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# On Uniqueness Conditions for Decreasing Solutions of Semilinear Elliptic Equations

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*Dedicated to my uncle Toam Chatue J.B.* (†14/08/1997)

**Abstract.** For $f \in C([0,\infty)) \cap C^1((0,\infty))$ and $b > 0$, existence and uniqueness of radial solutions $u = u(r)$ of the problem $\triangle u + f(u_+) = 0$ in $\mathbb{R}^n$ $(n > 2)$, $u(0) = b$ and $u'(0) = 0$ are well known. The uniqueness for the above problem with boundary conditions $u(R) = 0$ and $u'(0) = 0$ is less known beside the cases where $\lim_{r\to\infty} u(r) = 0$. It is our goal to give some sufficient conditions for the uniqueness of the solutions of the problem $D_\alpha u + f(u_+) = 0$ $(r > 0), u(\rho) = 0$ and $u'(0) = 0$ based only on the evolution of the functions $f(t)$ and $\frac{d}{dt}\frac{f(t)}{t}$.

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**AMS subject classification:** 35 J 65, 34 B 15

## 1. Introduction

Let the function $f \in C([0,\infty)) \cap C^1((0,\infty))$ be such that it remains positive, or it has a finite number of positive zeros and changes its sign accross any of them. For $a = n - 1 > 1$ $(n \in \mathbb{N})$ and any $u_0 > 0$, the problem

$$\left.\begin{aligned} D_a u := u'' + \frac{a}{r}u' &= -f(u_+) \\ u(0) &= u_0 \\ u'(0) &= 0 \end{aligned}\right\} \qquad (E)$$

is known to have a unique solution $u \in C^2([0,\infty))$ which is positive in some interval $[0,\rho)$ [3, 6]. For $\rho > 0$, *finite or not*, we will investigate some uniqueness conditions for the associated problem

$$\left.\begin{aligned} D_a u + f(u_+) = 0 \quad (r > 0) \\ u(\rho) = 0 \\ u'(0) = 0. \end{aligned}\right\} \qquad (BV)_\rho$$

For ease writing, the following notations will be used:

1) $u_+(r) = \max\{0, u(r)\}$ and $F(t) = \int_0^t f(s)\,ds$.

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2) $f^+ = \{t > 0 |\ f(t) > 0\}$ and $F^+ = \{t > 0 |\ F(t) > 0\}$.
3) $\psi(t) = \frac{d}{dt}\frac{f(t)}{t}$ and $\lambda$ denotes any zero of $\psi$.
4) $\Psi(t) = \frac{f(t)}{t}$, $\psi^- = \{t > 0 |\ \psi(t) < 0\}$ and $\psi^+ = \{t > 0 |\ \psi(t) > 0\}$.
5) For connected components of $\psi$, define $\psi_i^\pm = \{(t_i, t_{i+1}) \subset \psi^\pm | 0 < t_i < t_{i+1}$ and $\psi(t_i) = \psi(t_{i+1}) = 0\}$ and similarly $f_i^\pm$.
6) For $s \neq t$, $\{s, t\}$ will denote the elements lying between $s$ and $t$.
7) $f$ will be said to satisfy the condition $|f'; \psi^+|$ or $|f'; \psi^-|$ if there exists $\alpha'$ and $C > 0$ or $B > 0$ such that $|f'| > C$ and $\psi^+ > B$ or $\psi^- < -B$, respectively, in $(0, \alpha']$. $|f'; \psi|_I$ will have the same definition where $I \subset \mathbb{R}_+$ replaces $(0, \alpha']$.

## 2. Main results

Let $\rho > 0$ be finite or not.

**Theorem 1.** *If $f(t)$ is decreasing in $t > 0$, then problem $(BV)_\rho$ has at most one solution.*

**Theorem 2.** *If $\frac{d}{dt}\frac{f(t)}{t} > 0$ or $\frac{d}{dt}\frac{f(t)}{t} < 0$ in $t > 0$, then problem $(BV)_\rho$ has at most one solution.*

**Theorem 3.** *Assume that $\frac{d}{dt}\frac{f(t)}{t} > 0$ in $(0, \lambda)$ and $\frac{d}{dt}\frac{f(t)}{t} < 0$ in $(\lambda, \infty)$. If $f' \leq 0$ in some interval $(0, \alpha']$ or condition $|f'; \psi^+|$ holds, then problem $(BV)_\rho$ has at most one solution.*

**Theorem 4.**

**(i)** *Assume that $\psi^- = (0, \lambda)$, $\psi^+ = (\lambda, \infty)$, $f$ is strictly monotone in $(0, \lambda + k)$ for some $k > 0$ and condition $|f'; \psi^+|$ holds. Then problem $(BV)_\rho$ has at most one solution.*

**(ii)** *Assume that $\psi$ has a finite number of zeros, $f$ is strictly monotone in any $\overline{\psi_i^-}$ and condition $|f'; \psi^+|$ or $|f'; \psi^-|$ holds. Then problem $(BV)_\rho$ has at most one solution.*

## 3. Preliminaries

Let $u$ be a solution of problem (E), positive on $I = (r_0, r_2)$. After multiplying the equation in problem (E) by $u'$, integration on $I$ leads to the identity

$$\frac{u'(r_2)^2}{2} + F(u(r_2)) + a \int_{r_0}^{r_2} \frac{u'(s)^2}{s}\, ds = F(u(r_0)) + \frac{u'(r_0)^2}{2}. \tag{1}$$

**Lemma 1.** *Let $u$ be a solution of problem* (E)*, non-constant in some interval $(R, R + \tau)$ with $R > 0$ and $\tau > 0$.*

**(i)** *If*

$$u'(R) = 0 \quad \Longrightarrow \quad u(r) \neq u(R) \ \ \text{for all } r \in (R, R + \tau), \tag{2}$$

*then the solutions of problem* $(BV)_\rho$ *are strictly decreasing in* $(0,\rho)$.

**(ii)** *If* $u$ *is a solution of problem* $(BV)_\rho$, *then* $F(u(0)) > 0$ *and* $u(0) \in f^+ \cap F^+$.

**Proof.** (i) It suffices to notice that for $r_0 = R$ and $r_2 > R$ in (1), one cannot have $u(r) = u(R)$ for $r > R$. Let $u$ be a solution of problem $(BV)_\rho$. Then from (2), $u$ has to be decreasing in some interval $(0,r)$, $u' < 0$ and decreasing as long as $f(u) > 0$ in $(0,r)$. From the equation, if $u'(r_1) = 0$ for some $r_1 > 0$ ($r_1$ being the first such $r$), $u''(r_1) = -f(u(r_1))$. Identity (1) and (2) imply that $u(r_1)$ cannot be a local minimun and obviously nor a local maximun. This reaches a contradiction as $f$ has only simple zeros. Thus $u'(r_1) = 0$ cannot hold.

Statement (ii) is a direct consequence of identity (1) ∎

**Lemma 2.** *Let* $u$ *and* $v$ *be two distinct solutions of problem* $(E)$ *which are positive in the interval* $I = (R,\rho)$. *Then*

$$\left\{r^a v(r)^2 \frac{u'}{v}\right\}_R^\rho = \int_R^\rho s^a uv\{\Psi(v) - \Psi(u)\}ds. \tag{3}$$

*Consequently, if* $u$ *and* $v$ *are two distinct solutions of problem* $(E)$ *strictly positive in* $I = (R,r)$, *with* $u > v$ *in* $I$ *and* $(u'v - uv')(R) = 0$, *then:*

**(i)** $u(I) \cup v(I) \subset \psi^+ \implies \frac{u}{v}$ *is strickly decreasing on* $I$.

**(ii)** $u(I) \cup v(I) \subset \psi^- \implies \frac{u}{v}$ *is strickly increasing on* $I$.

*Note that the condition* $(u'v - uv')(R) = 0$ *can be replaced by* $(u'v - uv')(R) \le 0$ *for the case* (i) *and by* $(u'v - uv')(R) \ge 0$ *for the case* (ii).

**Proof.** It is enough to notice that the function $W = u'v - uv' = v^2(\frac{u}{v})'$ satisfies

$$(r^a W)' = r^a uv\{\Psi(v) - \Psi(u)\} = r^a uv\left\{\frac{f(v)}{v} - \frac{f(u)}{u}\right\}$$

in $(R,\rho)$. For statement (i), it is enough to notice that $\Psi(v) - \Psi(u) < 0$ on $I$ by (3) whence $W < 0$ on $I$. Statement (ii) follows from a similar argument ∎

**Lemma 3.** *Let* $u$ *and* $v$ *be two distinct solutions of problem* (E) *which are non-negative in* $I = (r_1, r_2)$.

**(i)** *If* $(u'v - uv')(r_1) = (u'v - uv')(r_2) = 0$ *and* $u'v - uv' \ne 0$ *in* $I$, *then either* $\psi$ *has a zero in* $\{u(r), v(r)\}$ *for* $r \in I$ *or* $u(r) = v(r)$ *has a solution in* $I$.

**(ii)** *If* $u(r_1) = v(r_1)$ *and* $u(r_2) = v(r_2)$, *then* $\psi$ *has a zero in* $\{u(r), v(r)\}$ *for* $r < r_2$.

**Proof.** (i) From identity (3),

$$\frac{f(v)}{v} - \frac{f(u)}{u}$$

changes the sign at some $R \in I$ and either $u(R) = v(R)$ or there exist $R_1, R_2 \in I$ such that

$$\frac{f(u(R_1))}{u(R_1)} = \frac{f(v(R_2)}{v(R_2)}.$$

The later case implies that $\psi$ has a zero in $\{u, v\}$ for $r \in \{R_1, R_2\}$ by the mean value theorem.

(ii) Without loss of generality, suppose that $u > v$ in $I$. For $z(r) = u(r) - v(r)$, there exists $R_1 \in I$ such that

$$z'(R_1) = u'(R_1) - v'(R_1) = 0.$$

As $u > v$ and $u' < 0$ in $I$, we have

$$(u'v - uv')(R_1) = u'(R_1)(v - u)(R_1) > 0$$

whence $u(r_2) = v(r_2)$ holds only if there exists $R_2 \in (R_1, r_2)$ such that $(u'v - uv')(R_2) = 0$. The conclusion follows from statement (i) as $(u'v - uv')(0) = 0$ ∎

If $u$ and $v$ are two distinct solutions of problem (E) and $s > 0$ is such that

$$U(r) = u(r) + s, \quad V(r) = v(r) + s, \quad Z(r) = u(r) - s, \quad Y(r) = v(r) - s$$

are positive, then

$$X'' + \frac{a}{r}X' = -f(X - s) \qquad \text{for } X = U, V \tag{4$_a$}$$

$$\Phi'' + \frac{a}{r}\Phi' = -f(\Phi + s) \qquad \text{for } \Phi = Z, Y. \tag{4$_b$}$$

The next lemma is easy to verify.

**Lemma 4.** *For $0 < s < t$, define $f_{\pm s}(t) = \frac{f(t \pm s)}{t}$. Then*

$$\left.\begin{aligned} \frac{\partial}{\partial t} f_s(t) &= \frac{(t+s)^2 \Psi'(t+s) - sf'(t+s)}{t^2} \\ \frac{\partial}{\partial t} f_{-s}(t) &= \frac{(t-s)^2 \Psi'(t-s) + sf'(t-s)}{t^2}. \end{aligned}\right\} \tag{5}$$

*Consequently, for $I_s(t) = [t, t+s]$ and $I_{-s}(t) = [t-s, t]$,*

$$I_\tau(t) \subset \psi^+ \cap \{f' \le 0\} \implies \frac{\partial f_s(t)}{\partial t} > 0 \tag{5$_a$}$$

$$I_{-\tau}(t) \subset \psi^+ \cap \{f' \ge 0\} \implies \frac{\partial f_{-s}(t)}{\partial t} > 0 \tag{5$_b$}$$

$$I_\tau(t) \subset \psi^- \cap \{f' \ge 0\} \implies \frac{\partial f_s(t)}{\partial t} < 0 \tag{5$_c$}$$

$$I_{-\tau}(t) \subset \psi^- \cap \{f' \le 0\} \implies \frac{\partial f_{-s}(t)}{\partial t} < 0 \tag{5$_d$}$$

*for $0 < s < \tau$.*

**Lemma 5.** *Let $u$ and $v$ be two distinct solutions of problem* (E) *with* $\text{meas}\{r > 0 | \, u(r) = v(r) > 0\} = 0$.

**(i)** *As long as $u$ and $v$ remain in the same connected component of $\psi$, the problem $u(r) = v(r) > 0$ has at most one solution.*

**(ii)** *Suppose that $\psi^+ = (\lambda, A)$ and $\psi^- = (0, \lambda)$. For $u$ and $v$ two solutions of problem* (E) *with $u(A) > v(A) > \lambda$ and $(u'v - uv')(A) \leq 0$, if $u(r_1) = v(r_1) \leq \lambda$, then $u(r) = v(r) > 0$ does not hold for $r > r_1$. If in addition $f' \geq 0$ in some interval $[0, \alpha']$, then $u(r) = v(r) \geq 0$ cannot hold for $r > r_1$. Consequently, if $\psi$ has a finite number of components, then $u(r) = v(r) \geq 0$ has a finite number of solutions.*

**Proof.** (i) The claim follows from the fact that remaining in the same component $\psi$, if $u - v$ has two distinct zeros, then $\frac{u}{v}$ is strictly monotone between them with the same value 1 in both ends. That cannot hold.

(ii) Let $u \geq v \geq \lambda$ in some subset of $\psi^+$. Suppose that $u(R) = v(R) \leq \lambda$ and $0 < u < v$ in some interval $I = (R, r)$. Then

$$(u'v - uv')(R) = u(R)(u - v)'(R) < 0$$

as $u' < v'$ at $R$. Therefore $\frac{v}{u}$ is increasing in some $r > R$ with the value 1 at $R$. We have $v > u$ as long as $u > 0$. If $u(\rho) = v(\rho) = 0$ and $f' \geq 0$ in some interval $[0, \alpha']$, then with $Z(r) = v(r) + s$ and $Y(r) = u(r) + s$ for some small $s > 0$ and $X = Y$ or $X = Z$ we have $D_a X + f(X - s) = 0$ in some interval $J = (R, \rho)$ and $Y(\rho) = Z(\rho) = s$. From $(4)_a$ and $(5)_b$, $(\frac{Z}{Y})' > 0$ in $J$ conflicting with the fact that $(\frac{Z}{Y})(R) > 1$ and $\frac{Z}{Y}(\rho) = 1$. Now statement (ii) follows from the fact that no component of $\psi^+$ neither any of $\psi^-$ can have more than two solutions of the problem $u(r) = v(r) \geq 0$ ∎

**Lemma 6.** *Let $u$ and $v$ be two distinct solutions of problem* (E), *$A = \overline{u(I) \cup v(I)}$ for $I = (r_0, r_1)$ and some $J = [t_0, t_1] \subset A$ with $t_0 > \inf A$.*

**(i)** *Suppose that $A \subset \psi^+$ and*

($\alpha$) *$u > v$ and $u'v - uv' \leq 0$ at $r_0$*

($\beta$) *$f' \leq 0$ in $J$ or condition $|f'; \psi^+|_J$ holds.*

*Then $u(r) = v(r) > \inf A$ has a solution $r_1$ in $I$ with $u'(r_1) \neq v'(r_1)$. If in addition $t_0 = 0$, then $u \neq v$ for $r > r_1$ as long as $u, v \geq 0$ in $A$.*

**(ii)** *Suppose that $A \subset \psi^-$ and*

($\alpha$) *$u > v$ and $u'v - uv' \geq 0$ at $r_0$*

($\beta$) *$f' \leq 0$ in $J$ or condition $|f'; \psi^-|_J$ holds.*

*Then $u > v \geq 0$ in $I$.*

**Proof.** (i) From identity (3), $\frac{u}{v}$ is decreasing in $\psi^+$ as long as $u > v > 0$ there. Assume that $u > v \geq \lambda := \inf \psi^+$. Let $s > 0$ and $t > 0$ be such that $t + s \in J$ and let $v(R') = s < u(R')$ for some $R'$. The functions $Y = v - s$ and $Z = u - s$ satisfy $Y(R') = 0$ and $Z(R') > 0$; for $X = Y$ and $X = Z$ we have $D_a X = -f(X + s)$ in $(r_0, R')$. From (5) and $(5)_a$, if $f' \leq 0$ in $J$, then $(\frac{\partial}{\partial t}) f_s(t) > 0$. Applying Lemma 2 to $Y$

and $Z$ we find that $\frac{Z}{Y}$ is decreasing in $(r_0, R')$ which conflicts with their values at $R'$. The assumption cannot hold. So there is an $R'' \in I$ such that $u(R'') = v(R'')$. As

$$(u'v - uv')(R'') = u(R'')(u - v)'(R'') < 0,$$

we have $u'(R'') < v'(R'')$.

The second part of statement (i) follows the same process as for $s \in (0, t^2\frac{B}{4C})$ and $s \in (0, \frac{t}{2})$,

$$\frac{\partial}{\partial t} f_s(t) > t^{-2}\left\{t^2\frac{B}{4} - sC\right\} > 0.$$

Let $u(r_1) = v(r_1)$ and $u > v$ in $(r_1, \rho)$. If $u = v = 0$ at $\rho$, for $W = U$ with $U = u + s$ and $W = V$ with $V = v + s$ we have $D_aW = -f(W - s)$ in $(r_1 + s, \rho)$ and $\frac{U}{V}(r_1) = \frac{U}{V}(\rho)$. As condition $|f'; \psi^+|_J$ holds, $\frac{U}{V}$ is monotone in $(r_1, \rho)$ and this cannot hold from their values at the both ends.

(ii) Identity (3) implies that $(\frac{u}{v})' > 0$ as long as $u > v$ in $I$ whence they cannot intersect there nor intersect at some $r_1$ with $u(r_1) = v(r_1) > 0$. Assume that $u(r_1) = v(r_1) = 0$. Let $s > 0$ and $t > 0$ such that $t - s \in J$. The functions $U = u + s$ and $V = v + s$ satisfy for some $R_1 > r_1$ and $W = U$ or $W = V$ the relation $D_aW = -f(W - s)$ in $(R_1, r_1) = K$ with $W(r_1) = s$. If $f' \leq 0$ in $J$, $\frac{U}{V}$ is increasing in $K$ with a value greater than 1 at $R_1$. This conflicts with their values at $r_1$.

The last part follows the same process as before. In fact, for $s \in (0, \frac{t}{2})$,

$$(t - s)^2\psi(t - s) + sf'(t - s) < -t^2\frac{B}{4} + s\sup_I |f'|$$

and it suffices to take $s \in (0, t^2\frac{B}{4C})$ for (5) and $(5)_d$ to apply ∎

**Lemma 7.** *Let $A < \lambda < B$, $\psi_0^- = (A, \lambda)$, $\psi_0^+ = (\lambda, B)$, $u$ and $v$ two distinct solutions of problem* (E) *such that for some $0 \leq r_1 < r_2$*

**(i)** *$u(r_1), v(r_1) > \lambda$ and $(u'v - uv')(r_1) < 0$*

**(ii)** *$u(r_2) = v(r_2) < \lambda$ with $u > v$ in $(r_1, r_2)$.*

*Then if $f'$ is strictly monotone in $\psi_0^-$, we have $u'(r_2) < v'(r_2)$.*

**Proof.** Let $v(r_\lambda) = \lambda$. As $u'v - uv' = u'(v - u) + u(u - v)'$, $(u - v)' < 0$ and strictly decreasing in $(r_1, r_\lambda)$ (see (3)). If $(u - v)'(r_2) = 0$, then by the mean value theorem, there is $R \in (r_\lambda, r_2)$ such that $(u - v)''(R) = 0$. In that case, from the equations of $u$ and $v$,

$$a(u - v)'(R) = R\{f(v(R)) - f(u(R))\} < 0$$

and this cannot hold if $f' < 0$ in $\psi_0^-$ whence $(u - v)'(r_2) < 0$ in this case. If $f' > 0$ in $\psi_0^-$, then $(r^a(u - v)')' < 0$ in $(R, r_2)$ and $(u - v)'(R) < 0$ which leads to $(u - v)'(R_2) \leq (u - v)'(R) < 0$ ∎

## 4. Proof of the theorems

The lemmae established in Section 3 enable us to prove now the theorems.

**Proof of Theorem 1.** Let $u$ and $v$ be two solutions of problem (E), with $u > v$ in some interval $[0, r)$, say. From the equations,

$$(u-v)'(r) = \int_0^r \left(\frac{s}{r}\right)^a \{f(v) - f(u)\}\, ds \geq 0$$

whence $u(r) - v(r) \geq u(0) - v(0) > 0$. Therefore they cannot intersect as long as $v \geq 0$ ∎

**Proof of Theorem 2.** 1. In any of the cases, if $u > v$ in $[0, \rho)$ and $u(\rho) = v(\rho) = 0$, then the left-hand side of identity (3) is 0 while the right-hand side is non-zero as the integrand there does not change sign. So $u(\rho) \neq v(\rho) = 0$.

2. If there is $R \in (0, \rho)$ with $u(R) = v(R) > 0$ and $R_1 \in (R, \rho]$ with $u(R_1) = v(R_1)$, then there is $R_2 \in (R, R_1)$ with $(u'v - uv')(R_2) = 0$ and this cannot hold following similar an argument as in part 1 ∎

**Proof of Theorem 3.** Let $u$ and $v$ be two distinct solutions of problem (E). If there is $r < \rho$ such that $u(r) = v(r)$, then $u(r) < \lambda$. Lemma 6/(i) implies that $r \neq \rho$ ∎

**Proof of Theorem 4.** 1. Let $u$ and $v$ be two distinct solutions of problem (E). The problem $u(r) = v(r) > \lambda$ has at most one solution by Lemma 5/(i). Lemmae 6/(ii) and 7 imply that $u(\rho) \neq v(\rho)$.

2. Lemmae 6/(i) and 7 imply that $u - v$ changes sign accross any $r$ where $u(r) = v(r) > 0$. The ends of Theorems 2 and 3 complete the proof ∎

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