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Quantum invariants of periodic three-manifolds

Patrick M Gilmer

Abstract Let p be an odd prime and r be relatively prime to p. Let G be a finite p–group. Suppose an oriented 3–manifold $\tilde{M}$ has a free G–action with orbit space M. We consider certain Witten–Reshetikhin–Turaev $SU(2)$ invariants $w_r(M)$ in $\mathbb{Z}[\frac{1}{2r}, e^{\frac{2\pi i}{8r}}]$. We will show that $w_r(\tilde{M}) \equiv \iota^{3\,\mathrm{def}(\tilde{M}\to M)}(w_r(M))^{|G|} \pmod{p}$. Here $\iota = e^{\frac{2\pi i(r-2)}{8r}}$, def denotes the signature defect, and $|G|$ is the number of elements in G. We also give a version of this result if M and $\tilde{M}$ contain framed links or colored fat graphs. We give similar formulas for non-free actions which hold for a specified finite set of values for r.

AMS Classification 57M10; 57M12

Keywords p–group action, lens space, quantum invariant, Turaev–Viro invariant, branched cover, Jones polynomial, Arf invariant

Dedicated to Rob Kirby on his sixtieth birthday

1 Introduction

Assume p is an odd prime, and that r is relatively prime to p, and $r \geq 3$. Let G be a finite p–group, with $|G|$ elements. We let ζ_a denote $e^{\frac{2\pi i}{a}}$.

Let M be an oriented closed 3–manifold with an embedded r–admissibly colored fat trivalent graph J. We include the case that J is empty. We consider the Witten–Reshetikhin–Turaev $SU(2)$ invariants $w_r(M, J) \in R_r = \mathbb{Z}[\frac{1}{2r}, \zeta_t]$ [18, 13] where t is $4r$ if r is even and $8r$ if r is odd. Here $w_r(M, J)$ is the version of the WRT–invariant which is denoted $I_{-\zeta_{4r}}(M, J)$ by Lickorish [9], assuming η (also known as κ in this paper) is chosen to be $(\zeta_{2r} - \zeta_{2r}^{-1})/(i\sqrt{2r})$. In terms of the Kirby–Melvin [7] normalization τ_r, one has $w_r(M) = \tau_r(M)/\tau_r(S^1 \times S^2)$.

Let $\tilde{M}$ be an oriented closed 3–manifold with a G action. The singular set of the action is the collection of points whose isotropy subgroup is non-trivial. Let $\tilde{J}$ be an equivariant r–admissibly colored fat graph $\tilde{J}$ in $\tilde{M}$ which is disjoint from the singular set. The action is assumed to preserve the coloring and thickening of $\tilde{J}$. Burnside's theorem asserts that the center of a finite p–group is non-trivial. It follows that the quotient map $\tilde M \to \tilde M/G$ can be factored as a sequence of $\mathbb{Z}_p$ (possibly branched) covering maps. It follows that the orbit space is also a closed 3–manifold with an r–admissibly colored fat graph, the image of $\tilde J$. In this case, we will denote the orbit space of $\tilde M$ by M, and the orbit space of $\tilde J$ by J. We find a relationship between $w_r(M, J)$, and $w_r(\tilde M, \tilde J)$.

When the action is free, a signature defect of $\tilde M \to M$ may be defined as follows. One can arrange that some number, say n, of disjoint copies of $\tilde M \to M$ form the boundary of a regular G covering space of a 4–manifold W, denoted by $\tilde W$. Then define

$$\operatorname{def}(\tilde M \to M) = \frac{1}{n}\left(|G| \operatorname{Sign}(W) - \operatorname{Sign}(\tilde W)\right).$$

The defect can be seen to be well-defined using Novikov additivity together with that fact the signature of an m–fold unbranched covering space of a closed manifold is m times the signature of the base manifold. This generalizes the definition of the signature defect for a finite cyclic group [5,]. As $\Omega_3(BG) = H_3(BG)$, and $H_3(BG)$ is annihilated by multiplication by $|G|$, n can be taken to be $|G|$ in the above definition. For this definition, it is not necessary that G be a p–group. We remark that 3 def $(\tilde M \to M)$ is an integer. We will give a proof in section 3.

We will be working with congruences modulo the odd prime p in the ring of all algebraic integers (over $\mathbb{Z}$) after we have inverted $2r$, where r is relatively prime to p. Let κ denote $\zeta_8\zeta_{4r}^{-1}$.

Theorem 1 *If G acts freely, then*

$$w_r(\tilde M, \tilde J) \equiv \kappa^{3\operatorname{def}(\tilde M \to M)}(w_r(M, J))^{|G|} \pmod p.$$

Our equations in Theorems 1, 2 and 3 take place in R_r/pR_r. We may think of R_r as polynomials in ζ_t with coefficients in $\frac{1}{2r}\mathbb{Z}$ of degree less than $\varphi(t)$. When multiplying such polynomial, one should use the t-th cyclotomic polynomial to rewrite the product as a polynomial in degree less than $\varphi(t)$. Such a polynomial lies in pR_r if and only if each of its coefficients maps to zero under the map $\frac{1}{2r}\mathbb{Z} \to \mathbb{Z}_p$ given by reduction modulo p. For example we consider Theorem 1 with $r = 5$ applied to the free Z_3 action on S^3 with quotient $L(3, 1)$. This action has defect 2/3. In this case $t = 40$, and the cyclotomic polynomial tells us to reduce to polynomials of degree less than 16 via $\zeta_t^{16} = \zeta_t^{12} - \zeta_t^{8} + \zeta_t^{4} - 1$. We have:

$$w_5(L(3,1)) = \frac{3\zeta_t + 2\zeta_t^3 - \zeta_t^5 - 4\zeta_t^7 + 4\zeta_t^9 + \zeta_t^{11} - 2\zeta_t^{13} - 3\zeta_t^{15}}{10},$$

$$w_5(S^3) = \quad = \frac{t + \ {}^3_t - 2\ {}^5_t + 3\ {}^7_t + 3\ {}^9_t - 2\ {}^{11}_t - 4\ {}^{13}_t + \ {}^{15}_t}{10}$$

$$w_5(S^3) - \ {}^2(w_5(L(3;1)))^3 = 3\ \frac{-\ {}^5_t + 2\ {}^7_t + 2\ {}^9_t - 2\ {}^{11}_t - 2\ {}^{13}_t + \ {}^{15}_t}{20}$$

We note that $R_r{=}pR_r = \mathbb{Z}[\ _t]{=}p\mathbb{Z}[\ _t]$. Let f be the smallest positive integer such that $p^f \quad 1 \pmod t$. $\mathbb{Z}[\ _t]{=}p\mathbb{Z}[\ _t]$ is the direct sum of $(t){=}f$ elds each with p^f elements [17; page 14]. It is important that the pth power map is the Frobenius automorphism of $R_r{=}pR_r$. So given $w_r(\tilde{M};\tilde{J})$ and def $(\tilde{M} \ ! \ M)$, we can always solve uniquely for $w_r(M;J)$. Theorem 1 by itself will provide no obstruction to the existence of a free G{action on a given manifold $\tilde{M}$.

If $w_r(\tilde{M};\tilde{J}) \not\equiv 0$ for an in nite collection of r prime to p then the values of $w_r(\tilde{M};\tilde{J})$, and $w_r(M;J)$ for this collection of r determine def $(\tilde{M} \ ! \ M)$. This is not apriori clear.

When the action is not free we have to restrict r to a few values, and we don't know an independent de nition of the exponent of . Theorems 2 and 3 by themselves will provide no obstruction to the existence of a G{action on a given manifold $\tilde{M}$.

Theorem 2 *If r divides $\frac{p\ 1}{2}$, then for some integer , one has*

$$w_r(\tilde{M};\tilde{J}) \qquad (w_r(M;J))^{jGj} \pmod p.$$

If G is cyclic and acts semifreely and r divides $\frac{jGj\ 1}{2}$, the same conclusion holds.

We tie down the factor in a special case.

Theorem 3 *Suppose r divides $\frac{p^s\ 1}{2}$, and $\tilde{M}$ is a p^s{fold branched cyclic cover of a knot K in a homology sphere M. Let J be a colored fat graph in M which misses K, and $\tilde{J}$ be the inverse image of J in $\tilde{M}$.*

$$w_r(\tilde{M};\tilde{J}) \qquad \frac{-2r}{p}^{\ s} \ {}^{-3}\ {}_{p^s(K)}(w_r(M;J))^{p^s} \pmod p;$$

Here ${}_{p^s}(K)$ denotes the total p^s{signature of K [8]. $\left(\frac{-2r}{p}\right)$ is a Legendre symbol.

As a corollary, we obtain the following generalization of a result of Murasugi's [11; Proposition 8]. Murasugi's hypothesis is that $\tilde{L}$ is p{periodic, is a *primitive* $\frac{p\ 1}{2}$ th root of unity and $p \quad 5$. Here $V_L(\ )$ denotes the Jones polynomial evaluated at $t = \ $.

Corollary 1 *Let $\tilde{L}$ be a p^s{periodic link in S^3 with quotient link L. If $\frac{p^s-1}{2} = 1$ and $\zeta \neq -1$, then*

$$V_{\tilde{L}}(\zeta) \equiv V_L(\zeta^{-1}) \pmod{p}.$$

L has an even number of components if and only if $\tilde{L}$ does also. In this case, we must choose the same $\sqrt{\zeta}$ when evaluating both sides of the above equation. The above equation is false, in general, for $\zeta = -1$: the trefoil is has period three with orbit knot the unknot. Making use of H. Murakami's formula [12] relating the Jones polynomial evaluated at i to the Arf invariant of a proper link we have the following corollary. First we observe: if $\tilde{L}$ is a n{periodic link in S^3 with quotient link L, and n is odd, then $\tilde{L}$ is proper if and only if L is proper.

Corollary 2 *Let $\tilde{L}$ be a n{periodic proper link in S^3 with quotient link L. Let $n = \prod p_i^{s_i}$ be the prime factorization of n. Suppose for each i, $p_i^{s_i} \equiv \pm 1$* (mod 8) *(for each i one may choose $\pm$ differently), then*

$$\mathrm{Arf}(L) \equiv \mathrm{Arf}(\tilde{L}) \pmod{2}.$$

The period three action on the trefoil also shows the necessity of the condition that $p_i^{s_i} \equiv \pm 1 \pmod{8}$.

In section 2 we establish versions of Theorems 1 and 2 for the related Turaev{Viro invariants by adapting an argument which Murasugi used to study the bracket polynomial of periodic links [11]. See also Traczyk's paper [15]. In fact these theorems (Theorems 4 and 5) are immediate corollaries of Theorems 1 and 2 but we prefer to give them direct proofs. The reason is that these proofs are simpler than the proofs of Theorems 1 and 2. These proofs can be used to obtain analogous results for other invariants defined by Turaev{Viro type state sums. Also Lemma 3, that we establish to prove Theorem 5, is used later in the proofs of Theorems 2 and 3.

In section 3, we relate $w_r(M; J)$ to the TQFT defined in [2]. We discuss p_1{structures. We also rephrase Theorem 1 in terms of manifolds with p_1{structure, and reduce the proof of Theorem 1 to the case $G = \mathbb{Z}_p$. We say that a regular $\mathbb{Z}_{p^s}${cover which is a quotient of a regular $\mathbb{Z}$ cover is a simple $\mathbb{Z}_{p^s}${cover. In section 4, we derive Theorem 1 for simple $\mathbb{Z}_{p^s}${covers of closed manifolds. This part of the argument applies generally to quantum invariants associated to any TQFT. We also obtain a version of Theorem 1 for simple $\mathbb{Z}_{p^s}${covers of manifolds whose boundary is a torus. In section 5, we derive Theorem 1 in the case M is a lens space and $G = \mathbb{Z}_p$. In section 6, we complete the proof of Theorem 1. One step is to show that if $\tilde{M}$ is a regular $\mathbb{Z}_p$–cover which is not a simple $\mathbb{Z}_p$–cover of M, then we may delete a simple closed curve γ in M so that the inverse image of $M-\gamma$ is a simple $\mathbb{Z}_p$–cover of $M-\gamma$. In section 7, we prove Theorem 2, Theorem 3, Corollary 1, and Corollary 2.

2 Turaev–Viro invariants

We also want to consider the associated Turaev–Viro invariants, which one may define by $\mathrm{tv}_r(M) = w_r(M)\overline{w_r(M)}$. Here conjugation is defined by the usual conjugation defined on the complex numbers. $\mathrm{tv}_r(M)$ was first defined as a state sum by Turaev and Viro [16], and later shown to be given by the above formula separately by Walker and Turaev. A very nice proof of this fact was given by Roberts [14]. We will use a state sum definition in the form used by Roberts. We pick a triangulation of M, and sum certain contributions over r–admissible colorings C of the triangulation. A coloring of a triangulation assigns to each 1–simplex a nonnegative integral color less than $r-1$. The coloring is admissible if: for each 2–simplex the colors assigned to the three edges a, b, and c satisfy $a+b+c$ is even, $a+b+c \le 2r-4$, and $|a-b| \le c \le a+b$.

$$\mathrm{tv}_r(M) = \sum_{c\in C} \prod_{v\in V} \eta^2 \prod_{e\in E} \Delta(c;e) \prod_{f\in F} \theta(c;f)^{-1} \prod_{t\in T} \mathrm{Tet}(c;t)$$

Here V is the set of vertices, E is the set of edges (or 1–simplexes), F is the set of faces (or 2–simplexes), and T is the set of tetrahedrons (or 3–simplexes) in the triangulation. The contributions are products of certain evaluations in the sense of Kaufman–Lins [6] of colored planar graphs. $\Delta(c;e) = \Delta_{c(e)}$, where Δ_i is the evaluation of a loop colored the color i. $\theta(c;f)$ is the evaluation of an unknotted theta curve whose edges are colored with the colors assigned by c to the edges of f. $\mathrm{Tet}(c;t)$ is the evaluation of a tetrahedron whose edges are colored with the colors assigned by c to the tetrahedron t. However we take all these evaluations in $\mathbb{C}$, taking A^2 to be $\zeta = \zeta_{2r}$. Also $\eta^2 = -\frac{(\zeta-\zeta^{-1})^2}{2r}$. So $\mathrm{tv}_r(M)$ lies in $\mathbb{Z}[\frac{1}{2r},\zeta]$. This follows from the following lemmas and the formulas in [6] for these evaluations.

Lemma 1 *For j not a multiple of $2r$, $(1-\zeta^j)^{-1} \in \mathbb{Z}[\frac{1}{2r},\zeta]$.*

Proof $\prod_{s=1}^{2r-1}(x-\zeta^s) = \sum_{i=0}^{2r-1} x^i$. Letting $x=1$, $\prod_{s=1}^{2r-1}(1-\zeta^s) = 2r$. □

Lemma 2 *For $n \le r-1$, the "quantum integers" $[n] = \frac{\zeta^n-\zeta^{-n}}{\zeta-\zeta^{-1}} = \zeta^{1-n}\frac{\zeta^{2n}-1}{\zeta^2-1}$ are units in $\mathbb{Z}[\frac{1}{2r},\zeta]$.*

Since it is xed by complex conjugation, $\mathrm{tv}_r(M) \; 2 \; \mathbb{Z}[\frac{1}{2r}; \quad + \quad ^{-1}]$.

Theorem 4 *If the action is free, then*

$$\mathrm{tv}_r(\tilde{M}) \qquad (\mathrm{tv}_r(M))^{jGj} \quad (\mathrm{mod}\ p):$$

Proof A chosen triangulation T of M lifts to a triangulation $\tilde{T}$ of $\tilde{M}$. Each admissible coloring of T lifts to an admissible coloring of $\tilde{T}$. As each simplex of M is covered by jGj simplexes of $\tilde{M}$, the contribution of a lifted coloring to the sum for $\mathrm{tv}_r(\tilde{M})$ is the jGjth power of the contribution of the original coloring to $\mathrm{tv}_r(M)$. jGj acts freely on the set of colorings of $\tilde{T}$ which are not lifts of some coloring of T. Moreover the contribution of each such triangulation in a given orbit of this G action is constant. Thus the contribution of the non-equivariant colorings is a multiple of p. Making use of the equation $x^{p^s} + y^{p^s} \quad (x+y)^{p^s}$ $(\mathrm{mod}\ p)$, the result follows. □

Lemma 3 *If r divides $\frac{p^s \; 1}{2}$, $\quad _i \quad (\quad _i)^{p^s}$ (mod p), for all i, and $\quad ^2 \quad (\quad ^2)^{p^s}$ (mod p).*

Proof Since r divides $\frac{p^s \; 1}{2}$, $2r$ divides $p^s \quad 1$. Thus $\quad ^{p^s \; 1} = 1$, and $\quad ^{p^s} = \quad ^1$. Thus

$$\quad _1^{p^s} = - \quad - \quad ^{-1} \; ^{p^s} \qquad - \quad ^{p^s} - \quad ^{-p^s} = - \quad - \quad ^{-1} = \quad _1 \quad (\mathrm{mod}\ p):$$

Also $\quad _0 = 1$. Thus $\quad _i \quad (\quad _i)^{p^s}$ (mod p), if i is zero or one. Using the recursion formula $\quad _{i+1} = \quad _1 \quad _i - \quad _{i-1}$, $\quad _i \quad (\quad _i)^{p^s}$ (mod p) follows by induction. Here is the inductive step:

$$(\quad _{i+1})^{p^s} = (\quad _1 \quad _i - \quad _{i-1})^{p^s}$$

$$\quad _1^{p^s} \quad _i^{p^s} - \quad _{i-1}^{p^s} \qquad _1 \quad _i - \quad _{i-1} = \quad _{i+1} \quad (\mathrm{mod}\ p):$$

It follows that $\sum_{i=0}^{r-2} \quad _i^2 \qquad \sum_{i=0}^{r-2} \quad _i^2 \; ^{p^s}$ (mod p). As $\quad ^2 \sum_{i=0}^{r-2} \quad _i^2 = 1$, and $\mathbb{Z}[\quad ; \frac{1}{2r}]{=}p(\mathbb{Z}[\quad ; \frac{1}{2r}])$ is a direct sum of elds, we have $(\quad ^2)^{p^s} \qquad ^2$ (mod p). □

Theorem 5 *If r divides $\frac{p \; 1}{2}$, then*

$$\mathrm{tv}_r(\tilde{M}) \qquad (\mathrm{tv}_r(M))^{jGj} \quad (\mathrm{mod}\ p):$$

If G is cyclic and acts semifreely and r divides $\frac{jGj \; 1}{2}$, the same congruence holds.

Proof We pick our triangulation of the base so that the image of the xed point set is a one dimensional subcomplex. By Lemma 3, whether a colored simplex in the base lies in the image of a simplex with a smaller orbit or not it contributes the same amount modulo p to a product associated to an equivariant coloring. Thus the proof of Theorem 4 still goes through. □

3 Quantum invariants, p_1–structures, and signature defects

Let M be a closed 3–manifold with a p_1–structure [2]. A fat colored graph in M is a trivalent graph embedded in M, with a specified 2–dimensional thickening (ie, banded in the sense of [2]) whose edges have been colored with nonnegative integers less than $r-1$. At each vertex the colors on the edges a, b, and c must satisfy the admissibility conditions: $a+b+c$ is even, $a+b+c \le 2r-4$, and $|a-b| \le c \le a+b$. Let J be such a graph (possibly empty) in M. Recall the quantum invariant $\langle M;J\rangle_{2r} \in k_{2r}$ defined in [2]. Consider the homomorphism [10; note page 134] $\sigma : k_{2r} \to \mathbb{C}$ which sends A to $-\zeta_{4r}$, and sends κ to $\zeta_8\,\zeta_{4r}^{-1}$. Let R_r denote the image of σ. By abuse of notation let ζ denote $\sigma(\zeta)$, and κ denote $\sigma(\kappa)$. Let $t = 4r$ if r is even and $t = 8r$ if r is odd. Then $R_r = \mathbb{Z}[\frac{1}{2r}, \zeta_t]$. Further abusing notation, we let $\langle M;J\rangle$ denote $\sigma(\langle M;J\rangle_{2r}) \in R_r$. If M is a 3–manifold without an assigned p_1–structure, we let $w_r(M;J)$ denote $\langle M';J\rangle$ where M' denotes M equipped with a p_1–structure with σ–invariant zero. If M already is assigned a p_1–structure, we let $w_r(M;J)$ denote $\langle M';J\rangle$ where M' denotes M equipped with a reassigned p_1–structure with σ–invariant zero. One has that $w_r(M;J) = \kappa^{-\sigma(M)}\langle M;J\rangle$. This agrees with $w_r(M;J)$ as defined in the introduction.

Assume now that M has been assigned a p_1–structure. Let $\tilde{M}$ be a regular G covering space. Give $\tilde{M}$ the induced p_1–structure, obtained by pulling back the structure on M. The following lemma generalizes [4; 3.5]. It does not require that G be a p–group.

Lemma 4 $3\operatorname{def}(\tilde{M} \to M) = |G|\,\sigma(M) - \sigma(\tilde{M})$. *In particular* $3\operatorname{def}(\tilde{M} \to M)$ *is an integer.*

Proof Pick a 4–manifold W with boundary $|G|$ copies of M such that the cover extends. We may connect sum on further copies of $\mathbb{C}P(2)$ or $\overline{\mathbb{C}P(2)}$ so that the p_1–structure on M also extends. Let $\tilde{W}$ be the associated cover of W with boundary $|G|$ copies of $\tilde{M}$. We have

$$|G|\,\sigma(M) = 3\operatorname{Sign}(W); \quad |G|\,\sigma(\tilde{M}) = 3\operatorname{Sign}(\tilde{W});$$
$$|G|\operatorname{def}(\tilde{M} \to M) = |G|\operatorname{Sign}(W) - \operatorname{Sign}(\tilde{W}).$$ □

Using this lemma, we rewrite Theorem 1 in an equivalent form. The conclusion is simpler. On the other hand, the hypothesis involves the notion of a p_1–structure. Since p_1–structures are sometimes a stumbling block to novices, we stated our results in the introduction without reference to p_1–structures.

Theorem 1′ *Let M have a p_1–structure, and $\tilde{M}$ be a regular G cover of M with the induced p_1–structure. Then*

$$\left\langle \tilde{M}, \tilde{J} \right\rangle \equiv \langle M, J\rangle^{|G|} \pmod{p}.$$

Note that we may define $\langle M, J\rangle$ for J a linear combination over R_r of fat colored graphs in M, by extending the function $\langle M, J\rangle$ linearly. If $J = \sum_i a_i J_i$, we define $\tilde{J}$ to be $\sum_i a_i^{|G|} \tilde{J}_i$. Since the pth power map is an automorphism of R_r/pR_r, we have that if Theorem 1′ is true for a given type manifold M, then it is true for such manifolds when we replace J and $\tilde{J}$ by linear combinations over R_r of colored fat graphs: J and $\tilde{J}$.

Finally we note that if Theorem 1′ is true for $G = \mathbb{Z}_p$, then it will follow for G a general finite p–group. In the next three sections we prove it for $G = \mathbb{Z}_p$.

4 Simple unbranched $\mathbb{Z}_{p^s}$–covers

A regular Z_{p^s}–covering space $\tilde{X}$ of X is classified by an epimorphism $\chi: H_1(X) \to \mathbb{Z}_{p^s}$. If χ factors through $\mathbb{Z}$, we say $\tilde{X}$ is a simple $\mathbb{Z}_{p^s}$–cover. In this section, we prove Theorem 1′ for simple $\mathbb{Z}_{p^s}$–covers. We also obtain a version for simple $\mathbb{Z}_{p^s}$–covers of manifolds whose boundary is a torus.

If $\chi: H_1(M) \to \mathbb{Z}$ is an epimorphism, let $\chi_{p^s}: H_1(M) \to \mathbb{Z}_{p^s}$ denote the composition with reduction modulo p^s. Suppose $\tilde{M}$ is classified by χ_{p^s}. Consider a Seifert surface for χ ie, a closed surface in M which is Poincare dual to χ. We may and do assume that this surface is in general position with respect to the colored fat graph J. Then the intersection of J with F defines some banded colored points. This surface also acquires a p_1–structure. Thus F is an object in the cobordism category $C^{p_1,c}_{2,r-1}$, [2; 4.6]. Let E be the cobordism from F to F obtained by slitting M along F. We view E as a morphism from F to F in the cobordism category $C^{p_1,c}_{2,r-1}$. Then M is the mapping torus of E, and $\tilde{M}$ is the mapping torus for E^{p^s}.

We may consider the TQFT which is a functor from $C^{p_1,c}_{2,r-1}$, to the category of modules over R_r obtained taking $p = 2r$ in [2] and applying the change of coefficients $\kappa: k_{2r} \to R_r$.

By [2; 1.2], we have

$$\langle M, J\rangle = \text{Trace}\,(Z(E)) \quad \text{and} \quad \left\langle \tilde{M}, \tilde{J} \right\rangle = \text{Trace}\left(Z\left(E^{p^s}\right)\right).$$

Let E be the matrix for $Z(E)$ with respect to some basis for $V(F)$. E has entries in $R_r = \mathbb{Z}[\frac{1}{2r}, \zeta_t]$. Write the entries as polynomials in ζ_t whose coefficients are quotients of integers by powers of $2r$. Let $v \in \mathbb{Z}$ such that $2rv \equiv 1 \pmod p$. Let E' denote the matrix over $\mathbb{Z}[\zeta_t]$ obtained by replacing all powers of $2r$ in the denominators of entries by powers of v in the numerators of these entries. We have

$$\text{Trace}\,(Z(E)) = \text{Trace}\,(E) \equiv \text{Trace}\,(E') \pmod p \text{ and},$$

$$\text{Trace}\left(Z(E)^{p^s}\right) = \text{Trace}\left(E^{p^s}\right) \equiv \text{Trace}\left(E'^{p^s}\right) \pmod p.$$

However all the eigenvalues of E' are themselves algebraic integers. The trace of E' is the sum of these eigenvalues counted with multiplicity. The trace of E'^{p^s} is the sum of p^sth powers of these eigenvalues counted with multiplicity. Therefore

$$\text{Trace}\left(E'^{p^s}\right) \equiv (\text{Trace}\,(E'))^{p^s} \pmod p.$$

Putting these equations together proves Theorem $1'$ for simple $\mathbb{Z}_{p^s}$ covers.

We now wish to obtain a version of Theorem $1'$ for manifolds whose boundary is a torus. Let N be a compact oriented 3–manifold with p_1–structure with boundary $S^1 \times S^1$. $S^1 \times S^1$ acquires a p_1–structure as the boundary. Let J be a colored fat graph in N which is disjoint from the boundary. Then (N, J) defines an element of $V(S^1 \times S^1)$ under the above TQFT. We denote this element by $[N, J]$.

If $\chi: H_1(N) \to \mathbb{Z}$, let $\chi_{p^s}: H_1(N) \to \mathbb{Z}$ denote the composition with reduction modulo $p^s: \mathbb{Z} \to \mathbb{Z}_{p^s}$. Suppose χ_{p^s} restricted to the boundary is an epimorphism. Suppose $\tilde{N}$ is a regular Z_{p^s} covering space given by χ_{p^s}. $\tilde{N}$ has an induced p_1–structure. Suppose that we have identified the boundary with $S^1 \times S^1$ so that χ restricted to the boundary is $\pi_1: H_1(S^1 \times S^1) \to H_1(S^1)$, followed by the standard isomorphism. Here π_1 denotes projection on the first factor. We can always identify the boundary in this way.

Let $\tilde{N}$ be the regular Z_{p^s} covering space given by χ_{p^s}, and $\tilde{J}$ the colored fat graph in $\tilde{N}$ given by the inverse image of J. The boundary of $\tilde{N}$ is naturally identified with $\tilde{S}^1 \times S^1$. Equip $S^1 \times D^2$ with a p_1–structure extending the p_1–structure that $S^1 \times S^1$ acquires as the boundary of N. Equip $(\tilde{S}^1) \times D^2$ with a p_1–structure extending the p_1–structure $(\tilde{S}^1) \times S^1$ acquires as the boundary of $\tilde{N}$. This is the same p_1–structure it gets as the cover of $S^1 \times S^1$. We have $[\tilde{N}, \tilde{J}] \in V(\tilde{S}^1 \times S^1)$. Also $\partial(S^1 \times D^2) = \partial N$, and $\partial(\tilde{S}^1 \times D^2) = \partial\tilde{N}$. For $0 \le i \le r-2$, let $e_i = [S_i] \in V(S^1 \times S^1)$, where S_i is $S^1 \times D^2$ with the core with the standard thickening colored i. Similarly let $\tilde{e}_i = [\tilde{S}_i] \in V(\tilde{S}^1 \times S^1)$, where $\tilde{S}_i$ is $\tilde{S}^1 \times D^2$ with the core with the standard thickening colored i.

Proposition 1 *If* $[N, J] = \sum_{i=0}^{r-2} a_i e_i$*, and* $[\tilde N, \tilde J] = \sum_{i=0}^{r-2} \tilde a_i \tilde e_i$*, then* $\tilde a_j \equiv a_j^{p^s} \pmod p$*, for all* j*, such that* $0 \le j \le r-2$.

Proof Note $(\tilde N, \tilde J) \cup_{S^1 \times S^1} -\tilde S_j$ is a simple cover of $(N, J) \cup_{S^1 \times S^1} -S_j$. Moreover $\langle (\tilde N, \tilde J) \cup_{S^1 \times S^1} -\tilde S_j \rangle = \tilde a_j$ and $\langle (N, J) \cup_{S^1 \times S^1} -S_j \rangle = a_j$. By Theorem $1'$, $\tilde a_j \equiv a_j^{p^s} \pmod p$. □

Proposition 2 *If* J *is a linear combination over* R_r *of colored fat graphs in* $S^1 \times D^2$ *then* $[S^1 \times D^2, J] \in V(S^1 \times S^1)$ *determines* $[\tilde S^1 \times D^2, \tilde J] \in V(\tilde S^1 \times S^1)$ *modulo* p.

Proof Just take $N = S^1 \times D^2$, and sum over the terms in J. □

See the paragraph following the statement of Theorem $1'$ for the definition of $\tilde J$.

5 $\mathbb{Z}_p$–covers of lens spaces

$L(m, q)$ can be described as $-m/q$ surgery to an unknot in S^3. A meridian of this unknot becomes a curve in $L(m, q)$ which we refer to as a meridian of $L(m, q)$. Below, we verify directly that Theorem 1 (and thus Theorem $1'$) holds when M is a lens space, J a meridian colored c, and $G = \mathbb{Z}_p$. By general position any fat graph J in a lens space can be isotoped into a tubular neighborhood of any meridian. Without changing the invariant of the lens space with J or the cover of the lens space with $\tilde J$ one can replace J by a linear combination of this meridian with various colorings and $\tilde J$ by the same linear combination of the inverse image of this meridian with the same colorings in the covering space. This follows from Proposition 2. Thus it will follow that Theorem 1 (and thus Theorem $1'$) will hold if M is a lens space, and J is any fat colored graph in M, and $G = \mathbb{Z}_p$. This is a step in the proof of Theorem $1'$ for $G = \mathbb{Z}_p$.

Consider the p–fold cyclic cover $L(m, q) \to L(mp, q)$. We assume m, q are greater than zero. q must be relatively prime to m and p.

Lemma 5

$$\text{def}\,(L(m, q) \to L(mp, q)) = \frac{1}{m}\left(\text{def}\left(S^3 \to L(mp, q)\right) - \text{def}\left(S^3 \to L(m, q)\right)\right).$$

Proof Suppose $Z \to X$ is a regular $\mathbb{Z}_{mp}$ covering of 4–manifolds with boundary, and on the boundary we have mp copies of the regular $\mathbb{Z}_{mp}$ covering $S^3 \to L(mp;q)$. Let Y denote Z modulo the action of $\mathbb{Z}_m \subset \mathbb{Z}_{mp}$. Then $Z \to Y$ is a regular $\mathbb{Z}_m$ covering, and on the boundary we have mp copies of the regular $\mathbb{Z}_m$ covering $S^3 \to L(m;q)$. Moreover $Y \to X$ is a regular $\mathbb{Z}_p$ covering and on the boundary we have mp copies of the regular $\mathbb{Z}_p$ covering $L(m;q) \to L(mp;q)$. We have:

$$mp\,(\operatorname{def}\,(L(m;q) \to L(mp;q)) = p\,\operatorname{Sign}(X) - \operatorname{Sign}(Y);$$

$$mp\,\left(\operatorname{def}\,(S^3 \to L(m;q)\right) = m\,\operatorname{Sign}(Y) - \operatorname{Sign}(Z) \quad \text{and}$$

$$mp\,\operatorname{def}\,(S^3 \to L(mp;q) = mp\,\operatorname{Sign}(X) - \operatorname{Sign}(Z).$$

The result follows. □

Suppose H is a normal subgroup of a finite group G which acts freely on M. By the above argument, one can show more generally that:

$$\operatorname{def}\,(M \to M/G) = \operatorname{def}\,(M \to M/H) + |H|\,\operatorname{def}\,(M/H \to M/G).$$

According to Hirzebruch [5] (with different conventions)

$$3\,\operatorname{def}\left(S^3 \to L(m;q)\right) = 12m\, s(q;m) \in \mathbb{Z}.$$

See also [8; 3.3], whose conventions we follow.

Thus

$$3\,\operatorname{def}\,(L(m;q) \to L(mp;q)) = \frac{1}{m}\,(12mp\, s(q;mp) - 12m\, s(q;m)) \in \mathbb{Z}.$$

We will need reciprocity for generalized Gauss sums in the form due to Siegel [1; Formula 2.8]. Here is a slightly less general form which suffices for our purposes:

$$\sum_{k=0}^{\gamma-1} \zeta_{2\gamma}^{\alpha k^2+\beta k} = \zeta_8\, \zeta_{8\alpha\gamma}^{-\beta^2} \sqrt{\frac{\gamma}{\alpha}} \sum_{k=0}^{\alpha-1} \zeta_{2\alpha}^{-(\gamma k^2+\beta k)}$$

where $\alpha;\beta;\gamma \in \mathbb{Z}$ and $\alpha, \gamma > 0$ and $\alpha\gamma + \beta$ is even.

We use this reciprocity to rewrite the sum

$$\sum_{n=1}^{m} \zeta_m^{(qr)n^2+(ql-1)n} = \sum_{n=1}^{m} \zeta_{2m}^{(2qr)n^2+2(ql-1)n}$$

$$= \zeta_8\, \zeta_{16mqr}^{-4(ql-1)^2} \sqrt{\frac{m}{2qr}} \sum_{n=1}^{2qr} \zeta_{4qr}^{-mn^2-2(ql-1)n}.$$

We substitute this into a formula from [3; section 2].

$$w_r(L(m;q); \ _c) = \frac{i(-1)^{c+1}}{p\overline{2rm}} \quad \frac{mb - mq \ \ (U) + q^2 l^2 + 1 \ \ 2ql}{4rmq} \sum_{n=1}^{m} \frac{(qr)n^2 + (ql \ \ 1)n}{m};$$

where $U = \begin{pmatrix} q & b \\ m & d \end{pmatrix}$, and $l = c + 1$. We remark that the derivation given in [3] is valid for $m \quad 1$. Also using

$$mb - mq \ \ (U) + q^2 l^2 + 1 = q^2(l^2 - 1) + 12mq \ s(q; m)$$

again from [3], we obtain:

$$w_r(L(m;q); \ _c) = \frac{\frac{3}{8}(-1)^{c+1}}{2r\, p\overline{q}} \ \frac{12m \ s(q;m)}{4rm} \ \frac{-q^2 - 1}{4rmq} \sum_{n=1}^{2qr} \frac{-mn^2 - 2(ql \ \ 1)n}{4qr}:$$

Similarly

$$w_r(L(mp;q); \ _c) = \frac{\frac{3}{8}(-1)^{c+1}}{2r\, p\overline{q}} \ \frac{12mp \ s(q;mp)}{4rmp} \ \frac{-q^2 - 1}{4rmpq} \sum_{n=1}^{2qr} \frac{-mpn^2 - 2(ql \ \ 1)n}{4qr}:$$

So we now work with congruences modulo p in the ring of algebraic integers after we have inverted $2r$ and q. We need to show that

$$w_r(L(m;q); \ _c) \quad \ ^{3 \ \mathrm{def}\ (L(m;q) \,!\, L(mp;q))} (w_r(L(mp;q); \ _c))^p \pmod p: \quad (5{:}1)$$

We have that:

$$\begin{aligned} {}^{3 \ \mathrm{def}\ (L(m;q) \,!\, L(mp;q))} &= \ _8^{-1} \ \ _{4r}^{\frac{1}{m}(12mp \ s(q;mp) - 12m \ s(q;m))} \\ &= \ _8^{\frac{1}{m}(12mp \ s(q;mp) - 12m \ s(q;m))} \ \ \frac{12m \ s(q;m) - 12mp \ s(q;mp)}{4rm}; \end{aligned}$$

$$\frac{-q^2 - 1}{4rmpq}^{\,p} = \frac{-q^2 - 1}{4rmq}$$

$$(-1)^p = -1 \ \text{ and};$$

$$(2r)^p \quad 2r \pmod p:$$

Note that $\frac{-mn^2 - 2(ql \ \ 1)n}{4qr}$ only depends on n modulo $2qr$.

Thus

$$\left(\sum_{n=1}^{2qr} \frac{-mpn^2 - 2(ql \ \ 1)n}{4qr}\right)^p \quad \sum_{n=1}^{2qr} \frac{-m(pn)^2 - 2(ql \ \ 1)pn}{4qr}$$

$$\sum_{n=1}^{2qr} \frac{-mn^2 - 2(ql \ \ 1)n}{4qr} \pmod p:$$

Here we have made use of the fact that as n ranges over all the congruence classes modulo $2qr$ so does pn. Let $\left(\frac{q}{p}\right)$ denote the Legendre–Jacobi symbol.

$$(\zeta_q^{\bar p})^p = q^{\frac{p-1}{2}}(\zeta_q^{\bar p}) \equiv \left(\frac{q}{p}\right)\zeta_q^{\bar p} \pmod p$$

Equation (5.1) will follow when we show:

$$\left(\frac{q}{p}\right)\zeta_8^3 = \zeta_8^{\frac{1}{m}(12mp\, s(q;mp) - 12m\, s(q;m))}\zeta_8^{3p},$$

or

$$\zeta_8^{3-3p+\frac{1}{m}(12m\, s(q;m)-12mp\, s(q;mp))} = \left(\frac{q}{p}\right). \tag{5.2}$$

We will make use of a congruence of Dedekind's [14; page 160 (73.8)]: for k positive and odd:

$$12k\, s(q;k) \equiv k + 1 - 2\left(\frac{q}{k}\right) \pmod 8.$$

We first consider the case that m is odd. We have:

$$\begin{aligned} 12m\, s(q;m) - 12mp\, s(q;mp) &\equiv m(1-p) + 2\left(\left(\frac{q}{mp}\right) - \left(\frac{q}{m}\right)\right) \\ &= m(1-p) + 2\left(\frac{q}{m}\right)\left(\left(\frac{q}{p}\right) - 1\right) \pmod 8. \end{aligned}$$

If m is odd, the equation (5.2) becomes:

$$(3-3p)m + m(1-p) + 2\left(\frac{q}{m}\right)\left(\left(\frac{q}{p}\right) - 1\right) \equiv \begin{cases} 0, & \text{for } \left(\frac{q}{p}\right) = 1 \\ 4, & \text{for } \left(\frac{q}{p}\right) = -1 \end{cases} \pmod 8,$$

which is easily checked. For the rest of this section, we consider the case that m is even. In this case, q must be odd. First we use Dedekind reciprocity [14; page 148 (69.6)] to rewrite $\frac{1}{m}(12m\, s(q;m) - 12mp\, s(q;mp))$ as:

$$\begin{aligned} \frac{1}{mq}(-12mq\, s(m;q) + m^2 + q^2 + 1 - 3mq + 12mqp\, s(mp;q) \\ - m^2p^2 - q^2 - 1 + 3mpq) \end{aligned}$$

$$\text{or } \frac{1}{q}\left(-12q\, s(m;q) + m - 3q + 12qp\, s(mp;q) - mp^2 + 3pq\right).$$

As q is odd, $q^2 = 1 \pmod 8$. So modulo eight the above expression is

$$q\left(-12q\, s(m;q) + m - 3q + 12qp\, s(mp;q) - mp^2 + 3pq\right).$$

As p is odd, $p^2 = 1 \pmod 8$, and $m - mp^2 = 0 \pmod 8$. The expression, modulo eight, becomes

$$-12q^2\, s(m;q) - 3q^2 + 12q^2 p\, s(mp;q) + 3pq^2.$$

So the exponent of $_8$ in Equation (5.2) modulo eight is:

$$q\,(-12q\, s(m;q) + 12qp\, s(mp;q)). \tag{5.3}$$

Again using Dedekind's congruence, we have:

$$12q\, s(m;q) \quad q + 1 - 2q \left(\frac{m}{q}\right) \pmod 8.$$

Similarly

$$12q\, s(mp;q) \quad q + 1 - 2q \left(\frac{mp}{q}\right) \pmod 8.$$

The expression (5.3) modulo eight, becomes

$$q(p-1)(q+1) + 2 \left(\frac{m}{q}\right) \left(1 - p \left(\frac{p}{q}\right)\right)$$

Thus we only need to see

$$_8^{\,q(p-1)(q+1)+2\left(\frac{m}{q}\right)\left(1-p\left(\frac{p}{q}\right)\right)} = \left(\frac{q}{p}\right).$$

Using quadratic reciprocity for Jacobi–Legendre symbols, this becomes

$$(-1)^{\frac{p-1}{2}\frac{q+1}{2}+\left(\frac{m}{q}\right)\frac{\left(1-p\left(\frac{p}{q}\right)\right)}{2}} = (-1)^{\frac{p-1}{2}\frac{q-1}{2}} \left(\frac{p}{q}\right);$$

or

$$(-1)^{\frac{p-1}{2}+\frac{\left(1-p\left(\frac{p}{q}\right)\right)}{2}} = \left(\frac{p}{q}\right);$$

which is easily checked.

6 Unbranched $\mathbb{Z}_p$–covers

Lemma 6 *If $2 H^1(M;\mathbb{Z}_p)$ is not the reduction of an integral class, then there is a simple closed curve γ in M such that the restriction of to $M - \gamma$ is the reduction of an integral class.*

Proof We consider the Bockstein homomorphism β associated to the short exact sequence of coefficients: $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}_p \to 0$. Let γ be a simple closed curve which represents the element which is Poincare dual to $\beta(\chi) \in H^2(M;\mathbb{Z})$. The restriction of $\beta(\chi)$ to $H^2(M-\gamma;\mathbb{Z})$ is zero. This is easily seen using the geometric description of Poincare duality which makes use of dual cell decompositions.

Naturality of the long exact Bockstein sequence completes the proof. □

It follows that any non-simple unbranched cyclic covering of a closed 3–manifold can be decomposed as the union of two simple coverings one of which is a covering of a solid torus. Let N denote M with a tubular neighborhood of γ deleted. We may replace N in M by a solid torus with a linear combination of colored cores obtaining a new manifold (M', J'), such that $\langle (M',J')\rangle = \langle (M,J)\rangle$. By Proposition 2, $\left\langle M', J' \right\rangle = \left\langle M, J \right\rangle$. M' is a union of two solid tori, and so is a lens space or $S^1 \times S^2$. Note that any cover of $S^1 \times S^2$ is simple. As Theorem 1′ when $G = \mathbb{Z}_p$ has already been established for M', we have now established Theorem 1′ when $G = \mathbb{Z}_p$.

7 Branched $\mathbb{Z}_{p^s}$–covers

We need to refine the last statement of Lemma 3:

Lemma 7 *If r divides $\frac{p^s-1}{2}$, $\kappa^{p^s} \equiv \left(\frac{-2r}{p}\right)^s \kappa \pmod p$.*

Proof We have that $\kappa = -\frac{\left(\frac{-1}{p}\right)}{\sqrt{2r}} i$, $\left(\frac{-1}{p}\right)^{p^s} \equiv \left(\frac{-1}{p}\right) \pmod p$, $i^{p^s} = \left(\frac{-1}{p}\right)^s i$, $(\sqrt{2r})^{p^s} \equiv \left(\frac{2r}{p}\right)^s \sqrt{2r} \pmod p$. □

We need the following which follows from Proposition 1, Lemma 3, and Lemma 7. In the notation developed at the end of section 4, let $\omega = \sum_{i=0}^{r-2} \lambda_i S_i$.

Proposition 3 *If r divides $\frac{p^s-1}{2}$, $\bar{\omega} \equiv \left(\frac{-2r}{p}\right)^s \sum_{i=0}^{r-2} \lambda_i \bar{S}_i \pmod p$.*

7.1 Proof of Theorem 2 for G cyclic

Suppose M is a 3–manifold, L is a link in this 3–manifold and there exists a homomorphism $\phi : H_1(M - L) \to \mathbb{Z}_{p^s}$ which sends each meridian of L to a unit of $\mathbb{Z}_{p^s}$. Then we may form a branched cover $\tilde{M}$ of M branched along L. Every semi-free $\mathbb{Z}_{p^s}$ action on an oriented manifold arises in this way. Then we may pick some parallel curve to each component of L whose homology class maps to zero. Perform integral surgery to M along L with framing given by these parallel curves, to form P. Then we may complete the regular unbranched cover of $M - L$ given by ϕ to a regular unbranched cover $\tilde{P}$ of P. If we then do surgery to P along an original meridian (with the framing given by a parallel meridian) of each component of L, we recover M. Similarly if we do surgery to $\tilde{P}$ along the inverse images of these meridians of L then we recover $\tilde{M}$. Note that the inverse image of each meridian of L is a single component in $\tilde{M}$ and $\tilde{P}$. We give M a p_1–structure with σ invariant zero. Then P receives a p_1–structure as the result of p_1–surgery on M [2; page 925]. $\tilde{P}$ receives p_1–structure as the cover of P. $\tilde{M}$ receives a p_1–structure as the result of p_1–surgery on $\tilde{P}$.

Now let J denote colored fat graph in M disjoint from L. Now let J^+ denote the linear combination of colored fat graphs in P given by J together the result of replacing the meridians of L by ω. As usual in this subject, the union of linear combinations is taken to be the linear combination obtained by expanding multilinearly. Then $\langle M, J\rangle = \langle P, J^+\rangle$, by [2]. By Theorem 1, $\left\langle \tilde{P}, \tilde{J}^+ \right\rangle \equiv \langle P, J^+\rangle^{p^s} \pmod{p}$. Using Proposition 3 and by [2], we have $\left\langle \tilde{M}, \tilde{J} \right\rangle = \left(\kappa_p^{-2r}\right)^{s\,\sigma_0(L)} \left\langle \tilde{P}, \tilde{J}^+ \right\rangle$. As some power of κ^3 is minus one, and changing the p_1–structure on $\tilde{M}$, has the effect of multiplying $\langle M, J\rangle$ by a power of κ, this yields Theorem 2 for semifree actions of cyclic groups. □

7.2 Proof of Theorem 2 for general p–groups.

Now we assume r divides $\frac{p-1}{2}$. Thus we have the congruence for every $\mathbb{Z}_p$ action by 7.1. However we can write the projection from $\tilde{M}$ to M as a sequence of quotients of $\mathbb{Z}_p$ actions. □

7.3 Proof of Theorem 3

In the argument of 7.1, if M is a homology sphere, and L is a knot K, then the longitude of K maps to zero under ϕ, and P is obtained by zero framed surgery along L. $\tilde{M}$ is a rational homology sphere, and $\tilde{P}$ is obtained by zero framed surgery along the lift of K. The trace of both surgeries have signature zero. If we give M a p_1–structure with σ invariant zero, then P also has a p_1–structure with σ invariant zero. Also $\tilde{M}$ and $\tilde{P}$ have p_1–structures with the same σ invariant. $\tilde{P}$ has a p_1–structure with σ invariant 3 def $(\tilde{P} \to P)$ by Lemma 4, but this is $-3\sigma_{p^s}(K)$. Thus $(w_r(M;J))^{p^s} = \langle M;J\rangle^{p^s} \equiv \left(\frac{-2r}{p}\right)^s \left\langle \tilde{M};\tilde{J}\right\rangle = \left(\frac{-2r}{p}\right)^s \kappa^{3\sigma_{p^s}(K)} w_r(\tilde{M};\tilde{J})$, modulo p. This proves Theorem 3. □

7.4 Proof of Corollary 1

We obtain Corollary 1 from Theorem 3 by taking M to be S^3, K to be the unknot and J to be L colored one with the framing given by a Seifert surface for J. Using [6; page 7], one has that

$$w_r(M;J) = \langle L\rangle_{A=-\zeta_{4r}} = V_L(\zeta_r^{-1})\left(-\zeta_{2r}-\zeta_{2r}^{-1}\right).$$

In evaluating the Jones polynomial at ζ_r^{-1}, we choose $\sqrt{\zeta_r^{-1}}$ to be ζ_{2r}^{-1} (for the time being.) Since the induced framing of $\tilde{L}$ is also given by the Seifert surface which is the inverse image of the Seifert surface for L,

$$w_r\left(\tilde{M};\tilde{J}\right) = \left\langle \tilde{L}\right\rangle_{A=-\zeta_{4r}} = V_{\tilde{L}}(\zeta_r^{-1})\left(-\zeta_{2r}-\zeta_{2r}^{-1}\right).$$

By Theorem 3, Lemma 3 and Lemma 7, $V_{\tilde{L}}(\zeta_r^{-1}) \equiv \left(V_L(\zeta_r^{-1})\right)^{p^s}$ (mod p). This means that the difference is p times an algebraic integer. $V_L(t)$ is a Laurent polynomial in $\sqrt{t}$ with integer coefficients. We have that

$$\sqrt{\zeta_r^{-1}}^{\,p^s} = \zeta_{2r}^{-p^s} = \zeta_{2r}^{1} = \sqrt{\zeta_r^{1}}$$

Thus:

$$\left(V_L(\zeta_r^{-1})\right)^{p^s} \equiv V_L(\zeta_r^{1}) \pmod p.$$

and so

$$V_{\tilde{L}}(\zeta_r^{-1}) \equiv V_L(\zeta_r^{1}) \pmod p.$$

As all primitive $2r$th roots of unity are conjugate over $\mathbb{Z}$,

$$V_{\tilde{L}}(\omega^{-1}) \equiv V_L(\omega^{1}) \pmod p$$

holds if ω is any primitive rth root such that r divides $\frac{p^s-1}{2}$, and $r > 2$. We must choose the same $\sqrt{\omega}$ when evaluating both sides.

For any link L, let $\#(L)$ denote the number of components of L. One has that $V_L(1) = (-2)^{\#(L)-1}$. One has that $\#(\tilde{L}) \equiv \#(L)$ (mod $p-1$). So $V_{\tilde{L}}(1) \equiv V_L(1)$ (mod p). Thus the stated congruence holds if ω is any $\frac{p^s-1}{2}$-th root of unity, and $\omega \neq -1$. □

7.5 Proof of Corollary 2

It suffices to prove the result for $n = p^s$ where $p^s \equiv 1 \pmod 8$. We apply Corollary 1 with $\ = i$ and choose $\sqrt{i} = e^{\frac{10\pi i}{8}}$. By H. Murakami [12], for any proper link $V_L(i) = (-1)^{\mathrm{Arf}(L)}(\sqrt{2})^{\#(L)-1}$, with the above choice of $\sqrt{i}$. Since 2 is a square modulo p, $\sqrt{2}^{p-1} = 1$. Thus $(\sqrt{2})^{\#(L)-1} \equiv (\sqrt{2})^{\#(L')-1} \pmod p$. Thus we conclude $(-1)^{\mathrm{Arf}(L)} = (-1)^{\mathrm{Arf}(L')} \pmod p$. Therefore $\mathrm{Arf}(L) \equiv \mathrm{Arf}(L') \pmod 2$. □

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Louisiana State University
Department of Mathematics
Baton Rouge, LA 70803, USA

Email: `gilmer@math.lsu.edu`

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