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On a Direct Decomposition of the Space $\mathcal{L}_p(\Omega)$

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Abstract. A direct decomposition of the space $\mathcal{L}_p(\Omega)$ is obtained as a generalization of the orthogonal decomposition of the space $\mathcal{L}_2(\Omega)$, where one of the subspaces is the space of all monogenic $\mathcal{L}_p$-functions. Basic results about the orthogonal decomposition are carried over to this more general context. In the end a boundary value problem of the Stokes equations will be studied by a method based on this direct decomposition.

Keywords: *Clifford analysis, L_p-decomposition, Stokes equations*

AMS subject classification: Primary 30 G 35, secondary 31 B 10, 35 Q 30, 46 B 25

1. Introduction

One of the most interesting facts of complex and hypercomplex function theory is the orthogonal decomposition of the space $\mathcal{L}_2(\Omega)$:

$$\mathcal{L}_2(\Omega) \;=\; \ker D(\Omega) \cap \mathcal{L}_2(\Omega) \oplus D(\mathring{\mathcal{W}}_2^1(\Omega)), \tag{1}$$

where $\ker D(\Omega)$ denotes the set of all holomorphic resp. monogenic functions over Ω. This decomposition has a lot of applications, especially to the theory of partial differential equations (e.g. [3] for the complex case and [6] for the hypercomplex case). In the second paper the authors use this orthogonal decomposition to study boundary value problems of mathematical physics over bounded domains in the scale of Hilbert spaces $\mathcal{W}_2^k(\Omega)$ in a self-contained theory. This was extended in [1, 5, 7] to the case of unbounded domains. But, a lot of applications, like investigations of Navier-Stokes equations in higher dimensions, require the usage of the scales of Banach spaces $\mathcal{W}_p^k(\Omega)$, where we do not have an orthogonal decomposition.

The aim of this paper is to extend the orthogonal decomposition (1) to the spaces $\mathcal{L}_p(\Omega)$ $(1 < p < \infty)$ in form of a direct decomposition in the hypercomplex case, c.f. in the case of Clifford analysis. In the latter part, we show the easy applicability of this direct decomposition with the example of a Stokes system.

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2. Preliminaries

Let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ be an orthonormal basis of $\mathbb{R}^n$. The Clifford algebra $C\ell_{0,n}$ is the free algebra over $\mathbb{R}^n$ generated modulo the relation

$$x^2 = -|x|^2 \mathbf{e}_0,$$

where $\mathbf{e}_0$ is the identity of $C\ell_{0,n}$. For the algebra $C\ell_{0,n}$ we have the anticommutative relationship

$$\mathbf{e}_i \mathbf{e}_j + \mathbf{e}_j \mathbf{e}_i = -2\delta_{ij}\mathbf{e}_0,$$

where δ_{ij} is the Kronecker symbol. In the following we will identify the Euclidean space $\mathbb{R}^n$ with $\bigwedge^1 C\ell_{0,n}$, the space of all vectors of $C\ell_{0,n}$. This means that each element x of $\mathbb{R}^n$ may be represented by

$$x = \sum_{i=1}^{n} x_i \mathbf{e}_i.$$

From an analysis viewpoint one extremely crucial property of the algebra $C\ell_{0,n}$ is that each non-zero vector $x \in \mathbb{R}^n$ has a multiplicative inverse given by $\frac{-x}{|x|^2}$. Up to a sign this inverse corresponds to the Kelvin inverse of a vector in Euclidean space.

For all what follows let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a sufficiently smooth boundary $\Gamma = \partial\Omega$. Then any function $f : \Omega \mapsto C\ell_{0,n}$ has a representation $f = \sum_A \mathbf{e}_A f_A$ with $\mathbb{R}$-valued components f_A. Thus notations $f \in \mathcal{C}^k(\Omega, C\ell_{0,n})$ $(k \in \mathbf{N} \cup \{0\})$ and $f \in \mathcal{L}_p(\Omega, C\ell_{0,n})$ $(1 \leq p)$ might be understood both coordinatewisely and directly. For instance, $f \in \mathcal{L}_p(\Omega, C\ell_{0,n})$ means that $\{f_k\} \subset \mathcal{L}_p(\Omega, \mathbb{R})$ or, equivalently, that $\int_\Omega |f(x)|^p dx < \infty$. All these sets are $C\ell_{0,n}$-bi-modules. $\mathcal{L}_2(\Omega, C\ell_{0,n})$ can be converted into a Hilbert $C\ell_{0,n}$-module, namely, an inner product can be defined as

$$\langle f, g \rangle := \int_\Omega \overline{f(x)} \cdot g(x)\, dx$$

and thus $\mathcal{L}_2(\Omega, C\ell_{0,n})$ becomes a right $C\ell_{0,n}$-module.

In the following we use the short notation $\mathcal{L}_p(\Omega)$, $\mathcal{C}^k(\Omega)$, etc., instead of $\mathcal{L}_p(\Omega, C\ell_{0,n})$, $\mathcal{C}^k(\Omega, C\ell_{0,n})$.

We now introduce the Dirac operator $D = \sum_{i=1}^n \mathbf{e}_i \frac{\partial}{\partial x_i}$. This operator is a hypercomplex analogue to the complex Cauchy-Riemann operator. In particular, we have $D^2 = -\Delta$, where Δ is the Laplacian over $\mathbb{R}^n$. A function $f : \Omega \mapsto C\ell_{0,n}$ is said to be *left-monogenic* if it satisfies the equation $(Df)(x) = 0$ for each $x \in \Omega$. A similar definition can be given for right-monogenic functions. Basic properties of the Dirac operator and left-monogenic functions one can find in [2, 6].

An important example of a left-monogenic function is the so-called generalized Cauchy kernel

$$e(x) = \frac{1}{\omega} \frac{-x}{|x|^n},$$

where ω denotes the surface area of the unit ball in $\mathbb{R}^n$. This function is a fundamental solution of the Dirac operator which we can use to introduce the following integral operators.

Definition 1. Let us denote by $\alpha(y)$ the outward pointing normal vector to Γ at y. For $f \in \mathcal{C}^{0,\beta}(\Omega)$ $(0 < \beta \leq 1)$ we have

- the Teodorescu transform (or T-operator)

$$Tf(x) = \int_\Omega e(x-y) f(y)\, d\Omega_y,$$

- the boundary operator F_Γ

$$F_\Gamma f(x) = \int_\Gamma e(x-y)\alpha(y) f(y)\, d\Gamma_y$$

with $x \notin \Gamma$, and

- the singular integral operator S_Γ

$$Sf(x) = 2\int_\Gamma e(x-y)\alpha(y) f(y)\, d\Gamma_y$$

where $x \in \Gamma$.

These integral operators have the following mapping properties in scales of Sobolev spaces ($k \in \mathbf{N} \cup \{0\}, 1 < p < \infty$):

- $T : \mathcal{W}_p^k(\Omega) \mapsto \mathcal{W}_p^{k+1}(\Omega)$
- $F_\Gamma : \mathcal{W}_p^{k+1-\frac{1}{p}}(\Gamma) \mapsto \mathcal{W}_p^{k+1}(\Omega)$
- $S_\Gamma : \mathcal{L}_p(\Gamma) \mapsto \mathcal{L}_p(\Gamma)$.

For the proof we refer to [6].

In this paper we need an additional mapping property for the T-operator.

Theorem 1. *The operator*

$$T : \mathcal{W}_p^{-1}(\Omega) \mapsto \mathcal{L}_p(\Omega) \qquad (1 < p < \infty)$$

is bounded.

Proof. Based on the fact that this operator is defined by the acting of its symbol to the Fourier transform of the function, the usual way for the proof would be the investigation of the symbol, i.e. the Fourier transform of the kernel, but here we use another way. This idea is more suitable for us due to the fact that our function spaces are modules, and it demonstrates the way one can work directly with this kind of spaces. Each element $f \in \mathcal{W}_p^{-1}(\Omega)$ has a representation in the form

$$f = f_0 + \sum_{k=1}^{n} \partial_k f_k$$

with $f_0, f_k \in \mathcal{L}_p(\Omega)$. If we choose

$$v = v_0 + \sum_{k=1}^{n} \partial_k v_k$$

with $v_0, v_k \in \mathcal{C}_0^\infty(\Omega)$, then we can consider

$$Tv = \int_\Omega e(x-y)v(y)\, d\Omega_y.$$

We have

$$\begin{aligned} Tv &= \int_\Omega e(x-y)\left(v_0(y) + \sum_{k=1}^n \partial_k v_k(y)\right) d\Omega_y \\ &= \int_\Omega e(x-y)v_0(y) d\Omega_y + \sum_{k=1}^n \int_\Omega \partial_k e(x-y) v_k(y)\, d\Omega_y. \end{aligned}$$

The integrals $\int_\Omega \partial_k e(x-y)v_k(y)d\Omega_y$ are strongly singular integrals of Calderon-Zygmund type for all k [6], i.e.

$$\begin{aligned} \|Tv\|_{\mathcal{L}_p(\Omega)} &\leq \left\| \int_\Omega e(x-y)v_0(y)\, d\Omega_y \right\|_{\mathcal{L}_p(\Omega)} + \sum_{k=1}^n \left\| \int_\Omega \partial_k e(x-y)v_k(y)\, d\Omega_y \right\|_{\mathcal{L}_p(\Omega)} \\ &\leq C_0 \|v_0\|_{\mathcal{L}_p(\Omega)} + \sum_{k=1}^n C_k \|v_k\|_{\mathcal{L}_p(\Omega)}, \end{aligned}$$

where $C_k \ (k = 0, \ldots, n)$ are constant. Using the density of $\mathcal{C}_0^\infty(\Omega)$ in $\mathcal{L}_p(\Omega)$ in the case of $f \in \mathcal{W}_p^{-1}(\Omega)$ we get

$$\|Tf\|_{\mathcal{L}_p(\Omega)} \leq C \left(\|f_0\|_{\mathcal{L}_p(\Omega)} + \sum_{k=1}^n \|f_k\|_{\mathcal{L}_p(\Omega)} \right)$$

for any arbitrary representation of f. From this $\|Tf\|_{\mathcal{L}_p(\Omega)} \leq \hat{C}\|f\|_{\mathcal{W}_p^{-1}(\Omega)}$ follows ∎

Proposition 1. *Due to the fact that the Laplace operator $\Delta : \mathcal{W}_p^1(\Omega) \mapsto \mathcal{W}_p^{-1}(\Omega)$ is bounded we get that the operator $D : \mathcal{L}_p(\Omega) \mapsto \mathcal{W}_p^{-1}(\Omega)$ is bounded because of $D = -\Delta T$.*

Furthermore, we have the following properties (for proofs we refer again to [6]):

Theorem 2. *Let $f \in \mathcal{L}_p(\Omega)$ $(1 < p < \infty)$. Then*

$$DTf = f,$$

i.e. the T-operator is a right inverse to D. In the case of $f \in \mathcal{C}^{0,\beta}(\overline{\Omega})$ $(0 < \beta \leq 1)$

$$DTf(x) = \begin{cases} f(x) & \text{for } x \in \Omega \\ 0 & \text{for } x \in \mathbb{R}^n \setminus \overline{\Omega} \end{cases}$$

holds.

Theorem 3 (Borel-Pompeiu formula). *For $f \in \mathcal{W}_p^1(\Omega)$ $(1 < p < \infty)$*

$$F_\Gamma f + TDf = f$$

holds.

Theorem 4 (Plemelj-Sokhotzki formulae). *Let* $f \in \mathcal{W}_p^{1-\frac{1}{p}}(\Gamma)$ $(1 < p < \infty)$. *Then*

$$\operatorname{tr} F_\Gamma f = \frac{1}{2} f + \frac{1}{2} S_\Gamma f \qquad \textit{and} \qquad \operatorname{tr}^- F_\Gamma f = -\frac{1}{2} f + \frac{1}{2} S_\Gamma f,$$

where tr^- *denotes the trace operator in the exterior domain* $\mathbb{R}^n \setminus \overline{\Omega}$.

From these Plemelj-Sokhotzki formulae we can derive the projections $P_\Gamma = \frac{1}{2} I + \frac{1}{2} S_\Gamma$ and $Q_\Gamma = -\frac{1}{2} I + \frac{1}{2} S_\Gamma$. Hereby, P_Γ is the projection onto the space of all $C\ell_{0,n}$-valued functions which are left-monogenic extendable into the domain Ω and Q_Γ is the projection onto the space of $C\ell_{0,n}$-valued functions which are left-monogenic extendable into the domain $\mathbb{R}^n \setminus \overline{\Omega}$ and vanish at infinity.

3. A direct $\mathcal{L}_p$-decomposition

As we have mentioned earlier the following theorem is proved in [6].

Theorem 5. *The Hilbert module* $\mathcal{L}_2(\Omega)$ *allows the orthogonal decomposition*

$$\mathcal{L}_2(\Omega) = \ker D(\Omega) \cap \mathcal{L}_2(\Omega) \oplus D(\mathring{\mathcal{W}}_2^1(\Omega)).$$

The proof of this theorem is based on the existence of an inner product in $\mathcal{L}_2(\Omega)$ and the properties of the boundary projections P_Γ and Q_Γ. Due to the fact that there is no inner product in $\mathcal{L}_p(\Omega)$ for $p \neq 2$ we cannot use the same basic ideas in our attempt to extend the above theorem to the case of $\mathcal{L}_p(\Omega)$. Nevertheless, by using the fact that the Dirichlet problem of the Poisson equation with homogeneous boundary data

$$\left.\begin{aligned} -\Delta u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \Gamma \end{aligned}\right\}$$

for $f \in \mathcal{W}_p^{-1}(\Omega)$ $(1 < p < \infty)$ has a unique solution [9], whereby the solution operator will be denoted with Δ_0^{-1}, we can prove the following theorem.

Theorem 6. *The space* $\mathcal{L}_p(\Omega)$ $(1 < p < \infty)$ *allows the direct decomposition*

$$\mathcal{L}_p(\Omega) = \ker D(\Omega) \cap \mathcal{L}_p(\Omega) \dot{+} D(\mathring{\mathcal{W}}_p^1(\Omega)). \tag{2}$$

Proof. As a first step let us take a look at the intersection of the two subspaces $\ker D(\Omega) \cap \mathcal{L}_p(\Omega)$ and $D(\mathring{\mathcal{W}}_p^1(\Omega))$. Suppose $f \in \ker D(\Omega) \cap \mathcal{L}_p(\Omega) \cap D(\mathring{\mathcal{W}}_p^1(\Omega))$. Obviously, we have $Df = 0$.

Moreover, because of $f \in D(\mathring{\mathcal{W}}_p^1(\Omega))$ there exists a function $g \in \mathring{\mathcal{W}}_p^1(\Omega)$ with $Dg = f$ and $\Delta g = 0$. From the uniqueness of Δ_0^{-1} we get $g = 0$ and, consequently, $f = 0$, i.e. the intersection of these subspaces contains only the zero function. Therefore, our sum is a direct one.

Now let $f \in \mathcal{L}_p(\Omega)$. Then $f_2 = D\Delta_0^{-1} Df \in D(\mathring{\mathcal{W}}_p^1(\Omega))$. Let us now apply D to the function $f_1 = f - f_2$. This results in

$$Df_1 = Df - Df_2 = Df - DD\Delta_0^{-1} Df = Df + \Delta\Delta_0^{-1} Df = Df - Df = 0,$$

i.e. $Df_1 \in \ker D(\Omega) \cap \mathcal{L}_p(\Omega)$. Because $f \in \mathcal{L}_p(\Omega)$ was arbitrary chosen (2) is a decomposition of the space $\mathcal{L}_p(\Omega)$ ∎

Starting from this decomposition we get the projections

$$\mathbf{P}:\ \mathcal{L}_p(\Omega) \mapsto \ker D \cap \mathcal{L}_p(\Omega)$$
$$\mathbf{Q}:\ \mathcal{L}_p(\Omega) \mapsto D(\mathring{\mathcal{W}}_p^1(\Omega)).$$

For $p = 2$ these are orthoprojections.

Now of interest is the question about representation formulae of our projections. Directly from our proof of decomposition (2) we get

$$\mathbf{Q}f = D\Delta_0^{-1}Df.$$

An interesting aspect is that this representation is independent of p. Because of $\mathcal{L}_q(\Omega) \subset \mathcal{L}_p(\Omega)$ for $q \geq p$ this means that $\mathbf{Q}$ as well as $\mathbf{P}$ maps the subset $\mathcal{L}_q(\Omega)$ of $\mathcal{L}_p(\Omega)$ into itself. Later on we will get other representations of $\mathbf{P}$ and $\mathbf{Q}$, but now let us take a look at the applications of this direct decomposition.

4. Applications

As a first application we can derive a very interesting property of our T-operator.

Theorem 7. *Let $f \in \mathcal{L}_p(\Omega)$ $(1 < p < \infty)$, then* $\operatorname{tr} Tf = 0$ *holds if and only if f belongs to the space* $\operatorname{im}\mathbf{Q}$.

Proof. If $f \in \operatorname{im}\mathbf{Q}$, then there exists $u \in \mathring{\mathcal{W}}_p^1(\Omega)$ with $f = Du$. From the Borel-Pompeiu formula we know $u = F_\Gamma u + TDu = TDu = Tf$. Because of $u \in \mathring{\mathcal{W}}_p^1(\Omega)$ we have $\operatorname{tr} Tf = 0$.

Now, let $\operatorname{tr} Tf = 0$. Using our projections we can decompose $f = \mathbf{P}f + \mathbf{Q}f$. This results in $\operatorname{tr} T\mathbf{P}f + \operatorname{tr} T\mathbf{Q}f = 0$. As we have previously shown, always $\operatorname{tr} T\mathbf{Q}f = 0$ holds. Therefore, $\operatorname{tr} T\mathbf{P}f = 0$. Moreover, $T\mathbf{P}f$ is harmonic and, due to the uniqueness of the Dirichlet problem of the Laplace equation, equals zero. Applying the Dirac operator D we get $0 = DT\mathbf{P}f = \mathbf{P}f$, i.e. $f \in \operatorname{im}\mathbf{Q}$ ∎

The basic ideas of this proof one can find already in [6], were the case of $p = 2$ was proved.

As we mentioned before, in [6] the orthogonal decomposition (1) is used to solve boundary value problems of mathematical physics. We will illustrate the same applicability of our direct decomposition with the example of a Stokes system.

As a first step we have to take a closer look to the Dirichlet problem of the Poisson equation. From [9] we know that the problem

$$\left.\begin{aligned} -\Delta u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \Gamma \end{aligned}\right\}$$

has a unique solution for $f \in \mathcal{W}_q^{-1}(\Omega)$ $(1 < q < \infty)$. This solution can be represented in the form $u = T\mathbf{Q}Tf$, i.e. we get the identity $\Delta_0^{-1}f = T\mathbf{Q}Tf$. Therefore, we get for the general problem

$$\left.\begin{aligned} -\Delta u &= f \quad \text{in } \Omega \\ u &= g \quad \text{on } \Gamma \end{aligned}\right\}$$

with $f \in \mathcal{W}_q^k(\Omega)$ and $g \in \mathcal{W}_q^{k+\frac{3}{2}}(\Gamma)$ $(k \geq -1, 1 < q < \infty)$ the representation

$$u = F_\Gamma g + T\mathbf{P}Dh + T\mathbf{Q}Tf,$$

where h is a $\mathcal{W}_q^{k+2}$-extension of g into the domain Ω. The proof is the same as in [6] for the case $q = 2$.

With the help of this representation we can prove the following

Theorem 8. *Suppose $k \in \mathbf{N}$ and $1 < q < \infty$. Then*

$$\mathrm{tr}\, TF_\Gamma : \mathcal{W}_q^{k-\frac{1}{q}}(\Gamma) \cap \mathrm{im} P_\Gamma \mapsto \mathcal{W}_q^{k+1-\frac{1}{q}}(\Gamma) \cap \mathrm{im} Q_\Gamma$$

is an isomorphism.

Proof. Obviously,

$$(\mathrm{tr}\, TF_\Gamma)(\mathcal{W}_q^{k-\frac{1}{q}}(\Gamma)) \subset \mathcal{W}_q^{k+1-\frac{1}{q}}(\Gamma).$$

Let us first show that $\ker \mathrm{tr}\, TF_\Gamma = \{0\}$. For this we suppose $v \in \mathcal{W}_q^{k-\frac{1}{q}}(\Gamma) \cap \mathrm{im} P_\Gamma$ with $\mathrm{tr}\, TF_\Gamma v = 0$. Because of $DDTF_\Gamma v = 0$ we get from the uniqueness of the solution of the Dirichlet problem of the Laplace equation $TF_\Gamma v = 0$. The operator T has only a trivial kernel, i.e. $F_\Gamma v = 0$, and due to $v \in \mathrm{im} P_\Gamma$ we have $v = 0$.

Now, let $w \in \mathrm{im} Q_\Gamma$, i.e. $w = Q_\Gamma v$. From the representation of the solution of the boundary value problem

$$\left.\begin{aligned} DDu &= 0 \quad \text{in } \Omega \\ u &= w \quad \text{on } \Gamma \end{aligned}\right\}$$

we obtain $u = F_\Gamma w + T\mathbf{P}Dh$, where h is a $\mathcal{W}_q^{k+2}$-extension of w in Ω. Because of $w \in \mathrm{im} Q_\Gamma$, it follows $F_\Gamma w = 0$ and $T\mathbf{P}Dh = TF_\Gamma \mathbf{P}Dh$. Therefore, we have $w \in \mathrm{im}\, \mathrm{tr}\, TF_\Gamma$.

From the proved injectivity and surjectivity of the mapping $\mathrm{tr}\, TF_\Gamma$ we get our statement ∎

The above used idea of generalizing the corresponding proof in the case of $\mathcal{W}_2^k(\Omega)$ to the case of $\mathcal{W}_p^k(\Omega)$ $(1 < p < \infty)$ was already outlined in [6].

As a proposition from this theorem we get for our projection $\mathbf{P}$ the representation

$$\mathbf{P}f = F_\Gamma (\mathrm{tr}\, TF_\Gamma)^{-1} \mathrm{tr}\, Tf$$

for $f \in \mathcal{W}_q^k(\Omega)$ $(k \in \mathbf{N}, 1 < q < \infty)$. In the case of $q = 2$ this representation is proved in [6].

Now, let us take a look at our example. We consider the system

$$-\Delta u + \frac{1}{\eta} Dp = \frac{\rho}{\eta} f \quad \text{in } \Omega \tag{3}$$

$$\mathrm{Sc}\, Du = 0 \quad \text{in } \Omega \tag{4}$$

$$u = 0 \quad \text{on } \Gamma. \tag{5}$$

This system describes the stationary flow of a homogeneous viscuous incompressible fluid for small Reynold numbers. Hereby, u is the velocity, p the pressure, ρ the density, and η the viscosity of the fluid.

In the case of $q = 2$ this system is investigated in [6], but in these investigations orthogonality conditions are widely used, which make a direct translation to our case impossible. Nevertheless, let us start our investigations with the following

Lemma 1. *Suppose* $f \in \mathcal{W}_q^{-1}(\Omega)$ *and* $p \in \mathcal{L}_q(\Omega)$ $(1 < q < \infty)$. *Then for each solution of system* (3) – (5) *we have the representation*

$$u = \frac{\rho}{\eta} T\mathbf{Q}Tf - \frac{1}{\eta} T\mathbf{Q}p.$$

Proof. Let $p_n \in \mathcal{W}_q^1(\Omega)$ with $p_n \to p$ in $\mathcal{L}_q(\Omega)$. Using the Borel-Pompeiu formula we get

$$T\mathbf{Q}T(Dp_n) = T\mathbf{Q}(p_n - F_\Gamma p_n) = T\mathbf{Q}p_n.$$

From the density of $\mathcal{W}_q^1(\Omega)$ in $\mathcal{L}_q(\Omega)$, $T\mathbf{Q}TDp = T\mathbf{Q}p$ follows. Therefore, for $u \in \mathring{\mathcal{W}}_q^1(\Omega)$ and $p \in \mathcal{L}_q(\Omega)$ we have

$$T\mathbf{Q}T\Big(\frac{\rho}{\eta} f\Big) = T\mathbf{Q}T\Big(DDu + \frac{1}{\eta} Dp\Big) = u + \frac{1}{\eta} T\mathbf{Q}p$$

whereas $u = \frac{\rho}{\eta} T\mathbf{Q}Tf - \frac{1}{\eta} T\mathbf{Q}p$ follows ∎

This result means that our system (3) - (5) is equivalent to the system

$$u + \frac{1}{\eta} T\mathbf{Q}p = \frac{\rho}{\eta} T\mathbf{Q}Tf \tag{6}$$

$$\frac{1}{\eta} \operatorname{Sc} \mathbf{Q}p = \frac{\rho}{\eta} \operatorname{Sc} \mathbf{Q}Tf \tag{7}$$

whereby D applied to the first equation results in

$$Du + \frac{1}{\eta} \mathbf{Q}p = \frac{\rho}{\eta} \mathbf{Q}Tf.$$

For solving our problem it is enough to consider system (6) - (7).

Suppose $u \in \mathring{\mathcal{W}}_q^1(\Omega) \cap \ker \operatorname{div}$, $f \in \mathcal{W}_q^{-1}(\Omega)$ and $p \in \mathcal{L}_q(\Omega, \mathbb{R})$ $(1 < q < \infty)$. First, let us consider the question if the sum in the above equation represents a direct sum. Hereby, it follows from $Du = \mathbf{Q}p$ that $\operatorname{Sc} \mathbf{Q}p = 0$ holds, because of $\mathbf{Q}p = Dv$ with $v \in \mathring{\mathcal{W}}_q^1(\Omega)$ and $\operatorname{Sc} Du = 0$. Therefore, it means that $u = 0$ and $p = 0$, i.e. $Du + \mathbf{Q}p$ is a direct sum, which is a subset of im$\mathbf{Q}$.

Therefore, the question is: Does a functional $H \in (\mathcal{L}_q(\Omega) \cap \operatorname{im}\mathbf{Q})'$ with $H(Du) = 0$ and $H(\mathbf{Q}p) = 0$, but $H(\mathbf{Q}Tf) \neq 0$, exists?

Equivalently, one can ask if there exists $h \in \mathcal{W}_r^{-1}(\Omega)$ $(\frac{1}{r} + \frac{1}{q} = 1)$ such that

$$(Du, \mathbf{Q}Th)_{\mathrm{Sc}} = 0 \qquad \text{for all } u \in \mathring{\mathcal{W}}_q^1(\Omega) \cap \ker \operatorname{div}$$

$$(\mathbf{Q}p, \mathbf{Q}Th)_{\mathrm{Sc}} = 0 \qquad \text{for all } p \in \mathcal{L}_q(\Omega)$$

but $(\mathbf{Q}Tf, \mathbf{Q}Th)_{\mathrm{Sc}} \neq 0$, whereby

$$(w, v)_{\mathrm{Sc}} = \operatorname{Sc} \int_\Omega \overline{w} v \, d\Omega$$

with $w \in \mathcal{L}_q(\Omega)$ and $v \in \mathcal{L}_r(\Omega)$ $(\frac{1}{q}+\frac{1}{r}=1)$? Hereby, we are using that $(\mathcal{L}_q(\Omega)\cap \mathrm{im}\mathbf{Q})' = \mathcal{L}_r(\Omega) \cap \mathrm{im}\mathbf{Q}$ $(\frac{1}{q}+\frac{1}{r}=1)$ holds.

Let us now consider the system

$$\left.\begin{aligned} (Du, \mathbf{Q}Th)_{\mathrm{Sc}} &= 0 \qquad \text{for all } u \in \mathring{\mathcal{W}}_q^1(\Omega) \cap \ker \mathrm{div} \\ (\mathbf{Q}p, \mathbf{Q}Th)_{\mathrm{Sc}} &= 0 \qquad \text{for all } p \in \mathcal{L}_q(\Omega, \mathbb{R}) \end{aligned}\right\}$$

with $h \in \mathcal{W}_r^{-1}(\Omega)$. Here

$$(Du, \mathbf{Q}Th)_{\mathrm{Sc}} = (u, D\mathbf{Q}Th)_{\mathrm{Sc}} = (u,h)_{\mathrm{Sc}} = 0$$

holds. From this $h = \mathrm{grad}\, g = Dg$ with $g \in \mathcal{L}_r(\Omega, \mathbb{R})$ follows (c.f. [4: Chapter III/Lemma 1.1]). Furthermore, we have

$$(\mathbf{Q}p, \mathbf{Q}Th)_{\mathrm{Sc}} = (\mathbf{Q}p, \mathbf{Q}TDg)_{\mathrm{Sc}} = (\mathbf{Q}p, \mathbf{Q}(g - F_\Gamma g))_{\mathrm{Sc}} = (\mathbf{Q}p, \mathbf{Q}g)_{\mathrm{Sc}} = 0$$

for all $p \in \mathcal{L}_q(\Omega, \mathbb{R})$. Therefore, $\mathbf{Q}g = 0$ as well as $h = D\mathbf{Q}g = 0$ and we get

$$(\mathbf{Q}Tf, \mathbf{Q}Th)_{\mathrm{Sc}} = 0 \qquad \text{for all } f \in \mathring{\mathcal{W}}_q^1(\Omega).$$

The conclusion is that we can decompose each function $\mathbf{Q}Tf$ $(f \in \mathcal{W}_q^{-1}(\Omega), 1 < q < \infty)$ in the form

$$Du + \frac{1}{\eta}\mathbf{Q}p = \frac{\rho}{\eta}\mathbf{Q}Tf$$

with $u \in \mathring{\mathcal{W}}_q^1(\Omega) \cap \ker \mathrm{div}$ and $p \in \mathcal{L}_q(\Omega, \mathbb{R})$. Moreover, applying the T-operator

$$u + \frac{1}{\eta}T\mathbf{Q}p = \frac{\rho}{\eta}T\mathbf{Q}Tf$$

follows.

We are now proceed to establish the following

Theorem 9. *The Stokes system* (3) – (5) *has a unique solution* $\{u,p\}$ *in the form*

$$u + \frac{1}{\eta}T\mathbf{Q}p = \frac{\rho}{\eta}T\mathbf{Q}Tf.$$

Hereby, the hydrostatic pressure p is unique up to a constant.

Remark 1. The uniqueness of the solution can be shown exactly in the same way as in [6].

Now, suppose $f \in \mathcal{W}_q^k(\Omega)$ $(k \geq 0, 1 < q < \infty)$. Then the unique solution $\{u,p\}$ of problem (3) - (5) has the representation

$$\left.\begin{aligned} u &= \frac{\rho}{\eta}T\,\mathrm{Vec}\,Tf - \frac{\rho}{\eta}T\,\mathrm{Vec}\,F_\Gamma(\mathrm{tr}\,T\,\mathrm{Vec}\,F_\Gamma)^{-1}\mathrm{tr}\,T\,\mathrm{Vec}\,Tf \\ p &= \rho\,\mathrm{Sc}\,Tf - \rho\,\mathrm{Sc}\,F_\Gamma(\mathrm{tr}\,T\,\mathrm{Vec}\,F_\Gamma)^{-1}\mathrm{tr}\,T\,\mathrm{Vec}\,Tf. \end{aligned}\right\}$$

Consequently, we have $u \in \mathcal{W}_q^{k+2}(\Omega) \cap \mathring{\mathcal{W}}_q^1(\Omega)$ and $p \in \mathcal{W}_q^{k+1}(\Omega, \mathbb{R})$. The proof of this representation is analogous to the proof of a similar result in [6] for the case $q = 2$.

Remark 2. System (3) - (5) should really be considered as an example for the general way of treating such kind of problems by the help of our direct decomposition. In the same way it is also possible to investigate all the other problems from [6 - 8], like the Navier-Stokes problem, Helmholtz equation, Lamé equations, or the biharmonic equation.

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