is

$$\tilde\eta_{|F_p} = \sum_{q\in Q} \rho_q(f(p))\,(\eta + d\beta_q)_{|F_p} = \sum_{q\in Q} \rho_q(f(p))\,\tilde\eta_{q|F_p},$$

ie, a convex combination of positive forms; hence $\tilde{\eta}$ induces a symplectic structure on each fibre of f (outside of the critical points). Hence, as in Thurston's original argument, for large enough $\lambda > 0$ the 2–form

$$\omega_\lambda = \tilde{\eta} + \lambda\, f^*\omega_{S^2}$$

is closed and non-degenerate over $X \setminus (A \cup \Gamma)$, and restricts positively to the fibres of f; moreover ω_λ vanishes transversely along Γ, as expected. However, ω_λ does not extend smoothly over the base locus A, and we need to apply a trick due to Gompf [8] in order to complete the construction.

Step 4 Near a base point of f, consider local co-ordinates in which f is the projectivisation map from $\mathbb{C}^2 \setminus \{0\}$ to $\mathbb{CP}^1$, and denote by r the radial co-ordinate and by α the pullback to $\mathbb{C}^2 \setminus \{0\} = \mathbb{R}^+ \times S^3$ of the standard contact form of S^3. Then we have

$$\omega_\lambda = \lambda\, f^*\omega_{S^2} + \omega_0 = (\lambda + r^2)\, f^*\omega_{S^2} + \frac{1}{2}\, d(r^2) \wedge \alpha.$$

Setting $R^2 = \lambda + r^2$, we have $\omega_\lambda = R^2\, f^*\omega_{S^2} + \frac{1}{2}\, d(R^2) \wedge \alpha$. Hence, the radially symmetric map $\varphi(z) = (\lambda + |z|^2)^{1/2}\, z/|z|$ defines a symplectic embedding of $(\mathbb{C}^2 \setminus \{0\}, \omega_\lambda)$ into $(\mathbb{C}^2, \omega_0)$, whose image is the complement of a ball of radius $\lambda^{1/2}$. Therefore, by replacing the ball of radius ϵ around each point of A in (X, ω_λ) by a standard ball of radius $(\lambda + \epsilon^2)^{1/2}$ in $(\mathbb{C}^2, \omega_0)$ we can obtain a globally defined near-symplectic structure ω. More precisely, ω is naturally defined on the 4–manifold Y obtained from X by this cut-and-paste process; however Y can easily be identified with X via a diffeomorphism which equals identity outside of an arbitrarily small neighbourhood of A.

Another viewpoint is to observe that ω_λ extends smoothly to the manifold $\hat{X}$ obtained by blowing up X at the base points; gluing in standard balls in place of the exceptional divisors amounts to a symplectic blowdown of $(\hat{X}, \omega_\lambda)$, and yields a well-defined near-symplectic form on X. In any case, one easily checks that the various requirements satisfied by ω_λ (vanishing along Γ, and positivity over the fibres of f) still hold for the modified form ω; this completes the main part of the argument.

The cohomology class of the constructed form ω is $h + \lambda\, f^*[\omega_{S^2}]$ (identifying implicitly $H^2(X)$ with $H^2(X \setminus A)$). If we assume that every component of every fibre contains a base point we can take h to be Poincaré dual to the class of the fibre. In that case $f^*[\omega_{S^2}] = h$ (up to a scalar factor), so after scaling by $\frac{1}{1+\lambda}$ we can ensure that $[\omega] = h$ is Poincaré dual to the fibre. (However, since we have no control over the relative class $[\omega] \in H^2(X, \Gamma)$, deformations of near-symplectic forms in the class h are not always generated by isotopies of X).

Before we can state more precisely our uniqueness result for the deformation class of ω, we consider again the positivity property for the restriction of ω to the fibres of f, and its implications for the local structure near a point of Γ. Recall that the first-order variation of ω at a point x of Γ yields canonically a linear map $\nabla_x\omega\colon N\Gamma_x \to \Lambda^2 T^*X_x$. Restrict locally f to a normal slice D to Γ through x obtained as the preimage of a transverse arc to $f(\Gamma)$ through $f(x)$. Then the 2–jet of $f_{|D}$ at x defines a non-degenerate quadratic form Q on $TD_x \simeq N\Gamma_x$; and, if one approaches x in the direction of a non-zero vector $v \in TD_x$, the plane field $\operatorname{Ker} df$ converges to $v^\perp = \operatorname{Ker} Q(v,\cdot) \subset TD_x$. Hence, the positivity condition on the restriction of the near-symplectic form ω to the fibres of f implies that, for every $v \in TD_x \setminus \{0\}$, the 2–form $\nabla_x\omega(v)$ evaluates non-negatively on the 2–dimensional subspace $v^\perp \subset TD_x$ (since it is the limit of the tangent spaces to the fibres when approaching x in the direction of v). However, in our case it is easy to check that the above construction of ω guarantees that $\omega_{|\operatorname{Ker} df}$ is bounded from below by a constant multiple of the distance to Γ (ie, a constant multiple of the norm of ω). Equivalently, the tangent spaces to the fibres near x do not tend to degenerate to isotropic subspaces as one approaches x. This implies that the restriction of $\nabla_x\omega(v)$ to the 2–plane $v^\perp$ is in fact *positive* for every $v \neq 0$. Now we have the following:

Lemma 21 *Let ω_0, ω_1 be two near-symplectic forms with the same zero set Γ and for which the smooth parts of the fibres of f are symplectic. Assume moreover that for all $x \in \Gamma$, $v \in N\Gamma_x \setminus \{0\}$, $j \in \{0,1\}$, the restriction of $\nabla_x\omega_j(v)$ to the limiting tangent plane $v^\perp$ is positive. Then ω_0 and ω_1 are deformation equivalent through near-symplectic forms with the same properties.*

Proof Start with the convex combinations $\omega_s = (1-s)\omega_0 + s\,\omega_1$. For all $s \in [0,1]$, ω_s is a closed 2–form which vanishes on Γ and evaluates positively on the fibres of f outside of the critical points, but it may be degenerate at some points of $X \setminus \Gamma$. We can avoid this problem by deforming ω_0 and ω_1 to make them standard over a small neighbourhood of A, choosing a large enough constant $\lambda > 0$, and considering the 2–forms $\tilde\omega_s$ obtained from $\omega_s + \lambda\, f^*\omega_{S^2}$ by inserting standard balls near the base points as described above.

The 2–forms $\tilde\omega_s$ are closed and positive on fibres, they vanish on Γ, and if λ is large enough they are non-degenerate outside of Γ (away from $A \cup B$ this follows from Thurston's classical argument; and at a point $x \in A \cup B$ this follows from positivity on the fibres, which implies that $\tilde\omega_s$ tames a naturally defined complex structure on T_xX). Moreover, $\tilde{\omega}_0$ and ω_0 are deformation equivalent through the family of near-symplectic forms obtained by blowing down $\omega_0 + t\, f^*\omega_{S^2}$ for $t \in [0, \lambda]$; and similarly for $\tilde{\omega}_1$ and ω_1. Hence, all that remains to be checked is the non-degeneracy of $\nabla\tilde{\omega}_s$ along Γ for all $s \in [0, 1]$.

By assumption, for all $x \in \Gamma$ and $v \in N\Gamma_x \backslash \{0\}$, the 2–forms $\nabla_x\omega_j(v)$ $(j = 0, 1)$, and consequently $\nabla_x\tilde{\omega}_j(v)$ too, evaluate positively on the limiting vertical tangent space $v^\perp$. Since this positivity condition is preserved by convex combinations, we conclude that $\nabla_x\tilde{\omega}_s(v)$ evaluates positively on $v^\perp$. Moreover this implies that $\nabla_x\tilde{\omega}_s(v) \neq 0$ for all $s \in [0, 1]$, $x \in \Gamma$, $v \in N\Gamma_x \setminus \{0\}$, which proves that $\tilde{\omega}_s$ vanishes transversely along Γ and hence is a near-symplectic form.

7.2 Proof of Proposition 1

Consider $\mathbb{R}^4$ with its standard Euclidean structure and orientation, inducing a splitting $\Lambda^2\mathbb{R}^4 = \Lambda^2_{+,0} \oplus \Lambda^2_{-,0}$. The wedge-product restricts to a given 3-dimensional subspace $P \subset \Lambda^2\mathbb{R}^4$ as a definite positive bilinear form if and only if P can be written as the graph $P = \{\alpha + L(\alpha),\ \alpha \in \Lambda^2_{+,0}\}$ of a linear map $L\colon \Lambda^2_{+,0} \to \Lambda^2_{-,0}$ with operator norm less than 1. Therefore, positive definite subspaces form a "convex" subset of the Grassmannian of 3-planes in $\Lambda^2\mathbb{R}^4$. Moreover, given an element $\beta \in \Lambda^2\mathbb{R}^4$ with $\beta \wedge \beta > 0$, the space of all positive definite 3-planes containing β is again convex (and hence contractible). In another guise, the set of positive definite subspaces can be identified with the set of conformal classes of Euclidean metrics on $\mathbb{R}^4$, ie, for each such subspace P there is a unique metric, up to scale, which realises P as its space of self-dual forms.

Given a near-symplectic form ω on X, our goal is to build a Riemannian metric with respect to which ω is self-dual; for this purpose, we first build a smooth rank 3 subbundle P of Λ^2T^*X, positive definite with respect to the wedge-product, and such that ω is a section of P. The smoothness assumption implies that, at every point $x \in \Gamma = \omega^{-1}(0)$, P_x must coincide with the image of the intrinsically defined derivative $\nabla\omega_x\colon T_xX \to \Lambda^2T^*X_x$. We can extend the construction of P first to a neighbourhood of Γ, and then to all of X, using the convexity property mentioned in the previous paragraph to patch together local constructions by means of a partition of unity.

By the discussion above, there is a unique conformal class $[g]$ which realises the subbundle P as the bundle of self-dual forms. For any metric g in this conformal class, the 2–form ω is self-dual, and then closedness implies harmonicity. This completes the proof of the first statement in the Proposition.

We now consider the claim that if X is compact and $b_2^+(X) \geq 1$ then for generic Riemannian metrics on X one can obtain near-symplectic structures from self-dual harmonic forms. This is proved by considering the space $\mathcal{C}$ of pairs (g,a), where g is a $C^{k,\alpha}$ Riemannian metric on X and $a \in H^2_{+,g}$ is a cohomology class such that $a^2 = 1$ and admitting a self-dual representative. The universal bundle Λ^+ over $X \times \mathcal{C}$, whose fibre at (x,g,a) is $\Lambda^2_{+,g}T^*X_x$, admits a universal section Ω whose restriction to $X \times \{(g,a)\}$ is the unique harmonic self-dual 2–form in the given cohomology class. It can be shown that Ω is transverse to the zero section of Λ^+ (see for example [11], Section 3). The statement follows by observing that the regular values of the projection of $\Omega^{-1}(0)$ to $\mathcal{C}$ form a dense subset of the second Baire category in $\mathcal{C}$. Detailed proofs have already appeared in the literature, and the reader is referred to [9] (Theorem 1.1) or [11] (Proposition 1).

The only remaining statement to prove is that $[\omega] \in H^2(X,\mathbb{R})$ can be chosen to be the reduction of a rational class. However, this follows readily from the observation that the set of all $(g,a) \in \mathcal{C}$ for which the self-dual harmonic form in the class a has transverse zeros is an open subset of $\mathcal{C}$, and therefore necessarily contains points such that a is proportional to a rational cohomology class.

8 Topological considerations and examples

8.1 Monodromy

Consider a near-symplectic 4–manifold (X,ω) with $\omega^{-1}(0) = \Gamma$, and a singular Lefschetz pencil $f\colon X \setminus A \to S^2$ such that each component of Γ maps bijectively to the equator as in Theorem 2. Up to a small perturbation we can assume that f is injective on the set B of isolated critical points, and that $f(B) \cap f(\Gamma) = \emptyset$. After blowing up the base points, we obtain a new manifold $\hat{X}$, and f extends to a well-defined map $\hat{f}\colon \hat{X} \to S^2$.

Let V be a tubular neighbourhood of the equator in S^2, disjoint from $f(B)$, and denote by $D_\pm$ the two components of $S^2 \setminus V$. Then we can decompose $\hat{X}$ into three pieces: $X_+ = \hat{f}^{-1}(D_+)$, $W = \hat{f}^{-1}(V)$, and $X_- = \hat{f}^{-1}(D_-)$. The zero locus Γ of the near-symplectic form is entirely contained in W. The manifolds $X_\pm$ are symplectic, and the restriction of $\hat{f}$ to $X_\pm$ yields two symplectic Lefschetz fibrations $f_\pm\colon X_\pm \to D_\pm$, with fibres $\Sigma_\pm$.

Consider the quadratic local model $(\underline{x},t) \mapsto (Q(\underline{x}),t)$ describing the behaviour of f near Γ: the fibres are locally given by hyperboloids in $\mathbb{R}^3$, two-sheeted for

$Q > 0$ and one-sheeted for $Q < 0$, with a conical singularity for $Q = 0$. Hence, the fibres for $Q < 0$ are obtained from those for $Q > 0$ by attaching a handle, which decreases the Euler characteristic by 2. Since the diffeomorphisms used to paste this local model into f are oriented in the same manner for all components of Γ, the induced normal orientations of the equator are consistent, and we can choose D_+ (resp. D_-) to correspond to positive (resp. negative) values of Q in the local models near all components of Γ. With this convention, $\chi(\Sigma_-) = \chi(\Sigma_+) - 2m$, where m is the number of components of Γ (if we assume that Σ_+ is connected of genus g, then the genus of Σ_- is $g + m$).

Since the restriction of $\hat{f}$ to W has no critical points outside of Γ, the 4–manifold W is a fibre bundle over S^1, whose fibre Y (the preimage of a small arc transverse to the equator) defines a cobordism between Σ_+ and Σ_-, consisting of a series of handle attachment operations (one for each component of Γ). Hence W relates the boundaries of X_+ and X_- to each other via a sequence of fibrewise handle additions.

More precisely, identify $\overline{V}$ with $S^1 \times [-\delta, \delta]$, and consider for each $\theta \in S^1$ the two boundary fibres $\Sigma_{\pm,\theta} = \hat{f}^{-1}(\theta, \pm\delta)$. Then $\Sigma_{-,\theta}$ is obtained from $\Sigma_{+,\theta}$ by deleting $2m$ small discs and identifying m pairs of boundary components. Conversely $\Sigma_{+,\theta}$ is obtained from $\Sigma_{-,\theta}$ by cutting it open along m disjoint simple closed curves, and capping the boundary components with discs.

Letting θ vary, the union of these discs forms the tubular neighbourhood U_L of a *link* $L \subset \partial X_+$. The link L intersects each fibre of ∂X_+ in $2m$ points (ie, it is in fact a braid with $2m$ strands in ∂X_+); these points are naturally partitioned into m pairs, according to the manner in which the boundary components of $\Sigma_{+,\theta} \setminus (\Sigma_{+,\theta} \cap U_L)$ are glued to each other in order to obtain $\Sigma_{-,\theta}$. Since each pair of points canonically corresponds to a component of Γ, the components of L are naturally labelled ("coloured") by components of Γ (or, less canonically, by integers $1, \dots, m$).

Moreover, L also carries naturally a *relative framing*, which keeps track of the manner in which the boundary components of $\partial X_+ \setminus U_L$ with the same colour are identified. More precisely, the relative framing is the choice of a smooth involution $\rho\colon \partial U_L \to \partial U_L$, preserving the fibration structure above S^1, the colouring and the orientation, but exchanging the two components with the same colour in each fibre, up to isotopy. Given two relative framings ρ, ρ', for each of the m colours the restrictions of ρ and ρ' to the corresponding components of ∂U_L differ by an element of $\pi_1 \mathrm{Diff}(S^1) \simeq \mathbb{Z}$. Hence, the set of relative framings is a $\mathbb{Z}^m$–torsor.

The monodromy of ∂X_+, the $2m$–strand braid $L \subset \partial X_+$, the colouring $c\colon L \to \{1,\dots,m\}$ and the relative framing ρ determine completely the topology of the fibred cobordism W.

Recall that the symplectic Lefschetz fibrations $f_\pm\colon X_\pm \to D_\pm$ are determined by their monodromies, which take values in the relative mapping class groups $\mathrm{Map}(\Sigma_\pm, A)$, ie, the set of isotopy classes of orientation-preserving diffeomorphisms of $\Sigma_\pm$ which coincide with identity over a small neighbourhood of the base locus A. If we assume that $\Sigma_\pm$ are connected of genus $g_\pm$ and the number of base points is n, then $\mathrm{Map}(\Sigma_\pm, A)$ is nothing but the mapping class group $\mathrm{Map}_{g_\pm,n}$ of a genus $g_\pm$ surface with n boundary components. The monodromy around each isolated singular fibre is a positive Dehn twist along a simple closed curve (the corresponding vanishing cycle), and the product of these Dehn twists is equal to the monodromy $\psi_\pm$ of the boundary fibration $\partial X_\pm \to S^1$.

The coloured braid L and the relative framing ρ determine a lift of ψ_+ from $\mathrm{Map}(\Sigma_+, A)$ to $\mathrm{Map}(\Sigma_-, A)$, which we denote by $\hat\psi_+$. More precisely, starting from the mapping torus ∂X_+ of ψ_+, by deleting a tubular neighbourhood of the braid L one obtains a new fibre bundle over S^1, whose fibre has genus g_+ and $2m$ boundary components (if L is trivial, this lifts ψ_+ from $\mathrm{Map}_{g_+,n}$ to $\mathrm{Map}_{g_+,n+2m}$). The colouring and the relative framing then specify a manner in which the $2m$ boundary components are glued to each other, to obtain a bundle over S^1 with closed fibres of genus $g_+ + m = g_-$, and whose monodromy is by definition $\hat\psi_+ \in \mathrm{Map}(\Sigma_-, A)$. Because this 3-manifold coincides with ∂X_- up to a change of orientation, $\hat\psi_+\cdot\psi_-$ belongs to the kernel of the natural morphism $\mathrm{Map}(\Sigma_-, A) \to \mathrm{Map}(\Sigma_-)$. However, because each exceptional section of $\hat f$ obtained by blowing up A has a normal bundle of degree -1, the product $\hat\psi_+ \cdot \psi_-$ is not Id, but rather the boundary twist $\delta_A \in \mathrm{Map}(\Sigma_-, A)$, ie, the product of the Dehn twists along small loops encircling the various points of A.

If we assume that the identity components in $\mathrm{Diff}(\Sigma_\pm, A)$ are simply connected (eg, if $\Sigma_\pm$ both have genus at least 2), then the manner in which the boundaries of $X_\pm$ and W are glued to each other is determined uniquely up to isotopy. The above data (the monodromies of $X_\pm$, and the coloured link L with its relative framing) then determine completely the topology of f. Otherwise, the possible gluings of $\partial X_\pm$ to the boundary of W are parametrised by elements of $\pi_1\mathrm{Diff}(\Sigma_\pm, A)$.

Example To make the above discussion more concrete, we briefly consider the case where X_+ has no singular fibres ($X_+ \simeq \Sigma_+ \times D^2$) and Γ is connected. Then L intersects each fibre of ∂X_+ in two points, and Σ_- is obtained from Σ_+ by cutting it open at these two points and attaching a handle in the manner prescribed by the relative framing of L. The core of this handle is a simple closed loop $\gamma \subset \Sigma_-$, which can be thought of as the "vanishing cycle" associated to the equator.

The link L is an arbitrary element of the braid group $B_2(\Sigma_+)$, ie, the fundamental group of the complement of the diagonal in the second symmetric product of Σ_+. Depending on whether the monodromy preserves or exchanges the two points of $\Sigma_+ \cap L$ (ie, whether L has one or two components), the S^1–bundle over S^1 formed by the "vanishing cycle" inside ∂X_- can be either a torus or a Klein bottle. These two cases correspond respectively to the two local models N_+ and N_- described in the Introduction for the behaviour of ω in a neighbourhood of Γ.

We finish with a simple remark illustrating the importance of the relative framing of L. Even when the braid L is trivial, the boundary of X_- need not be diffeomorphic to $S^1 \times \Sigma_-$: in general, the monodromy of ∂X_- can be an arbitrary power of the Dehn twist along the vanishing cycle $\gamma \subset \Sigma_-$.

8.2 Examples

Example 1 The simplest non-trivial examples of singular Lefschetz fibrations $f\colon X \to S^2$ are those where Γ is connected, with a neighbourhood modelled on N_+, there are no isolated singular fibres, and the fibres are connected of genus 0 over D_+ and genus 1 over D_- (see Figure 4).

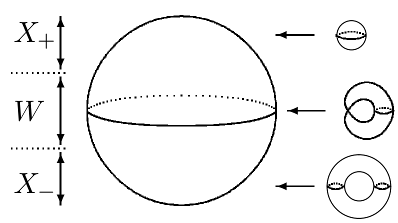

Figure 4: A genus 0/1 singular fibration

The total space of the fibration is a smooth 4–manifold X obtained by gluing together the three open pieces $X_- \simeq T^2 \times D^2$ lying over the southern hemisphere D_-, W lying over a neighbourhood of the equator, and $X_+ \simeq S^2 \times D^2$ lying over the northern hemisphere D_+. The manifold W is a product of S^1 with the standard cobordism from the torus T^2 to sphere S^2, which is diffeomorphic to a solid torus with a small ball removed. Hence, $W \simeq S^1 \times (S^1 \times D^2 \setminus B^3)$.

Because the diffeomorphism groups of S^2 and T^2 are not simply connected, there are various possible choices for the identification diffeomorphisms $\phi_\pm$ between the boundaries $S^1\times S^2$ (resp. $S^1\times T^2$) of $X_\pm$ and W. Since $\phi_\pm$ must be compatible with the fibration structure over S^1, they are described by families of diffeomorphisms of the boundary fibres, ie, elements of $\pi_1\mathrm{Diff}(S^2)\simeq\mathbb{Z}/2$ and $\pi_1\mathrm{Diff}(T^2)\simeq\mathbb{Z}^2$ (compare with the case of ordinary sphere or torus bundles over S^2).

Let us consider eg, the "untwisted" fibration $f\colon X\to S^2$, corresponding to trivial choices for both gluings. This fibration admits a section with trivial normal bundle (considering a point lying away from the "vanishing cycle" in each T^2 fibre, and the corresponding point in each S^2 fibre), and its fundamental group is $\mathbb{Z}$ (generated by a loop transverse to the vanishing cycle in a T^2 fibre). Its total space is diffeomorphic to the connected sum $(S^1\times S^3)\#(S^2\times S^2)$. Indeed, using the decomposition of S^3 into two solid tori, it is easy to see that $X_-\cup W$ is diffeomorphic to the complement of an embedded loop γ in $S^1\times S^3$ (the S^1 factor corresponds to the direction transverse to the vanishing cycles in the T^2 fibres). In the untwisted case, the loop γ projects to a single point in the S^1 factor, and represents an unknot in S^3; in particular, it can be contracted into an arbitrarily small ball in $S^1\times S^3$, and the attachment of the handle X_+ can be viewed as a connected sum operation performed on $S^1\times S^3$. Observing that S^4 splits into $(S^1\times B^3)\cup(D^2\times S^2)$ and hence that the corresponding handle attachment operation turns $S^4\setminus S^1$ into a S^2–bundle over S^2 (in this case $S^2\times S^2$), we conclude that X is as claimed.

If we still glue X_- via the trivial element in $\pi_1\mathrm{Diff}(T^2)$ but glue X_+ using the non-trivial element in $\pi_1\mathrm{Diff}(S^2)$, then we obtain $(S^1\times S^3)\#\mathbb{CP}^2\#\overline{\mathbb{CP}}^2$ instead. However, if eg, we twist the fibration by a loop of diffeomorphisms of T^2 corresponding to a unit translation in the direction transverse to the vanishing cycle, we lose the existence of a section, and the total space becomes simply connected. In fact, the new total space X' is diffeomorphic to S^4. Indeed, $X_-\cup W$ is still the complement of a closed loop in $S^1\times S^3$, but the missing loop γ' now projects non-trivially to the S^1 factor, and is isotopic to $S^1\times\{pt\}\subset S^1\times S^3$. Therefore, we now have $X_-\cup W\simeq S^1\times B^3$, and by gluing $X_+=D^2\times S^2$ along the boundary we obtain $X'\simeq S^4$. Theorem 3 fails to apply in this case, because the cohomological assumption fails to hold (the fibres are homologically trivial).

Example 2 – Isotropic blow-up There are several different operations that can be performed on a singular Lefschetz fibration $f\colon X^4\to S^2$ in order to modify its total space by a topological blow-up operation (ie, connected sum with $\overline{\mathbb{CP}}^2$). Keeping symplectic Lefschetz fibrations in mind, the "usual" blow-up construction amounts to the insertion of an isolated singular fibre with a homotopically trivial vanishing cycle. The exceptional sphere is then obtained as a component of the singular fibre, and is hence naturally symplectic with respect to any 2–form compatible with the fibration structure. If we perform the blow-up near a point $p \in \Gamma$, we can instead modify f according to the local operation represented on Figure 5.

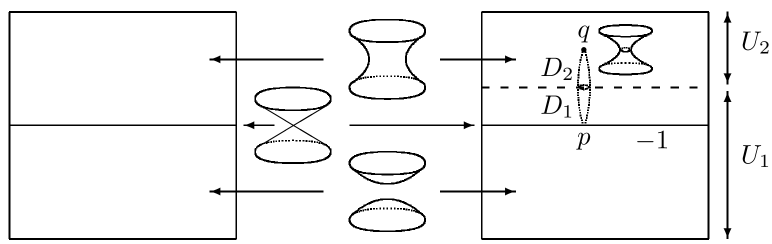

Figure 5: Blowing up near Γ: f (left) and f' (right)

We start from a small ball B^4 centred at p, over which f is as shown in the left half of Figure 5, and replace it with the total space of the fibration f' represented in the right half of the figure. The map f' differs from f in two respects: (1) it has an additional isolated critical point $q \in X_-$, where the vanishing cycle γ is the same as at p; (2) the relative framing of the link $L \subset \partial X_+$ is modified by -1. As explained at the end of the previous Section, changing the relative framing modifies the lift $\hat{\psi}_+$ of the monodromy of ∂X_+ to $\mathrm{Map}(\Sigma_-, A)$ by the inverse of the Dehn twist along γ; this compensates the modification of the monodromy of ∂X_- by the same Dehn twist due to the new isolated singular fibre.

The total space of f' is the union of two subsets U_1 and U_2 (see figure), both diffeomorphic to 4–balls. Over U_1 the map f' is modelled on $(t, x, y, z) \mapsto (t, x^2 + y^2 - z^2)$, while over U_2 it is modelled on $(z_1, z_2) \mapsto z_1^2 + z_2^2$. The total space of U_1 can be viewed as a disc bundle over a disc $D_1 = \{z = t = 0\}$, while the total space of U_2 is a disc bundle over a disc $D_2 = \{\mathrm{Im}\, z_1 = \mathrm{Im}\, z_2 = 0\}$. The boundaries of the two discs D_1 and D_2 match with each other, so that the total space of f' is a disc bundle over a sphere $S = D_1 \cup D_2$ (dotted in Figure 5). Moreover, it is easy to check that the normal bundle of S has degree -1.

From the near-symplectic point of view, this type of blow-up is not equivalent to the usual one. Indeed, in this setup the exceptional sphere S arises from a matching pair of vanishing cycles above an arc joining the critical values $f'(p)$

and $f'(q)$, and for a suitable choice of the compatible near-symplectic form ω on the total space of f' it will be ω–isotropic (or "near-Lagrangian").

Example 3 Consider an isolated Lefschetz-type critical point of a singular fibration, with vanishing cycle a loop γ in the nearby generic fibre. We can remove a neighbourhood of this singular fibre and insert in its place a configuration where the critical values form a simple closed loop δ, with fibre genus decreased by 1 inside δ, and using the same loop γ as "vanishing cycle", as shown in Figure 6. This adds a new component to Γ (this component is not mapped to the equator of S^2; here we consider singular Lefschetz fibrations more general than those given by Theorem 2).

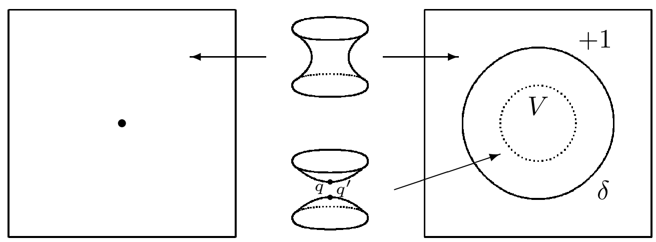

Figure 6: Inserting a critical circle: f (left) and f' (right)

The fibres outside δ are obtained from those inside by attaching a handle joining two points q, q' as shown in the figure. Along δ the points q, q' describe a trivial braid, but the relative framing differs from the trivial one by $+1$, so that on the outer side the monodromy around δ consists of a single positive Dehn twist along γ (which balances the loss of the isolated singular fibre).

The total space of the local model for f given on Figure 6 (left) is simply a 4–ball. On the other hand, the total space of the new fibration f' contains a smoothly embedded sphere S, obtained by considering the two points q and q' in each of the fibres inside δ (yielding the two hemispheres of S), and the singular points in the fibres above δ (yielding the equator). Using the fact that the monodromy around δ is a positive Dehn twist along γ, it can be checked easily that S has self-intersection $+1$. Moreover, the preimage of the interior region V is the disjoint union of two $D^2 \times D^2$'s, and hence a disc bundle over $S \cap f'^{-1}(V)$. On the other hand, the preimage of the outer region is diffeomorphic to $S^1 \times B^3$, and is again a disc bundle over a neighbourhood of the equator in S. Therefore, the total space of f' is a disc bundle over the sphere S, and it is diffeomorphic to the complement of a ball in $\mathbb{CP}^2$.

It follows that the operation we have described amounts to a connected sum with $\mathbb{CP}^2$ – an operation whose result is never a symplectic 4–manifold unless the original manifold had $b_2^+ = 0$, by the work of Taubes. In particular, if the configuration f' occurs inside a singular Lefschetz fibration satisfying the assumptions of Theorem 3, then its total space has $b_2^+ \geq 2$ and splits off a $\mathbb{CP}^2$ summand, and hence does not admit any symplectic structure (more generally, this also holds for similar configurations with arbitrarily positive relative framings, since these contain $+n$–spheres which can be blown up to produce a $\mathbb{CP}^2$ summand).

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