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On Products of Admissible Liftings and Densities

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Dedicated to Prof. Dr. B. Volkmann on the occasion of his 70^{th} birthday

Abstract. We introduce the classes of admissible densities and show that there exist densities respecting coordinates, and product densities with marginals prescribable within these classes. For liftings there is the corresponding class of admissibly generated liftings. We apply these results to improve theorems on product liftings and liftings respecting coordinates and to provide them a unifying approach.

Keywords: *Admissible densities, product liftings, liftings respecting coordinates*

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0. Introduction

For liftings in products of probability spaces the following problems turned up.

1^0 The problem of the existence of liftings compatible with the product structure such as "consistent liftings" (see [12]), "product liftings and densities" (see [8 - 10]), and "liftings respecting coordinates" (see [1,2]), where the last one is the most far-reaching concept. These liftings apply to the regularization of stochastic processes, to measurability problems for empirical processes (see [14, 15]), to the construction of strong liftings on products (see [7, 8]), and to stable measurable sets (see [13]).

As far as the existence of such liftings in full generality is unknown one asks for densities with the corresponding properties instead. For incomplete probability spaces (such as those based on Borelian and Baire-σ-algebras which occur very often in applications to probability theory) where liftings possibly will not exist (see [11]), one must resort to densities which still exist there by [3]. Our main result, Theorem 2.5, tells us that this is possible for densities respecting coordinates. For this reason our concern in this paper is with (possibly) incomplete probability spaces.

2^0 The problem of the existence of liftings and densities listed in Problem 1^0 with marginals prescribable to some extent. It is well known from [8] and [9] that the latter

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provides a tool for the construction of strong liftings in products from liftings in the factors, thus responding to a problem posed by Kupka [6].

While Problem 1^0 naturally and historically comes first, any solution of Problem 2^0 implies a solution of Problem 1^0. It has already been observed by Talagrand [15] that not all liftings have good properties from the product point of view (compare also [8]). Namely, there are complete probability spaces $(\Omega_i, \Sigma_i, \mu_i)$ for $i \in I \neq \emptyset$ such that for given liftings τ_i for μ_i there exists in general no lifting φ for the product measure $\otimes_{i\in I}\mu_i$ respecting coordinates or even being only a product lifting, having for each $i \in I$ the lifting τ_i as its marginal.

This raises the question, whether for a given probability space there exists a class of densities whose elements can always be prescribed as marginals for densities respecting coordinates or for product densities? In this paper we describe a large class of densities of this sort, called "admissible densities". For liftings a corresponding problem exists too and we answer it by introducing the class of "admissibly generated liftings".

The basic Theorem 2.5 then tells us that for densities respecting coordinates there is a completely free choice of marginals in one coordinate fixed in advance, while in all other coordinates there is a free choice within the class of admissible densities. This result improves the corresponding result for product densities of [10] (and clearly implies (with different method of proof) the existence results from [2] for densities respecting coordinates).

The class of the admissibly generated liftings (see Section 3) enables us to provide a unifying approach to all known results about product liftings as well as to the partial results on liftings respecting coordinates. In particular, we present a new proof of the existence of the consistent lifting of Talagrand.

1. Preliminaries

For a given probability space (Ω, Σ, μ) a set $N \in \Sigma$ with $\mu(N) = 0$ is called a *μ-null set*, and for $f, g \in \mathcal{L}^\infty(\mu)$ and $A, B \in \Sigma$ we write $f = g$ a.e. (μ) or $A = B$ a.e. (μ) if $\{\omega \in \Omega : f(\omega) \neq g(\omega)\}$ or $A \triangle B$ (the symmetric difference of A and B) is a μ-null set, respectively. By Σ_0 will be denoted the set of all μ-null subsets of Ω. If η is a σ-subalgebra of Σ, we write $E_\eta(f)$ for a version of the conditional expectation of $f \in \mathcal{L}^\infty(\mu)$ (the space of all bounded Σ-measurable functions on Ω) with respect to η. By $(\Omega, \hat{\Sigma}, \hat{\mu})$ there is denoted the (Carathéodory) completion of (Ω, Σ, μ).

By $\mathbb{N}$ and $\mathbb{R}$ we denote the sets of all natural and real numbers, respectively. If γ is an ordinal, then we will identify it with the set of all ordinals less than γ.

We use the notion of lifting and lower density (or density, for short) in the sense of Definitions 3 and 4, respectively, of [5: Chapter III, Section 1], and for each probability space (Ω, Σ, μ) we denote by $\Lambda(\mu)$ and $\vartheta(\mu)$ the systems of all liftings and densities, respectively. For each lifting ρ on $\hat{\Sigma}$ there exists exactly one (multiplicative) lifting $\tilde{\rho}$ (in the sense of [5: Chapter III, Section 1, Definition 2] on $\mathcal{L}^\infty(\mu)$ such that $\tilde{\rho}(\chi_A) = \chi_{\rho(A)}$ for all $A \in \Sigma$ (χ_A denotes the characteristic function of A) and vice versa (see [5: pp. 35 - 36]). For simplicity we write $\rho = \tilde{\rho}$ throughout.

If I is a non-empty set and $\langle(\Omega_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ is a family of arbitrary (possibly incomplete) probability spaces, then for each $\emptyset \neq J \subseteq I$ we denote by $(\Omega_J, \Sigma_J, \mu_J)$ the *uncompleted* product measure space $\otimes_{i\in J}(\Omega_i, \Sigma_i, \mu_i)$. It must be carefully noted that in this paper we are looking for the *uncompleted* product in contrast to former papers such as [2] and that the completion of $(\Omega_J, \Sigma_J, \mu_J) = \otimes_{i\in J}(\Omega_i, \Sigma_i, \mu_i)$ will always be written $(\Omega_J, \hat{\Sigma}_J, \hat{\mu}_J) = \hat{\otimes}_{i\in J}(\Omega_i, \Sigma_i, \mu_i)$. For any $J, K \subseteq I$ with $J \subseteq K$ we denote by p_{JK} the canonical projection from Ω_K onto Ω_J. For any $J \subseteq I$ the canonical projection of Ω_I onto Ω_J is denoted by p_J and the σ-algebra $p_J^{-1}(\Sigma_J) \subseteq \Sigma_I$ is denoted by Σ_J^*. For every non-empty index set I, I^* will denote the family of all non-empty proper subsets of I.

We say that $\varphi \in \vartheta(\mu_I)$ satisfies condition $(*)$ (see [2]), if for any $J, K \subseteq I$ with $J \cap K = \emptyset$ we have

$$\varphi(E \cup F) = \varphi(E) \cup \varphi(F) \qquad \text{for any } \; E \in \Sigma_J^* \; \text{ and } \; F \in \Sigma_K^*. \tag{$*$}$$

For a family $((\Omega_i, \Sigma_i, \mu_i))_{i\in I}$ of probability spaces and a probability space (Ω, Σ, μ) such that $\Omega = \Omega_I$, $\Sigma \supseteq \Sigma_I$, $\mu|\Sigma_I = \mu_I$ we call a lifting π for μ a *product lifting* of the liftings ρ_i for μ_i $(i \in I)$, and we write $\pi \in \otimes_{i\in I}\rho_i$, if the equation

$$\pi([A_{i_1}, ..., A_{i_n}]) = [\rho_{i_1}(A_{i_1}), ..., \rho_{i_n}(A_{i_n})] \tag{P}$$

holds true for all $n \in \mathbb{N}$, $i_1, ..., i_n \in I$ and all $A_{i_k} \in \Sigma_{i_k}$ $(k = 1, ..., n)$, where $[A_{i_1}, ..., A_{i_n}]$ denotes the cylinder set $\prod_{i\in I} B_i$ for $B_{i_k} = A_{i_k}$ $(k = 1, ..., n)$ and $B_i = \Omega_i$ $(i \in I \setminus \{i_1, ..., i_n\})$. If $I = \{1, \ldots, n\}$, then we write $\pi \in \rho_1 \otimes \cdots \otimes \rho_n$.

We say that $\varphi \in \vartheta(\mu_I)$ (or $\pi \in \Lambda(\mu_I)$) *respects coordinates* if for each $J \subseteq I$ the inclusion $\varphi(\Sigma_J \times \Omega_{J^c}) \subseteq \Sigma_J \times \Omega_{J^c}$ holds true.

If (Ω, Σ, μ) is a probability space and I is a non-empty set, we write μ^I for the product measure on Ω^I and Σ^I for its domain. A lifting $\rho \in \Lambda(\hat{\mu})$ is *consistent* if for every $n \in \mathbb{N}$ there exists $\rho^n \in \Lambda(\hat{\otimes}^n\mu)$ such that

$$\rho^n(A_1 \times \cdots \times A_n) = \rho(A_1) \times \cdots \times \rho(A_n) \tag{C}$$

for all $A_1, \ldots, A_n \in \Sigma$ (see Talagrand [12: Theorem 12]). We use a similar definition for densities instead of liftings.

2. Densities respecting coordinates

In the next definition we single out a class of densities with good properties from the product point of view.

Definition 2.1. Let (Θ, T, ν) be a probability space. A density $v \in \vartheta(\nu)$ is called *admissible* if it can be constructed with the help of the transfinite induction in the way described below.

(A) Let $\mathcal{D}$ be the smallest cardinal with the property that there exists a collection $\mathcal{M} \subset T$ such that $\sigma(\mathcal{M})$ is dense in T in the pseudometric generated by ν. Let $\mathcal{M} = (M_\alpha)_{\alpha<\kappa}$ be numbered by ordinals less than κ, where κ is the first ordinal of the cardinality $\mathcal{D}$. Denote by η_0 the σ-algebra $\sigma(T_0)$ and for each $1 \leq \alpha \leq \kappa$ denote by

η_α the σ-algebra generated by the family $\{M_\gamma\}_{\gamma<\alpha} \cup \eta_0$. We assume that $M_\alpha \notin \eta_\alpha$ for each α. It is clear that without loss of generality, we may do so.

(B) $\upsilon_0 \in \vartheta(\nu|\eta_0)$ is the only existing density on $(\Theta, \eta_0, \nu|\eta_0)$, i.e.

$$\upsilon_0(B) = \begin{cases} \emptyset & \text{if } B \in T_0 \\ \Theta & \text{if } B \notin T_0. \end{cases}$$

(C) If $\gamma \le \kappa$ is a limit ordinal of uncountable cofinality, then $\eta_\gamma = \bigcup_{\alpha<\gamma} \eta_\alpha$ and we define $\upsilon_\gamma \in \vartheta(\nu|\eta_\gamma)$ by setting

$$\upsilon_\gamma(B) := \upsilon_\alpha(B) \qquad \text{if } \ B \in \eta_\alpha \ \text{ and } \ \alpha < \gamma.$$

(D) Assume now that there exists an increasing sequence (γ_n^γ) of ordinals that is cofinal to $\gamma \le \kappa$. For simplicity put $\upsilon_n := \upsilon_{\gamma_n^\gamma}$ and $\eta_n := \eta_{\gamma_n^\gamma}$ for all $n \in \mathbb{N}$. Then $\eta_\gamma = \sigma(\cup_{n\in\mathbb{N}}\eta_n)$ and we can define υ_γ by setting

$$\upsilon_\gamma(B) := \bigcap_{k\in\mathbb{N}} \bigcup_{n\in\mathbb{N}} \bigcap_{m\ge n} \upsilon_m\Big(\Big\{E_{\eta_m}(\chi_B) > 1 - \frac{1}{k}\Big\}\Big) \qquad \text{for } \ B \in \eta_\gamma.$$

It follows by [3: Lemma 1] that $\upsilon_\gamma \in \vartheta(\nu|\eta_\gamma)$ and $\upsilon_\gamma|\eta_n = \upsilon_n$ for each $n \in \mathbb{N}$.

(E) Let now $\gamma = \beta + 1$. To simplify the notations let $M := M_\beta$. It is well known that

$$\eta_\gamma = \Big\{(G \cap M) \cup (H \cap M^c) : \, G, H \in \eta_\beta\Big\}.$$

Let $M_1 \supseteq M$ and $M_2 \supseteq M^c$ be $\eta_\beta-$envelopes of M and M^c, respectively, i.e. $M_1, M_2 \in \eta_\beta$, $(\nu|\eta_\beta)_*(M_1 \setminus M) = 0$ and $(\nu|\eta_\beta)_*(M_2 \setminus M^c) = 0$ $\quad$ ($(\nu|\eta_\beta)_*$ is the inner measure induced by $\nu|\eta_\beta$). Define

$$\upsilon_\gamma\Big((G \cap M) \cup (H \cap M^c)\Big) :=$$

$$\Big(M \cap \upsilon_\beta\big((G \cap M_1) \cup (H \cap M_1^c)\big)\Big) \cup \Big(M^c \cap \upsilon_\beta\big((G \cap M_2) \cup (H \cap M_2^c)\big)\Big)$$

for $G, H \in \eta_\beta$. By [3: Lemma 2] it then follows that $\upsilon_\gamma \in \vartheta(\nu|\eta_\gamma)$ and $\upsilon_\gamma|\eta_\beta = \upsilon_\beta$.

(F) We define $\upsilon \in \vartheta(\nu)$ just by setting $\upsilon = \upsilon_\kappa$. Throughout, the collection of all admissible densities on (Θ, T, ν) will be denoted by $A\vartheta(\nu)$ and each $\upsilon \in A\vartheta(\nu)$ will be considered together with all elements involved into the above construction without any additional remarks.

Proposition 2.2. *For each probability space* (Θ, T, ν) *we have* $A\vartheta(\nu) \neq \emptyset$.

This follows by converting the above definition into an inductive proof.

Theorem 2.3. *Let* (Θ, T, ν) *be an arbitrary probability space. If* $\upsilon \in A\vartheta(\nu)$, *then for each* (Ω, Σ, μ) *and each* $\tau \in \vartheta(\mu)$ *there exists* $\varphi \in \vartheta(\mu \otimes \nu)$ *such that*

$$\varphi(A \times B) = \tau(A) \times \upsilon(B) \qquad \textit{for all } \ A \in \Sigma \ \textit{ and } \ B \in T.$$

If $(\Omega, \Sigma, \mu) = \otimes_{i\in I}(\Omega_i, \Sigma_i, \mu_i)$ *and* $\tau \in \otimes_{i\in I}\tau_i$ *respects coordinates, then* φ *can be chosen to respect coordinates also. If moreover* τ *satisfies condition* $(*)$*, then also* φ *can be chosen to satisfy* $(*)$.

Proof. Let there be given $\tau \in \vartheta(\mu)$ and $\upsilon \in A\vartheta(\nu)$ alltogether with other elements involved in the construction of $\upsilon \in A\vartheta(\nu)$. In particular, the family $\mathcal{M} = (M_\alpha)_{\alpha<\kappa}$, the σ-subalgebras $(\eta_\alpha)_{\alpha<\kappa}$, and the sequences (γ_n) cofinal with limit ordinals are fixed.

Using transfinite induction, we shall be constructing now a transfinite sequence $(\hat\varphi_\alpha)_{\alpha<\kappa}$ with $\hat\varphi_\alpha \in \vartheta(\mu\otimes\nu|\Sigma\otimes\eta_\alpha)$ such that

$$\hat\varphi_\alpha(A\times B) = \tau(A)\times\upsilon_\alpha(B) \qquad \text{for all } \ A\in\Sigma \ \text{ and } \ B\in\eta_\alpha \tag{1}$$

and

$$\hat\varphi_\beta|\Sigma\otimes\eta_\alpha = \hat\varphi_\alpha \qquad \text{for } \ \alpha<\beta<\kappa. \tag{2}$$

Moreover, we assume that if $(\Omega, \Sigma, \mu) = \otimes_{i\in I}(\Omega_i, \Sigma_i, \mu_i)$ and $\tau \in \otimes_{i\in I}\tau_i$ respects coordinates and satisfies condition $(*)$, then each $\hat\varphi_\alpha$ respects the coordinates of the product space $\otimes_{i\in I}(\Omega_i, \Sigma_i, \mu_i)\otimes(\Theta, \eta_\alpha, \nu|\eta_\alpha)$ and satisfies condition $(*)$.

For $E\in\Sigma\otimes\eta_0$ we have $(\mu\otimes\nu)(E) = \int \nu(E_\omega)\,d\mu(\omega)$. Then $\widetilde{E} := \{\omega\in\Omega : \nu(E_\omega) = 1\}\in\Sigma$ by [4: Formulas (21.4) and (21.8)], and $E = \widetilde{E}\times\Theta$ a.e. $(\mu\otimes\nu)$. Hence if we define

$$\hat\varphi_0(E) = \tau(\widetilde{E})\times\Theta \qquad \text{for all } \ E = \widetilde{E}\times\Theta \ \textit{a.e.}\ (\mu\otimes\nu),$$

we have $\hat\varphi_0\in\vartheta(\mu\otimes\nu)|\Sigma\otimes\eta_0)$ and $\hat\varphi_0(A\times B) = \tau(A)\times\upsilon_0(B)$ for $A\in\Sigma$ and $B\in\eta_0$. Note that $\hat\varphi_0$ respects coordinates of the space $\otimes_{i\in I}(\Omega_i, \Sigma_i, \mu_i)\otimes(\Theta, \eta_0, \nu|\eta_0)$ and satisfies condition $(*)$.

Assume now that given $\gamma<\kappa$, a system $(\hat\varphi_\alpha)$ satisfying the required conditions (1) and (2) has been constructed for all $\alpha<\gamma$.

We have to distinguish three cases.

A) γ is a limit ordinal of uncountable cofinality: Then

$$\Sigma\otimes\eta_\gamma = \bigcup_{\alpha<\gamma}(\Sigma\otimes\eta_\alpha). \tag{3}$$

Setting

$$\hat\varphi_\gamma(E) = \hat\varphi_\alpha(E) \qquad \text{if } \ E\in\Sigma\otimes\eta_\alpha,$$

we get unambiguously defined densities $\hat\varphi_\gamma\in\vartheta(\mu\otimes\nu|\Sigma\otimes\eta_\gamma)$ such that $\hat\varphi_\gamma|\Sigma\otimes\eta_\alpha = \hat\varphi_\alpha$ for all $\alpha<\gamma$. It is a direct consequence of relation (3) that condition (1) is satisfied. Clearly, $\hat\varphi_\gamma$ respects the coordinates of the space $(\otimes_{i\in I}(\Omega_i, \Sigma_i, \mu_i))\otimes(\Theta, \eta_\gamma, \nu|\eta_\gamma)$ and satisfies condition $(*)$.

B) γ is of countable cofinality: For simplicity put $\upsilon_n := \upsilon_{\gamma_n^\gamma}$, $\hat\varphi_n := \hat\varphi_{\gamma_n^\gamma}$ and $\eta_n := \eta_{\gamma_n^\gamma}$ for all $n\in\mathbb{N}$. Then

$$\Sigma\otimes\eta_\gamma = \sigma(\cup_{n\in\mathbb{N}}(\Sigma\otimes\eta_n)).$$

Hence, we can define

$$\hat{\varphi}_\gamma(P) := \bigcap_{k\in\mathbb{N}} \bigcup_{n\in\mathbb{N}} \bigcap_{m\geq n} \hat{\varphi}_m\Big(\Big\{E_{\Sigma\otimes\eta_m}(\chi_P) > 1 - \frac{1}{k}\Big\}\Big) \qquad \text{for} \quad P \in \Sigma\otimes\eta_\gamma.$$

It follows by [3: Lemma 1] that $\hat{\varphi}_\gamma \in \vartheta(\mu\otimes\nu|\Sigma\otimes\eta_\gamma)$ and $\hat{\varphi}_\gamma|\Sigma\otimes\eta_n = \hat{\varphi}_n$ for each $n\in\mathbb{N}$. Now for $A\in\Sigma$ and $B\in\eta_\gamma$ we have

$$\hat{\varphi}_\gamma(A\times B) := \bigcap_{k\in\mathbb{N}} \bigcup_{n\in\mathbb{N}} \bigcap_{m\geq n} \hat{\varphi}_m\Big(\Big\{E_{\Sigma\otimes\eta_m}(\chi_{A\times B}) > 1 - \frac{1}{k}\Big\}\Big).$$

But in virtue of [8: Lemma 2.1] we have

$$\begin{aligned}\Big\{E_{\Sigma\otimes\eta_m}(\chi_{A\times B}) > 1 - \frac{1}{k}\Big\} &= \Big\{E_{\Sigma\otimes\eta_m}(\chi_A\otimes\chi_B) > 1 - \frac{1}{k}\Big\}\\ &= \Big\{(E_\Sigma(\chi_A)\otimes E_{\eta_m}(\chi_B)) > 1 - \frac{1}{k}\Big\}\\ &= \Big\{(\chi_A\otimes E_{\eta_m}(\chi_B)) > 1 - \frac{1}{k}\Big\}\\ &= A\times\Big\{E_{\eta_m}(\chi_B) > 1 - \frac{1}{k}\Big\} \ \textit{a.e.}\ (\mu\otimes\nu|\Sigma\otimes\eta_m)\end{aligned}$$

for $m,k\in\mathbb{N}$. This implies that

$$\begin{aligned}\hat{\varphi}_\gamma(A\times B) &= \bigcap_{k\in\mathbb{N}} \bigcup_{n\in\mathbb{N}} \bigcap_{m\geq n} \hat{\varphi}_m\Big(A\times\Big\{E_{\eta_m}(\chi_B) > 1 - \frac{1}{k}\Big\}\Big)\\ &= \bigcap_{k\in\mathbb{N}} \bigcup_{n\in\mathbb{N}} \bigcap_{m\geq n} \Big(\tau(A)\times\upsilon_m\Big(\Big\{E_{\eta_m}(\chi_B) > 1 - \frac{1}{k}\Big\}\Big)\\ &= \tau(A)\times\Big(\bigcap_{k\in\mathbb{N}} \bigcup_{n\in\mathbb{N}} \bigcap_{m\geq n} \upsilon_m\Big(\Big\{E_{\eta_m}(\chi_B) > 1 - \frac{1}{k}\Big\}\Big)\Big)\\ &= \tau(A)\times\upsilon_\gamma(B),\end{aligned}$$

i.e. $\hat{\varphi}_\gamma(A\times B) = \tau(A)\times\upsilon_\gamma(B)$ for all $A\in\Sigma$ and $B\in\eta_\gamma$.

In order to check whether $\hat{\varphi}_\gamma$ respects coordinates, it sufficies to consider sets $P\in\Sigma\otimes\eta_\gamma$ of the form $P = Q\times\Omega_{J^c}$ where $J\in I^*$, and $Q\in\Sigma_J\otimes\eta_\gamma$ of the form $P = Q\times\Omega_{J^c}\times\Theta$ where $Q\in\Sigma_J$. Since τ respects coordinates and $\hat{\varphi}_\gamma$ is the product of τ and υ_γ, we have in the second case

$$\hat{\varphi}_\gamma(P) = \hat{\varphi}_\gamma(Q\times\Omega_{J^c}\times\Theta) = \tau(Q\times\Omega_{J^c})\times\Theta = A\times\Omega_{J^c}\times\Theta$$

where $A\in\Sigma_J$, i.e. $\hat{\varphi}_\gamma$ respects the coordinates of P. In the first case, we have for each $m\in\mathbb{N}$

$$\begin{aligned}\Big\{E_{\Sigma\otimes\eta_m}(\chi_{Q\times\Omega_{J^c}}) > 1 - \frac{1}{k}\Big\} &= \Big\{\big(E_{\Sigma_J\otimes\eta_m}(\chi_Q)\cdot(\chi_{\Omega_{J^c}})\big) > 1 - \frac{1}{k}\Big\}\\ &= \Omega_{J^c}\times\Big\{E_{\Sigma_J\otimes\eta_m}(\chi_Q) > 1 - \frac{1}{k}\Big\} \ \textit{a.e.}\ (\mu\otimes\nu|\Sigma\otimes\eta_m).\end{aligned}$$

Since all $\hat{\varphi}_m$ respect coordinates, we have

$$\hat{\varphi}_m\Big(\Big\{E_{\Sigma\otimes\eta_m}(\chi_{Q\times\Omega_{J^c}}) > 1-\frac{1}{k}\Big\}\Big) = \hat{\varphi}_m\Big(\Omega_{J^c}\times\Big\{E_{\Sigma_J\otimes\eta_m}(\chi_Q) > 1-\frac{1}{k}\Big\}\Big)$$
$$= \Omega_{J^c}\times A_m$$

where $A_m \in \Sigma_{J^c}\otimes\eta_m$. It follows that $\hat{\varphi}_\gamma(P)\in\Sigma_{J^c}\otimes\eta_\gamma$, i.e. $\hat{\varphi}_\gamma$ respects coordinates.

To show that $\hat{\varphi}_\gamma$ satisfies condition $(*)$, we need the following

Claim 1. *For arbitrary $J, K \subseteq I_\gamma := I\cup\{\gamma\}$ with $J\cap K=\emptyset$, $A\in\Sigma^*_J\otimes\eta_\gamma$, $B\in\Sigma^*_K\otimes\eta_\gamma$, and $m,k\in\mathbb{N}$ we have a.e. $(\mu\otimes\nu|\Sigma\otimes\eta_m)$ the condition*

$$\Big\{E_{\Sigma\otimes\eta_m}(\chi_{A\cup B}) > 1-\frac{1}{k}\Big\} = \Big\{E_{\Sigma\otimes\eta_m}(\chi_A) > 1-\frac{1}{k}\Big\}\cup\Big\{E_{\Sigma\otimes\eta_m}(\chi_B) > 1-\frac{1}{k}\Big\}.$$

Proof. Let J, K, A, B, m, k be as in the claim. For $A=\emptyset$ or $B=\emptyset$ the claim is obvious. Suppose that $A\neq\emptyset$ and $B\neq\emptyset$. It is clear that *a.e.* $(\mu\otimes\nu|\Sigma\otimes\eta_m)$ we have

$$\Big\{E_{\Sigma\otimes\eta_m}(\chi_{A\cup B}) > 1-\frac{1}{k}\Big\} \supseteq \Big\{E_{\Sigma\otimes\eta_m}(\chi_A) > 1-\frac{1}{k}\Big\}\cup\Big\{E_{\Sigma\otimes\eta_m}(\chi_B) > 1-\frac{1}{k}\Big\}.$$

To prove the converse relation, notice that without loss of generality one may assume that $\gamma\in J\setminus K$. Then $B = B_K\times\Omega_{I\setminus K}\times\Theta\in\Sigma\otimes\eta_m$ and so

$$E_{\Sigma\otimes\eta_m}(\chi_B) = \chi_B \qquad a.e.\ (\mu\otimes\nu|\Sigma\otimes\eta_m).$$

Consequently, we get *a.e.* $(\mu\otimes\nu|\Sigma\otimes\eta_m)$ that

$$\Big\{E_{\Sigma\otimes\eta_m}(\chi_{A\cup B}) > 1-\frac{1}{k}\Big\} = \Big\{E_{\Sigma\otimes\eta_m}(\chi_A) + \chi_B - E_{\Sigma\otimes\eta_m}(\chi_A)\chi_B > 1-\frac{1}{k}\Big\}. \tag{4}$$

If $(\omega,\theta)\in B$, then

$$E_{\Sigma\otimes\eta_m}(\chi_A)(\omega,\theta) + \chi_B(\omega,\theta) - E_{\Sigma\otimes\eta_m}(\chi_A)(\omega,\theta)\chi_B(\omega,\theta) = \chi_B(\omega,\theta),$$

hence we get *a.e.* $(\mu\otimes\nu|\Sigma\otimes\eta_m)$ that

$$\begin{aligned}&\Big\{E_{\Sigma\otimes\eta_m}(\chi_A) + \chi_B - E_{\Sigma\otimes\eta_m}(\chi_A)\chi_B > 1-\frac{1}{k}\Big\}\cap B\\ &\quad= \Big\{\chi_B > 1-\frac{1}{k}\Big\}\cap B\\ &\quad\subseteq \Big(\Big\{E_{\Sigma\otimes\eta_m}(\chi_A) > 1-\frac{1}{k}\Big\}\cap B\Big)\cup\Big(\Big\{E_{\Sigma\otimes\eta_m}(\chi_B) > 1-\frac{1}{k}\Big\}\cap B\Big).\end{aligned} \tag{5}$$

If $(\omega,\theta)\notin B$, then

$$E_{\Sigma\otimes\eta_m}(\chi_A)(\omega,\theta) + \chi_B(\omega,\theta) - E_{\Sigma\otimes\eta_m}(\chi_A)(\omega,\theta)\chi_B(\omega) = E_{\Sigma\otimes\eta_m}(\chi_A)(\omega,\theta),$$

hence we have *a.e.* $(\mu\otimes\nu|\Sigma\otimes\eta_m)$ that

$$\begin{aligned}&\Big\{E_{\Sigma\otimes\eta_m}(\chi_A) + \chi_B - E_{\Sigma\otimes\eta_m}(\chi_A)\chi_B > 1-\frac{1}{k}\Big\}\cap B^c\\ &\quad= \Big\{\chi_B > 1-\frac{1}{k}\Big\}\cap B^c\\ &\quad\subseteq \Big(\Big\{E_{\Sigma\otimes\eta_m}(\chi_A) > 1-\frac{1}{k}\Big\}\cap B^c\Big)\cup\Big(\Big\{E_{\Sigma\otimes\eta_m}(\chi_B) > 1-\frac{1}{k}\Big\}\cap B^c\Big).\end{aligned} \tag{6}$$

From (4) - (6) it follows that *a.e.* $(\mu\otimes\nu|\Sigma\otimes\eta_m)$ we have

$$\Big\{E_{\Sigma\otimes\eta_m}(\chi_{A\cup B}) > 1-\frac{1}{k}\Big\} \subseteq \Big\{E_{\Sigma\otimes\eta_m}(\chi_A) > 1-\frac{1}{k}\Big\}\cup\Big\{E_{\Sigma\otimes\eta_m}(\chi_B) > 1-\frac{1}{k}\Big\}.$$

This completes the proof of Claim 1 ∎

Claim 2. *If a set $D \in \Sigma_I \otimes \eta_\gamma$ depends on the coordinates $J \cup \{\gamma\}$, then the same is true for $E_{\Sigma_I \otimes \eta_m}(\chi_D)$.*

Proof. If $D = D_J \times \Omega_{J^c}$ with $D_J \in \Sigma_J \otimes \eta_\gamma$, then

$$E_{\Sigma\otimes\eta_m}(\chi_{D_J\times\Omega_{J^c}}) = E_{(\Sigma_J\otimes\eta_m)\times\Omega_{J^c}}(\chi_{D_J} \otimes \chi_{\Omega_{J^c}}) = E_{\Sigma_J\otimes\eta_m}(\chi_{D_J}) \otimes \chi_{\Omega_{J^c}}$$

and the claim is proved ∎

Let J, K, A, B be as in Claim 1. Assume that

$$A = A_J \times \Omega_{I\setminus J} \qquad \text{and} \qquad B = B_K \times \Omega_{I\setminus K} \times \Theta$$

where $A_J \in \Sigma_J \otimes \eta_\gamma$ and $B_K \in \Sigma_K$. Applying the inductive assumption and the above claims we have

$$\begin{aligned}
&\hat{\varphi}_m\Big(\Big\{E_{\Sigma\otimes\eta_m}(\chi_{A\cup B}) > \frac{1}{k}\Big\}\Big)\\
&= \hat{\varphi}_m\Big(\Big\{E_{\Sigma\otimes\eta_m}(\chi_A) > \frac{1}{k}\Big\} \cup \Big\{E_{\Sigma\otimes\eta_m}(\chi_B) > \frac{1}{k}\Big\}\Big)\\
&= \hat{\varphi}_m\Big(\Big\{E_{\Sigma_J\otimes\eta_m}(\chi_{A_J}) \otimes \chi_{\Omega_{I\setminus J}} > \frac{1}{k}\Big\} \cup \Big\{E_{\Sigma_K}(\chi_{B_K} \otimes \chi_{\Omega_{I\setminus K}} \otimes \chi_\Theta > \frac{1}{k}\Big\}\Big)\\
&= \hat{\varphi}_m\Big(\Big\{E_{\Sigma_J\otimes\eta_m}(\chi_{A_J}) \otimes \chi_{\Omega_{I\setminus J}} > \frac{1}{k}\Big\}\Big) \cup \hat{\varphi}_m\Big(\Big\{E_{\Sigma_K}(\chi_{B_K}) \otimes \chi_{\Omega_{I\setminus K}} \otimes \chi_\Theta > \frac{1}{k}\Big\}\Big)\\
&= \hat{\varphi}_m\Big(\Big\{E_{\Sigma\otimes\eta_m}(\chi_A) > \frac{1}{k}\Big\}\Big) \cup \hat{\varphi}_m(B).
\end{aligned}$$

Then

$$\begin{aligned}
\hat{\varphi}_\gamma(A\cup B) &= \bigcap_{k\in\mathbb{N}} \bigcup_{n\in\mathbb{N}} \bigcap_{m\geq n} \hat{\varphi}_m\Big(\Big\{E_{\Sigma\otimes\eta_m}(\chi_{A\cup B}) > 1 - \frac{1}{k}\Big\}\Big)\\
&= \bigcap_{k\in\mathbb{N}} \bigcup_{n\in\mathbb{N}} \bigcap_{m\geq n} \Big[\hat{\varphi}_m\Big(\Big\{E_{\Sigma\otimes\eta_m}(\chi_A) > \frac{1}{k}\Big\}\Big) \cup \hat{\varphi}_m(B)\Big]\\
&= \bigcap_{k\in\mathbb{N}} \bigcup_{n\in\mathbb{N}} \bigcap_{m\geq n} \Big[\hat{\varphi}_m\Big(\Big\{E_{\Sigma\otimes\eta_m}(\chi_A) > \frac{1}{k}\Big\}\Big) \cup \hat{\varphi}_\gamma(B)\Big]\\
&= \hat{\varphi}_\gamma(A) \cup \hat{\varphi}_\gamma(B),
\end{aligned}$$

i.e. $\hat{\varphi}_\gamma$ satisfies condition $(*)$.

C) $\gamma = \beta + 1$: To simplify the notations let $M := M_\beta$. It is well known that

$$\Sigma \otimes \eta_\gamma = \Big\{\big(K \cap (\Omega \times M)\big) \cup \big(L \cap (\Omega \times M^c)\big) : K, L \in \Sigma \otimes \eta_\beta\Big\}.$$

Let $M_1 \supseteq M$ and $M_2 \supseteq M^c$ be η_β-envelopes of M and M^c, respectively, used in the process of describing υ_γ. An easy calculation shows that

$$E_1 = \Omega \times M_1 \qquad \text{and} \qquad E_2 = \Omega \times M_2 \tag{7}$$

are $\Sigma \otimes \eta_\beta$-envelopes of $\Omega \times M$ and $\Omega \times M^c$, respectively. Define

$$\hat{\varphi}_\gamma\Big[\Big(K \cap (\Omega \times M)\Big) \cup \Big(L \cap (\Omega \times M^c)\Big)\Big] :=$$

$$\Big((\Omega \times M) \cap \hat{\varphi}_\beta\big((K \cap E_1) \cup (L \cap E_1^c)\big)\Big) \cup \Big((\Omega \times M^c) \cap \hat{\varphi}_\beta\big((K \cap E_2) \cup (L \cap E_2^c)\big)\Big)$$

for $K, L \in \Sigma \otimes \eta_\beta$. By [3: Lemma 2] it then follows that $\hat{\varphi}_\gamma \in \vartheta(\mu \otimes \nu | \Sigma \otimes \eta_\gamma)$ and $\hat{\varphi}_\gamma | \Sigma \otimes \eta_\beta = \hat{\varphi}_\beta$. For $A \in \Sigma$ and $G, H \in \eta_\beta$ write $B = (G \cap M) \cup (H \cap M^c)$. Then $B \in \eta_\gamma$ and

$$\begin{aligned} A \times B &= A \times \big((G \cap M) \cup (H \cap M^c)\big) \\ &= \big((A \times G) \cap (\Omega \times M)\big) \cup \big((A \times H) \cap (\Omega \times M^c)\big) \end{aligned}$$

together with $K := A \times G$ and $L := A \times H \in \Sigma \otimes \eta_\beta$. For simplicity put $E_0 := \Omega \times M$. By definition we have

$$\hat{\varphi}_\gamma(A \times B) = \Big(E_0 \cap \hat{\varphi}_\beta\big((K \cap E_1) \cup (L \cap E_1^c)\big)\Big) \cup \Big(E_0^c \cap \hat{\varphi}_\beta\big((K \cap E_2) \cup (L \cap E_2^c)\big)\Big).$$

By an application of (7) this can be rewritten as

$$\hat{\varphi}_\gamma(A \times B) = \big(E_0 \cap \hat{\varphi}_\beta(A \times R)\big) \cup \big(E_0^c \cap \hat{\varphi}_\beta(A \times S)\big)$$

if

$$R := (G \cap M_1) \cup (H \cap M_1^c) \qquad \text{and} \qquad S := (G \cap M_2) \cup (H \cap M_2^c).$$

Since $R, S \in \eta_\beta$, this implies

$$\hat{\varphi}_\gamma(A \times B) = \Big(E_0 \cap \big(\tau(A) \times \upsilon_\beta(R)\big)\Big) \cup \Big(E_0^c \cap \big(\tau(A) \times \upsilon_\beta(S)\big)\Big).$$

By means of $E_0 = \Omega \times M$ the latter formula can be transformed into

$$\hat{\varphi}_\gamma(A \times B) = \tau(A) \times \upsilon_\gamma(B) \qquad \text{for all } \ A \in \Sigma \ \text{ and } \ B \in \eta_\gamma.$$

Therefore $\hat{\varphi}_\gamma$ satisfies condition (1).

In order to check whether $\hat{\varphi}_\gamma$ respects coordinates, it is sufficient to consider sets $P \in \Sigma \otimes \eta_\gamma$ of the form $P = Q \times \Omega_{J^c}$ where $J \in I^*$ and $Q \in \Sigma_J \otimes \eta_\gamma$, and of the form $P = Q \times \Omega_{J^c} \times \Theta$ where $Q \in \Sigma_J$. Since τ respects coordinates and $\hat{\varphi}_\gamma$ is a product of τ and υ_γ, we have in the second case

$$\hat{\varphi}_\gamma(P) = \hat{\varphi}_\gamma(Q \times \Omega_{J^c} \times \Theta) = \tau(Q \times \Omega_{J^c}) \times \Theta = A \times \Omega_{J^c} \times \Theta$$

where $A \in \Sigma_J$, i.e $\hat{\varphi}_\gamma$ respects the coordinates of P.

In the first case we take $K = K' \times \Omega_{J^c}$ and $L = L' \times \Omega_{J^c}$ with $K', L' \in \Sigma_J \otimes \eta_\beta$ such that $P = (K \cap E_0) \cup (L \cap E_0^c)$. Then

$$P = (K \cap E_0) \cup (L \cap E_0^c) = \Big[\big(K' \cap (\Omega_J \times M)\big) \cup \big(L' \cap (\Omega_J \times M^c)\big)\Big] \times \Omega_{J^c}.$$

Since $\hat{\varphi}_\beta$ respects coordinates, we have

$$\begin{aligned}
&\hat{\varphi}_\beta\big((K\cap E_1)\cup(L\cap E_1^c)\big)\\
&\qquad=\hat{\varphi}_\beta\Big[\big((K'\times\Omega_{J^c})\cap(\Omega\times M_1)\big)\cup\big((L'\times\Omega_{J^c})\cap(\Omega\times M_1^c)\big)\Big]\\
&\qquad=\hat{\varphi}_\beta\Big[\Big(\big(K'\cap(\Omega_J\times M_1)\big)\cup\big(L'\cap(\Omega_J\times M_1^c)\big)\Big)\times\Omega_{J^c}\Big]\\
&\qquad=A\times\Omega_{J^c}
\end{aligned}$$

where $A\in\Sigma_J\otimes\eta_\beta$. Similarly,

$$\hat{\varphi}_\beta\big((K\cap E_2)\cup(L\cap E_2^c)\big)=B\times\Omega_{J^c}$$

with $B\in\Sigma_J\otimes\eta_\beta$. Consequently,

$$\begin{aligned}
&\hat{\varphi}_\gamma\big((K\cap E_0)\cup(L\cap E_0^c)\big)\\
&\quad=\Big[E_0\cap\hat{\varphi}_\beta\big((K\cap E_1)\cup(L\cap E_1^c)\big)\Big]\cup\Big[E_0^c\cap\hat{\varphi}_\beta\big((K\cap E_2)\cup(L\cap E_2^c)\big)\Big]\\
&\quad=\big(E_0\cap(A\times\Omega_{J^c})\big)\cup\big(E_0^c\cap(B\times\Omega_{J^c})\big)\\
&\quad=\Big[\big((\Omega_J\times M)\cap A\big)\cup\big((\Omega_J\times M^c)\cap B\big)\Big]\times\Omega_{J^c}.
\end{aligned}$$

In the case of K or L from $\Sigma_J^*\times\Theta$ the calculations are similar. As a consequence $\hat{\varphi}_\gamma$ respects coordinates.

To show that $\hat{\varphi}_\gamma$ satisfies condition $(*)$, consider $J,H\subseteq I_\gamma$ with $J\cap H=\emptyset$. If

$$A=(K\cap E_0)\cup(L\cap E_0^c)\in\Sigma_J^*\otimes\eta_\gamma\quad\text{and}\quad B=(B\cap E_0)\cup(B\cap E_0^c)\in\Sigma_H^*$$

where $K,L\in\Sigma_J^*\otimes\eta_\beta$, then applying the inductive assumption we get

$$\begin{aligned}
\hat{\varphi}_\gamma(A\cup B)&=\hat{\varphi}_\gamma\Big((K\cap E_0)\cup(L\cap E_0^c)\cup(B\cap E_0)\cup(B\cap E_0^c)\Big)\\
&=\hat{\varphi}_\gamma\Big[\big((K\cup B)\cap E_0\big)\cup\big((L\cup B)\cap E_0^c\big)\Big]\\
&=\Big[E_0\cap\hat{\varphi}_\beta\Big(\big((K\cup B)\cap E_1\big)\cup\big((L\cup B)\cap E_1^c\big)\Big)\Big]\\
&\quad\bigcup\Big[E_0^c\cap\hat{\varphi}_\beta\Big(\big((K\cup B)\cap E_2\big)\cup\big((L\cup B)\cap E_2^c\big)\Big)\Big]\\
&=\Big[E_0\cap\hat{\varphi}_\beta\Big((K\cap E_1)\cup(L\cap E_1^c)\cup B\Big)\Big]\\
&\quad\bigcup\Big[E_0^c\cap\hat{\varphi}_\beta\Big((K\cap E_2)\cup(L\cap E_2^c)\cup B\Big)\Big]\\
&=\Big[E_0\cap\hat{\varphi}_\beta\Big((K\cap E_1)\cup(L\cap E_1^c)\Big)\cup\hat{\varphi}_\beta(B)\Big]\\
&\quad\bigcup\Big[E_0^c\cap\hat{\varphi}_\beta\Big((K\cap E_2)\cup(L\cap E_2^c)\Big)\cup\hat{\varphi}_\beta(B)\Big]\\
&=\hat{\varphi}_\gamma\Big((K\cap E_0)\cup(L\cap E_0^c)\Big)\cup\hat{\varphi}_\beta(B)\\
&=\hat{\varphi}_\gamma(A)\cup\hat{\varphi}_\gamma(B),
\end{aligned}$$

i.e. $\hat{\varphi}_\gamma$ satisfies condition $(*)$.

We can define now $\varphi\in\vartheta(\mu\otimes\nu)$ possessing the required properties just by setting $\varphi=\hat{\varphi}_\kappa$. The densities are properly defined, since each element of $\Sigma\otimes T$ is measurable with respect to some $\Sigma\otimes\eta_\alpha$, with $\alpha\leq\kappa$ ∎

Lemma 2.4. *Let* (Ω_i, Ξ_i, μ_i) $(i = 1,2,3)$ *be probability spaces and let* $f : \Omega_1 \times \Omega_2 \times \Omega_3 \to [0,1]$ *be a bounded* $\Xi_1 \otimes \Omega_2 \otimes \Xi_3$*-measurable function. Then there exists a* $\Xi_1 \otimes \Omega_2 \times \Omega_3$*-measurable version of* $E_{\Xi_1 \otimes \Xi_2 \times \Omega_3}(f)$.

Proof. Let $D \in \Xi_1 \otimes \Xi_2$ be an arbitrary set. By the assumption, we may assume that there is a bounded $\Xi_1 \otimes \Xi_3$-measurable function g satisfying everywhere the equality $f(\omega_1, \omega_2, \omega_3) = g(\omega_1, \omega_3)$. Let a function h be given by the equality $h(\omega_1) := \int_{\Omega_3} g(\omega_1, \omega_3)\, d\mu_3(\omega_3)$. Since g can be uniformly approximated by measurable simple functions, we get the Ξ_1-measurability of h. Then, applying the Fubini theorem (cf. [4: Theorem 21.12]), we have

$$\begin{aligned} \int_{D\times\Omega_3} f\, d(\mu_1 \otimes \mu_2 \otimes \mu_3) &= \int_D \left(\int_{\Omega_3} g(\omega_1, \omega_3)\, d(\mu_3) \right) d(\mu_1 \otimes \mu_2) \\ &= \int_D h(\omega_1)\, d(\mu_1 \otimes \mu_2)(\omega_1, \omega_2) \\ &= \int_{D\times\Omega_3} h\, d(\mu_1 \otimes \mu_2 \otimes \mu_3). \end{aligned}$$

This means that $h \otimes \chi_{\Omega_2} \otimes \chi_{\Omega_3}$ is a $\Xi_1 \otimes \Omega_2 \times \Omega_3$-measurable version of $E_{\Xi_1 \otimes \Xi_2 \times \Omega_3}(f)$ ∎

To some extent we can prescribe the marginals of a density respecting coordinates.

Theorem 2.5. *Let* $\langle(\Omega_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ *(*I *a non-empty index set) be a family of probability spaces with product* (Ω, Σ, μ)*. If* $i_0 \in I$ *is fixed, then for each* $\tau_{i_0} \in \vartheta(\mu_{i_o})$ *and for arbitrary* $\tau_i \in A\vartheta(\mu_i)$ *with* $i \in I \setminus \{i_0\}$ *there exists* $\varphi \in \vartheta(\mu)$ *such that* φ *respects coordinates, and* $\varphi \in \otimes_{i\in I}\tau_i$.

Proof. Let κ be the first ordinal of cardinality equal to card(I). Without loss of generality, we may assume that $I = \kappa$ and $i_0 = 0$. Put also $(X_\gamma, T_\gamma, \nu_\gamma) := \otimes_{\alpha<\gamma}(\Omega_\alpha, \Sigma_\alpha, \mu_\alpha)$ for $1 \le \gamma \le \kappa$ and let $p_{\alpha\gamma}$ be the canonical projection from X_γ onto X_α whenever $\alpha < \gamma$.

We shall be constructing inductively densities $\varphi_\gamma \in \vartheta(\nu_\gamma)$ respecting coordinates, and such that

$$\varphi_\gamma \in \otimes_{\delta<\gamma}\tau_\delta \qquad \text{for } 1 \le \gamma \le \kappa \tag{8}$$

and

$$\varphi_\gamma \circ p_{\alpha\gamma}^{-1} = p_{\alpha\gamma}^{-1} \circ \varphi_\alpha \qquad \text{for } 1 \le \alpha < \gamma. \tag{9}$$

To start the induction define $\varphi_1 := \tau_0$. Suppose that for some $\gamma \le \kappa$ and all $\alpha < \gamma$ the densities $\varphi_\alpha \in \vartheta(\nu_\alpha)$ respecting coordinates and satisfying (8) and (9) are already known. We have to distinguish three cases.

A) $\gamma = \alpha + 1$: By Theorem 2.3 there exists a density $\varphi_{\alpha+1} \in \vartheta(\nu_{\alpha+1})$ respecting coordinates and satisfying the condition $\varphi_\gamma \in \otimes_{\beta<\gamma}\tau_\beta$. Clearly, condition (9) is also satisfied.

B) γ is of countable cofinality: We assume here that if $H \subset \gamma$, then $H^c := \gamma \setminus H$. For each α with $1 \le \alpha < \gamma$ we put $T_\alpha^* := p_{\alpha\gamma}^{-1}(T_\alpha)$ and $\nu_\alpha^* := \nu_\gamma | T_\alpha^*$. Clearly, for $1 \le \alpha \le \beta < \gamma$

$$T_\alpha^* \subseteq T_\beta^* \qquad \text{and} \qquad \nu_\beta^* | T_\alpha^* = \nu_\alpha^*.$$

For each $\alpha < \gamma$ define a density $\varphi_\alpha^* \in \vartheta(\nu_\alpha^*)$ by means of

$$\varphi_\alpha^*(A^*) := p_{\alpha\gamma}^{-1}(\varphi_\alpha(A))$$

where $A \in T_\alpha$ and $A^* = p_{\alpha\gamma}^{-1}(A)$ a.e. (ν_α^*). It is easily seen that $\varphi_\beta^*|T_\alpha^* = \varphi_\alpha^*$ for all α and β with $1 \le \alpha \le \beta < \gamma$.

Let (γ_n) be an increasing sequence of ordinals cofinal with γ. Then for each $\alpha < \gamma$ there exists $n \in \mathbb{N}$ such that $T_\alpha^* \subseteq T_n^*$ and $\varphi_n^*|T_\alpha^* = \varphi_\alpha^*$ where $T_n^* := T_{\gamma_n}^*$ and $\varphi_n^* := \varphi_{\gamma_n}^*$. Hence

$$T_\gamma = \sigma\Big(\bigcup_{n\in\mathbb{N}} T_n^*\Big) \qquad \text{and} \qquad \varphi_{n+1}^*|T_n^* = \varphi_n^*$$

for each $n \in \mathbb{N}$. Thus, we can define

$$\varphi_\gamma(P) := \bigcap_{k\in\mathbb{N}} \bigcup_{n\in\mathbb{N}} \bigcap_{m\ge n} \varphi_m^*\Big(\Big\{E_{T_m^*}(\chi_P) > \frac{1}{k}\Big\}\Big) \qquad \text{for each } \ P \in T_\gamma.$$

It follows by [3: Lemma 1] that $\varphi_\gamma \in \vartheta(\nu_\gamma)$ and $\varphi_\gamma|T_n^* = \varphi_n^*$ for all $n \in \mathbb{N}$, hence $\varphi_\gamma|T_\alpha^* = \varphi_\alpha^*$ for all $\alpha < \gamma$.

We have to show yet that φ_γ respects coordinates. To do it, take a non-empty set $J \subset \gamma$ and assume that $P = Q \times \Omega_{J^c}$ where $Q \in \Sigma_J$. Notice that for each $m \in \mathbb{N}$ we have the equality

$$\gamma = (\gamma_m \cap J) \cup (\gamma_m \cap J^c) \cup \gamma_m^c.$$

Let $\Xi_1 := \Sigma_{\gamma_m \cap J}$, $\Xi_2 := \Sigma_{\gamma_m \cap J^c}$ and $\Xi_3 := \Sigma_{\gamma_m^c \cap \gamma}$. Applying Lemma 2.4 to $f = \chi_P$, we see that the function $E_{T_m^*}(\chi_P)$ can be assumed to be $\Xi_1 \otimes \Omega_{\gamma_m^c \cup J^c}$-measurable. Since φ_m respects coordinates, we get a similar measurability of the set $\varphi_m^*\big(\big\{E_{T_m^*}(\chi_P) > 1 - \frac{1}{k}\big\}\big)$. In particular, the set $\varphi_m^*\big(\big\{E_{T_m^*}(\chi_P) > 1 - \frac{1}{k}\big\}\big)$ is $\Sigma_J \times \Omega_{J^c}$-measurable. Consequently, the set $\varphi_\gamma(P)$ is $\Sigma_J \times \Omega_{J^c}$-measurable. This proves that φ_γ respects coordinates.

C) γ is of uncountable cofinality: In this case $T_\gamma = \bigcup_{1\le\alpha<\gamma} T_\alpha^*$. Now define for each $1 \le \alpha < \gamma$ a density $\varphi_\alpha^* \in \vartheta(\nu_\alpha^*)$, where $\nu_\alpha^* := \nu_\gamma|T_\alpha^*$, by

$$\varphi_\alpha^*(A^*) := p_{\alpha\gamma}^{-1}(\varphi_\alpha(A))$$

for each $A \in T_\alpha$ with $A^* = p_{\alpha\gamma}^{-1}(A)$ a.e. (ν_α^*). Since $\varphi_\beta^*|T_\alpha^* = \varphi_\alpha^*$ for all α and β with $1 \le \alpha \le \beta < \gamma$ there exists a density $\varphi_\gamma \in \vartheta(\nu_\gamma)$ defined for each $A \in T_\alpha^*$ by

$$\varphi_\gamma(A) = \varphi_\alpha^*(A).$$

Clearly, φ_γ respects coordinates and $\varphi_\gamma|T_\alpha^* = \varphi_\alpha^*$ for arbitrary $1 \le \alpha < \gamma$. Hence

$$\varphi_\gamma \circ p_{\alpha\gamma}^{-1} = p_{\alpha\gamma}^{-1} \circ \varphi_\alpha \qquad \text{for arbitrary } \ 1 \le \alpha < \gamma.$$

It follows from steps (B) and (C) that for each limit ordinal $1 < \gamma \le \kappa$ there exists always a density $\varphi_\gamma \in \vartheta(\nu_\gamma)$ satisfying condition (9). As an immediate consequence of (9) we have

$$\varphi_\gamma\Big(A \times \prod_{\alpha\le\beta<\gamma} \Omega_\beta\Big) = \varphi_\alpha(A) \times \prod_{\alpha\le\beta<\gamma} \Omega_\beta \tag{10}$$

for all $1 \le \alpha \le \gamma \le \kappa$ and $A \in T_\alpha$. The condition $\varphi_\gamma \in \otimes_{\beta<\gamma}\tau_\beta$ is a direct consequence of (10) and of the inductive assumption about each φ_α with $\alpha < \gamma$. We can define now $\varphi \in \vartheta(\mu_I)$ posessing the required properties just by setting $\varphi := \varphi_\kappa$ ∎

The following corollary, containing the main result of [10] as well as Theorem 3 of [8: Section 2] is an immediate consequence of Theorem 2.5.

Corollary 2.6. *Let $\langle(\Omega_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a family of probability spaces. Fix $i_0 \in I$. Then for each $\tau_{i_0} \in \vartheta(\mu_{i_o})$ there exist $\tau_i \in \vartheta(\mu_i)$ for $i \in I\backslash\{i_0\}$ and $\varphi \in \vartheta(\mu_I)$ respecting coordinates and satisfying the condition $\varphi \in \otimes_{i\in I}\tau_i$.*

Corollary 2.7. *Let (Ω, Σ, μ) be a probability space. For a non-empty set I write $p_{Ii}(\omega) := \omega_i$ for $\omega \in \Omega^I$ and $i \in I$. Then for each $\tau \in A\vartheta(\mu)$ and for each non-empty set I there exists $\tau_I \in \vartheta(\mu_I)$ such that*

$$\tau_I(p_{Ii}^{-1}(A)) = p_{Ii}^{-1}(\tau(A))$$

for each $A \in \Sigma$ and $i \in I$. In particular, each admissible density is a consistent density.

The following result is a particular case of Fremlin's Theorem 346G in [2], proved in a different way. But here the completeness assumption for the probability spaces, used in Fremlin's result, is avoided.

Corollary 2.8. *Let $\langle(\Omega_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a family of probability spaces. Then there exists $\varphi \in \vartheta(\mu_I)$ respecting coordinates.*

3. Admissibly generated liftings

The next definition is a counterpart of Definition 2.1 above and singles out a class of liftings with good properties from the product point of view.

Definition 3.1. Let $\upsilon \in \vartheta(\nu)$ be an arbitrary density on (Θ, T, ν). It is well known by [16] that

$$\mathcal{F}(\theta) := \{B \in T : \theta \in \upsilon(B)\}$$

is a filterbasis on Θ so that for each $\theta \in \Theta$ one can choose an ultrafilter $\mathcal{U}(\theta)$ on Θ finer than $\mathcal{F}(\theta)$. We define then

$$\sigma(B) := \{\theta \in \Theta : B \in \mathcal{U}(\theta)\} \qquad \text{for all } B \in T.$$

It has been proven in [16] that $\sigma \in \Lambda(\hat{\nu})$ and

$$\upsilon(B) \subseteq \sigma(B) \subseteq [\upsilon(B^c)]^c \qquad \text{for all } B \in T. \tag{11}$$

An arbitrary lifting σ constructed in the above manner from a density υ will be called a *lifting generated by* υ. If the lifting σ described above is generated by an admissible density υ, then it is called *admissibly generated* (by υ) and the family of all admissibly generated liftings on (Θ, T, ν) is denoted by $AG\Lambda(\nu)$.

Remark. It follows from Proposition 2.2 and Definition 3.1 that $AG\Lambda(\nu) \neq \emptyset$.

Theorem 3.2. *Let (Ω, Σ, μ) and (Θ, T, ν) be complete probability spaces and let $\tau \in \vartheta(\mu)$, $\upsilon \in \vartheta(\nu)$, and $\varphi \in \vartheta(\mu \otimes \nu)$ be densities such that the condition $\varphi \in \tau \otimes \upsilon$*

holds true and φ satisfies condition $(*)$. Then, for each $\rho \in \Lambda(\mu)$ and each $\sigma \in \Lambda(\nu)$ generated by τ and υ, respectively, there exists $\pi \in \Lambda(\mu \hat{\otimes} \nu)$ such that

$$\begin{aligned} \pi(A \times B) &= \rho(A) \times \sigma(B) && \textit{for all } A \in \Sigma \textit{ and } B \in T \\ \varphi(E) \subseteq \pi(E) &\subseteq [\varphi(E^c)]^c && \textit{for all } E \in \Sigma \otimes T \\ \pi(f \otimes g) &= \rho(f) \otimes \sigma(g) && \textit{for all } f \in \mathcal{L}^\infty(\mu) \textit{ and } g \in \mathcal{L}^\infty(\nu). \end{aligned} \tag{P}$$

Proof. For each $\omega \in \Omega$ and each $\theta \in \Theta$ let

$$\mathcal{F}(\omega) := \{A \in \Sigma : \omega \in \tau(A)\} \qquad \text{and} \qquad \mathcal{F}(\theta) := \{B \in T : \theta \in \upsilon(B)\}$$

be filterbases generated by τ and υ, respectively. Then let

$$\mathcal{U}(\omega) := \{A \in \Sigma : \omega \in \rho(A)\} \qquad \text{and} \qquad \mathcal{U}(\theta) := \{B \in T : \theta \in \sigma(B)\}$$

be the ultrafilters generated by ρ and σ, respectively, so that

$$\mathcal{F}(\omega) \subseteq \mathcal{U}(\omega) \subset \Sigma \qquad \text{and} \qquad \mathcal{F}(\theta) \subseteq \mathcal{U}(\theta) \subset T.$$

For each $(\omega, \theta) \in \Omega \times \Theta$ define a filterbase by

$$\mathcal{F}(\omega, \theta) := \{E \in \Sigma \otimes T : (\omega, \theta) \in \varphi(E)\}.$$

Claim 1. *For each $(\omega, \theta) \in \Omega \times \Theta$, $A \in \mathcal{U}(\omega)$, $B \in \mathcal{U}(\theta)$, and $E \in \mathcal{F}(\omega, \theta)$ we have*

$$E \cap (A \times B) \neq \emptyset.$$

Proof. Let $(\omega, \theta), E$ and A, B be as in Claim 1. Assume that $E \cap (A \times B) = \emptyset$. Then we get by using condition $(*)$

$$\begin{aligned} \varphi(E) &\subseteq \varphi([A^c \times \Theta] \cup [\Omega \times B^c]) \\ &= \varphi(A^c \times \Theta) \cup \varphi(\Omega \times B^c) \\ &= [\upsilon(A^c) \times \Theta] \cup [\Omega \times \tau(B^c)] \\ &\subseteq [\rho(A^c) \times \Theta] \cup [\Omega \times \sigma(B^c)] \\ &= [\rho(A) \times \sigma(B)]^c, \end{aligned}$$

i.e. $\varphi(E) \cap [\rho(A) \times \sigma(B)] = \emptyset$, which contradicts to the assumption $(\omega, \theta) \in \varphi(E) \cap [\rho(A) \times \sigma(B)]$ of Claim 1 ∎

By the above claim there exists an ultrafilter $\mathcal{U}(\omega, \theta) \subseteq \Sigma \hat{\otimes} T$ finer than $\mathcal{F}(\omega, \theta)$ such that

$$A \times B \in \mathcal{U}(\omega, \theta) \qquad \text{for all } A \in \mathcal{U}(\omega) \text{ and } B \in \mathcal{U}(\theta). \tag{12}$$

For each $E \in \Sigma \hat{\otimes} T$ put

$$\pi(E) := \Big\{(\omega, \theta) \in \Omega \times \Theta : E \in \mathcal{U}(\omega, \theta)\Big\}.$$

It follows by [16], for example, that $\pi \in \Lambda(\mu \hat{\otimes} \nu)$.

Claim 2. *For each $A \in \Sigma$ and $B \in T$ we have*

$$\rho(A) \times \sigma(B) = \pi(A \times B).$$

Proof. For $A \in \Sigma$ and $B \in T$ we get from (12) that

$$\begin{aligned}(\omega,\theta) \in \rho(A) \times \sigma(B) &\Longleftrightarrow \omega \in \rho(A) \ \text{ and } \ \theta \in \sigma(B)\\ &\Longleftrightarrow A \in \mathcal{U}(\omega) \& B \in \mathcal{U}(\theta)\\ &\Longrightarrow A \times B \in \mathcal{U}(\omega,\theta)\\ &\Longleftrightarrow (\omega,\theta) \in \pi(A \times B),\end{aligned}$$

i.e. $\rho(A) \times \sigma(B) \subseteq \pi(A \times B)$. It remains to show that $\pi(A \times B) \subseteq \rho(A) \times \sigma(B)$. Applying the first part of the proof we get for each $A \in \Sigma$ and $B \in T$ that

$$\begin{aligned}(\rho(A) \times \sigma(B))^c &= [\rho(A^c) \times \Theta] \cup [\Omega \times \sigma(B^c)]\\ &\subseteq \pi(A^c \times \Theta) \cup \pi(\Omega \times B^c)\\ &= [\pi(A \times B)]^c.\end{aligned}$$

This completes the proof of Theorem 3.2 ∎

Corollary 3.3. *Let (Θ, T, ν) be an arbitrary complete probability space. If $\sigma \in AG\Lambda(\nu)$, then for each complete probability space (Ω, Σ, μ) and for each $\rho \in \Lambda(\mu)$ there exists $\pi \in \Lambda(\mu \hat{\otimes} \nu)$ such that $\pi \in \rho \otimes \sigma$.*

Proof. Let (Ω, Σ, μ) and ρ be as as in the corollary. Since $AG\Lambda(\nu) \neq \emptyset$, it follows by Theorem 2.3 that there exists $\varphi \in \vartheta(\mu \otimes \nu)$ satisfying condition $(*)$ and such that $\varphi \in \rho \otimes \tau$. The result now follows from Theorem 3.2 ∎

The following corollary is the main result of [8].

Corollary 3.4 (see [8: Section 2, Theorem 4]). *If (Ω, Σ, μ) and (Θ, T, ν) are complete probability spaces, then for each $\rho \in \Lambda(\mu)$ there exist $\sigma \in \Lambda(\nu)$ and $\pi \in \Lambda(\mu \hat{\otimes} \nu)$ such that*

$$\begin{aligned}\pi(A \times B) &= \rho(A) \times \sigma(B) \qquad \textit{for all } \ A \in \Sigma \ \textit{ and } \ B \in T \qquad (P)\\ \pi(f \otimes g) &= \rho(f) \otimes \sigma(g) \qquad \textit{for all } \ f \in \mathcal{L}^\infty(\mu) \ \textit{ and } \ g \in \mathcal{L}^\infty(\nu).\end{aligned}$$

Theorem 3.5. *Let κ be an ordinal and $\langle(\Omega_\alpha, \Sigma_\alpha, \mu_\alpha)\rangle_{\alpha<\kappa}$ be a family of complete probability spaces. Moreover, for each $1 \leq \lambda \leq \kappa$ put $(X_\lambda, T_\lambda, \nu_\lambda) := \hat{\otimes}_{\alpha<\lambda}(\Omega_\alpha, \Sigma_\alpha, \mu_\alpha)$. Then for each $\rho_0 \in \Lambda(\mu_0)$ and each collection $\{\rho_\alpha \in AG\Lambda(\mu_\alpha) : 0 < \alpha < \kappa\}$ there exists a family $\{\pi_\lambda \in \Lambda(\nu_\lambda) : 1 \leq \lambda \leq \kappa\}$ such that:*

(i) *$\pi_\lambda \in \otimes_{\alpha<\lambda}\rho_\alpha$ for each $1 \leq \lambda \leq \kappa$.*

(ii) *$\pi_\kappa\big(A \times \prod_{\lambda\leq\gamma<\kappa} \Omega_\gamma\big) = \pi_\lambda(A) \times \prod_{\lambda\leq\gamma<\kappa} \Omega_\gamma$ for each $1 \leq \lambda < \kappa$ and $A \in T_\lambda$.*

Proof. For each $1 \leq \lambda \leq \kappa$ let $p_{\alpha\lambda}$ be the canonical projection from X_λ onto X_α whenever $\alpha < \lambda$. We are going to construct inductively liftings $\pi_\lambda \in \Lambda(\nu_\lambda)$ such that

$$\pi_\lambda \in \otimes_{\delta<\lambda}\rho_\delta \qquad \text{for } \ 0 < \lambda \leq \kappa \tag{13}$$

and

$$\pi_\lambda \circ p_{\alpha\lambda}^{-1} = p_{\alpha\lambda}^{-1} \circ \pi_\alpha \qquad \text{for } \ 0 \leq \alpha < \lambda. \tag{14}$$

To start the induction let $\pi_1 := \rho_0 \in \Lambda(\nu_1)$. Then assume that for $1 \leq \lambda < \kappa$ liftings $\pi_\alpha \in \Lambda(\nu_\alpha)$ satisfying conditions (13) and (14) are already known. We have to distinguish three cases.

A) $\lambda = \alpha + 1$: By Corollary 3.3 there exists a lifting $\pi_{\alpha+1} \in \Lambda(\nu_{\alpha+1})$ satisfying condition (13). Clearly, condition (14) is also satisfied by $\pi_{\alpha+1}$.

B) λ is of countable cofinality: For each α with $0 \leq \alpha < \lambda$ denote by T_α^* the σ-algebra $T_\alpha^* \cup (T_\lambda)_0$, and $\nu_\alpha^* := \nu_\lambda | T_\alpha^*$. Clearly, for $0 \leq \alpha \leq \beta < \lambda$, $T_\alpha^* \subseteq T_\beta^*$ and $\nu_\beta^*(A) = \nu_\alpha^*(A)$ holds for any $A \in T_\alpha^*$. For each $\alpha < \lambda$ define a lifting $\pi_\alpha^* \in \Lambda(\nu_\alpha^*)$ by

$$\pi_\alpha^*(A^*) := p_{\alpha\lambda}^{-1}(\pi_\alpha(A))$$

where $A^* \in T_\alpha^*$ and $A \in T_\alpha$ with $A^* = p_{\alpha\lambda}^{-1}(A)$ a.e. (ν_α^*). It is easily seen that, for all α and β with $0 \leq \alpha \leq \beta < \lambda$, $\pi_\beta^* | T_\alpha^* = \pi_\alpha^*$ holds. But since λ is of countable cofinality for each $\alpha < \lambda$ there exists $n \in \mathbb{N}$ such that $T_\alpha^* \subseteq T_n^* := T_{\alpha_n}^*$ and $\pi_n^* | T_\alpha^* = \pi_\alpha^*$ where $\pi_n^* := \pi_{\alpha_n}^*$ and (α_n) is an increasing sequence cofinal with λ. Hence $\pi_{n+1} | T_n^* = \pi_n$ for each $n \in \mathbb{N}$.

It follows easily from [5: Theorem IV.2] that there exists $\pi_\lambda \in \Lambda(\nu_\lambda)$ being the common extension of all π_α^* with $\alpha < \lambda$. Consequently, the family $(\pi_\alpha)_{\alpha \leq \lambda}$ satisfies condition (14). The relation $\pi_\lambda \in \otimes_{\delta < \lambda} \rho_\delta$ is an immediate consequence of the inductive assumption about π_α with $\alpha < \lambda$, since λ is a limit ordinal.

C) The limit ordinal λ is of uncountable cofinality: With the notation from the Case B) we have now $T_\lambda = \bigcup_{0 \leq \alpha < \lambda} (T_\alpha^*)$. Since, for all α and β with $0 \leq \alpha \leq \beta < \lambda$, $\pi_\beta^* | T_\alpha^* := \pi_\alpha^*$ holds, there exists a lifting $\pi_\lambda \in \Lambda(\nu_\lambda)$ such that $\pi_\lambda | T_\alpha^* = \pi_\alpha^*$ for each $1 < \lambda \leq \kappa$. Consequently, the family $(\pi_\alpha)_{\alpha \leq \lambda}$ satisfies condition (14). The relation $\pi_\lambda \in \otimes_{\delta < \lambda} \rho_\delta$ follows in the same way as before ∎

The following corollary is the main result of [9].

Corollary 3.6 (see [9: Section 2, Theorem 1]). *Let be given a family* $((\Omega_i, \Sigma_i, \mu_i))_{i \in I}$ *(I a non-empty index set) of complete probability spaces. Fix $i_0 \in I$. Then for each $\rho_{i_0} \in \Lambda(\mu_{i_o})$ there exist $\rho_i \in \Lambda(\mu_i)$ for $i \in I \setminus \{i_0\}$ and $\pi \in \Lambda(\hat{\mu}_I)$ such that $\pi \in \otimes_{i \in I} \rho_i$.*

Proof. Let κ be the first ordinal of cardinality equal to card(I). Without loss of generality we may assume that $I = \kappa$ and $i_0 = 0$. Since $AG\Lambda(\mu_i) \neq \emptyset$ for $i \in I \setminus \{i_0\}$ the result follows immediately by Theorem 3.5 ∎

Corollary 3.7 (see [2: Theorem 346H]). *Let κ be an ordinal and $\langle(\Omega_\alpha, \Sigma_\alpha, \mu_\alpha)\rangle_{\alpha<\kappa}$ be any family of complete probability spaces with completed product (Ω, Σ, μ). Then there exists a $\pi \in \Lambda(\mu)$ such that $\pi(A) \in \Sigma_J^*$ whenever $A \in \Sigma_J^*$ and $J \subseteq \kappa$ is either a singleton or an initial segment of κ.*

The following corollary is a generalization of [2: Theorem 346I].

Corollary 3.8. *Let (Ω, Σ, μ) be a complete probability space. For any non-empty set I write $p_{Ii}(\omega) := \omega_i$ for $\omega \in \Omega^I$ and $i \in I$. Then for each $\rho \in AG\Lambda(\mu)$ and for each non-empty set I there exists $\rho_I \in \Lambda(\mu^I)$ such that*

$$\rho_I(p_{Ii}^{-1}(A)) = p_{Ii}^{-1}(\rho(A))$$

for each $A \in \Sigma$ and $i \in I$. In particular, each admissibly generated lifting is a consistent lifting.

Corollary 3.9 (see [12: Theorem 12]). *Let (Ω, Σ, μ) be a complete probability space. Then there exists a consistent lifting $\rho \in \Lambda(\mu)$.*

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