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Extension of the Bernstein Condition to Systems of Ordinary Differential Equations of General Form

A. Ja. Lepin and A. D. Myshkis

Abstract. The Bernstein condition of boundedness of the derivatives of an a priori bounded solution of a 2nd order ordinary differential equation is extended to systems in which each equation has its own order.

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1. Introduction

The Bernstein theorem for the equation

$$x''(t) = f(t, x(t), x'(t))$$

is well-known [1: Section 1.2]. According to it, the inequality

$$|f(t,x,x_1)| \le A\,x_1^2 + B \qquad (A, B \text{ constants})$$

guarantees the boundedness of x', if the solution x of the equation above is bounded. This theorem was extended in several directions. So, the vector equation

$$x^{(n)}(t) = f\big(t, x(t), x'(t), ..., x^{(n-1)}(t)\big) \qquad \big(x(t) \in \mathbb{R}^m \;\; (m \ge 1), n \ge 2\big) \tag{1}$$

was considered in [2] with f continuous. There was proven that, if the function f satisfies the estimation

$$\big|f(t, x, x_1, \ldots, x_{n-1})\big| \le A\left(|x_1|^n + |x_2|^{\frac{n}{2}} + \ldots + |x_{n-1}|^{\frac{n}{n-1}}\right) + B \tag{2}$$

for $|x| \le a \;\; (a > 0)$ and $A, B > 0$, then any solution $x : [t_0, T] \to \mathbb{R}^m$ of (1) which satisfies the a priori estimation $|x(t)| \le \alpha$ with sufficiently small α depending only

A. Ja. Lepin: Latvian Univ., Inst. Math. & Inf. Sci., bul. Rainis 29, Riga, Latvia
A. D. Myshkis: Moscow State Univ. of Comm., ul. Obraztsova 15, Moscow 101475, Russia;
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on A, B and m, n can be continued onto the whole semiaxis $[t_0, \infty)$ and has bounded derivatives $x', \ldots, x^{(n-1)}$ on it. But if condition (2) is replaced by

$$\sup_{t\in[t_0,\infty)} \max_{|x|\le a} \big|f(t,x,x_1,\ldots,x_{n-1})\big| = o\big(|x_1|^n + \ldots + |x_{n-1}|^{\frac{n}{n-1}}\big) \tag{3}$$

as

$$\left.\begin{matrix} |x_1| \\ \vdots \\ |x_{n-1}| \end{matrix}\right\} \to \infty,$$

for any fixed $a > 0$, then the condition of sufficient smallness of α is eliminated, i.e. any a priory bounded solution x has bounded derivatives $x', \ldots, x^{(n-1)}$ (this statement holds under estimation (2) only if $n = 2$ and $m = 1$, i.e. in the case covered by the Bernstein theorem).

The transition to a right-hand side of equation (1) which satisfies the Carathéodory conditions (see, e.g., [3: Section 18.4]), the replacement of boundedness of the solution x on its uniform L_p-boundedness on segments of fixed length, and some other generalizations are contained in [4, 5]. The results of [5] can be applied especially to the system of scalar equations

$$x_i^{(n_i)}(t) = f_i\big(t,\ldots,x_j^{(k)}(t),\ldots\big) \qquad \Big({\textstyle{i,j=1,\ldots,m \atop k=0,\ldots,n_j-1}}\Big). \tag{4}$$

The aim of the present paper is to give effective sufficiency conditions on the functions f_i for the possibility of a continuation onto the whole semi-axis of any a priory bounded solution of system (4) and the boundedness of all its derivatives $x_j^{(k)}(t)$ $(k \le n_j - 1)$

2. General plan of the estimation of derivatives

2.1. We consider solutions of the system of scalar equations (4), whose right-hand sides are given for $t \in [0,\infty)$ and arbitrary values of other arguments and satisfy the Carathéodory condition. Uniqueness of the solution of any Cauchy problem is not supposed. Let the solution

$$t \mapsto x(t) = \big(x_1(t), \ldots, x_m(t)\big)$$

of system (4) be built starting from $t = 0$ in the direction of growth of t, and let be known that the values of this solution, being arbitrarily continued, cannot leave some domain

$$Q = [-\alpha_1, \alpha_1] \times \cdots \times [-\alpha_m, \alpha_m] \qquad (\alpha_1, \ldots, \alpha_m \in (0,\infty)).$$

The problem is to find conditions on the functions f_i under which all derivatives of the solution of system (4) indicated in the right-hand sides of that system remain bounded. In particular, it follows from here that any such solution can be continued on the whole semi-axis $[0,\infty)$.

We shall use the Kolmogorov-Gorny inequality (see, e.g., [6: Supplement 37]) for any function $\psi \in C^s([a,b];\mathbb{R})$

$$\|\psi^{(k)}\| \le a_{s,k}\,\|\psi\|^{\frac{s-k}{s}} \Big[\max\Big\{\|\psi^{(s)}\|, \frac{s!}{(b-a)^s}\|\psi\|\Big\}\Big]^{\frac{k}{s}} \qquad (k=0,\dots,s-1) \tag{5}$$

where $\|\cdot\| = \max_{[a,b]}|\cdot|$ while $a_{s,k}>0$ are absolute constants with $a_{s,0}=1$.

The following simple lemma will be needed for us:

Lemma 1. *For any $s \in \mathbb{N}$ there exists $r_s > 0$ such that the implication*

$$a \in \mathbb{R}, b \in (a,\infty), \varphi \in C^s([a,b],\mathbb{R}) \quad \Longrightarrow \quad (b-a)^s \min|\varphi^{(s)}| \le r_s \max|\varphi|$$

holds.

2.2. Let $x : [0,t] \to Q$ $(0 < t < \infty)$ be a solution of system (4) and denote

$$M_i(t) = \max_{\tau\in[0,t]} |x_i^{(n_i-1)}(\tau)| \qquad (i=1,\dots,m).$$

We find conditions under which all functions $M_i(t)$ remain bounded in the continuation process of any such solution of system (4). Then the boundedness of its derivatives of lower orders will follow from (5).

Consider the i-th equation of system (4). If

$$|x_i^{(n_i-1)}(\tau)| > \tfrac{1}{2}M_i(t) \qquad (\forall\,\tau \in [0,t]),$$

then from Lemma 1

$$\tfrac{1}{2}t^{n_i-1}M_i(t) \le r_{n_i-1}\alpha_i, \qquad \text{i.e. } M_i(t) \le 2\frac{r_{n_i-1}\alpha_i}{t^{n_i-1}} \tag{6}$$

follows. Let now be

$$\min_{\tau\in[0,t]} |x_i^{(n_i-1)}(\tau)| \le \tfrac{1}{2}M_i(t).$$

Then values $t_{i1}, t_{i2} \in [0,t]$ depending on t exist such that

$$\left.\begin{aligned} |x_i^{(n_i-1)}(t_{i1})| &= M_i(t) \\ |x_i^{(n_i-1)}(t_{i2})| &= \tfrac{1}{2}M_i(t) \\ |x_i^{(n_i-1)}(\tau)| &\in (\tfrac{1}{2}M_i(t), M_i(t)) \ \forall\,\tau \text{ between } t_{i1} \text{ and } t_{i2}. \end{aligned}\right\}$$

Integrating both parts of equation (4) from t_{i1} up to t_{i2}, we obtain

$$M_i(t) = 2\left|\int_{t_{i1}}^{t_{i2}} f_i\big(\tau,\dots,x_j^{(k)}(\tau),\dots\big)\,d\tau\right|. \tag{7}$$

Moreover, from Lemma 1

$$\tfrac{1}{2}|t_{i2}-t_{i1}|^{n_i-1}M_i(t) \le r_{n_i-1}\alpha_i, \qquad \text{i.e. } |t_{i2}-t_{i1}| \le \Big(2\frac{r_{n_i-1}\alpha_i}{M_i(t)}\Big)^{\frac{1}{n_i-1}} \tag{8}$$

follows.

In order to estimate the right-hand side of (7), denote

$$\Phi_i(\ldots, b_{jk}, \ldots; \delta, t) = \sup\left\{\left|\int_{t_1}^{t_1+h} f_i(\tau, \ldots, \varphi_{jk}(\tau), \ldots)\, d\tau\right| : \begin{array}{c} 0 \le t_1 \le t_1 + h \le t, h \le \delta \\ \varphi_{jk} \in C\big([0,t], [-b_{jk}, b_{jk}]\big) \end{array}\right\} \tag{9}$$

for $b_{jk} > 0$ $(j = 1, \ldots, m; k = 0, \ldots, n_i - 1)$. Then we obtain from (5) - (7) (with $s = n_j - 1$) and (8) that

$$M_i(t) \le 2 \max\left\{\frac{r_{n_i-1}\alpha_i}{t^{n_i-1}}, \Phi_i\left(\ldots, a_{n_j-1,k}\alpha_j^{\frac{n_j-1-k}{n_j-1}}\left[\max\left\{M_j(t), (n_j-1)!\,\frac{\alpha_j}{t^{n_j-1}}\right\}\right]^{\frac{k}{n_j-1}}, \ldots; \left[2\frac{r_{n_i-1}\alpha_i}{M_i(t)}\right]^{\frac{1}{n_i-1}}, t\right)\right\} \tag{10}$$

$(i = 1, \ldots, m)$. Here one must take $M_j(t)$ instead of the inner maximum if $k = n_j - 1$. If some $n_i = 1$, then the corresponding equation (10) is not considered and $M_i(t)$ is replaced by α_i in all other equations.

Thus we have obtained system (10) which contains m inequalities connecting m non-decreasing non-negative functions $t \mapsto M_i(t)$. According to that what has been said we obtain the following

Theorem 1. *If from the inequality system* (10) *the boundedness of all functions* M_i *for any* $\{\alpha_j\}$ *or any sufficiently small* $\{\alpha_j\}$ *follows, then by continuation of any a priori bounded solution or respectively any bounded with sufficiently small constants solution of system* (4)*, all derivatives indicated in the right-hand sides of system* (4) *remain bounded and the continuation is possible for arbitrary large values of* t.

3. Examples

3.1. Consider equation (1) with $m = 1$, i.e. the scalar case, where condition (2) holds. Then we obtain from (9)

$$\Phi(b_0, \ldots, b_{n-1}; \delta, t) \le A\,\delta\big(b_1^n + \ldots + b_{n-1}^{\frac{n}{n-1}}\big) + B\delta$$

for all $0 \le t < \infty$ and $|b_0| \le a$. We can assume that $t \ge t_0$ for some $t_0 > 0$ as it is possible to apply the existence theorem for the Cauchy problem on the interval $[0, t_0]$ for sufficiently small t_0. Then inequality (10) for $M(t) > 0$ takes the form

$$M(t) \le 2 \max\left\{\frac{r_{n-1}\alpha}{t_0^{n-1}}, \left[2\frac{r_{n-1}\alpha}{M(t)}\right]^{\frac{1}{n-1}} \times \left[B + A\sum_{k=1}^{n-1}\left(a_{n-1,k}\alpha^{\frac{n-1-k}{n-1}}\left[\max\left\{M(t), (n-1)!\,\frac{\alpha}{t_0^{n-1}}\right\}\right]^{\frac{k}{n-1}}\right)^{\frac{n}{k}}\right]\right\}$$

From here

$$M(t) \leq \max\left\{C_1\alpha, C_2\Big[\frac{\alpha}{M(t)}\Big]^{\frac{1}{n-1}} + C_3 \max\left\{\sum_{k=1}^{n-1} \alpha^{\frac{n-k}{k}} M(t), \alpha^{\frac{n^2-n-1}{k(n-1)}}\right\}\right\} \tag{11}$$

follows where the constants C_1, C_2, C_3 do not depend on α and $M(t)$. We see that the function $t \mapsto M(t)$ is bounded for all $t \geq t_0$ and sufficiently small α, and therefore Theorem 1 is applicable in the 2nd variant.

If condition (2) is repalced by (3), then the boundedness of $M(t)$ for any α follows from the fact that the value A and therefore C_3 in estimate (11) can be chosen arbitrarily small for sufficiently large $M(t)$. Therefore Theorem 1 is applicable in the 1st variant in this case.

3.2. Let be $m = 1$ and let the right-hand side of equation (1) admit the estimate

$$\big|f(t, x, x_1, \ldots, x_{n-1})\big| \leq \sum_{r=1}^{p} g_r(t)\, |x_1|^{\alpha_{r,1}} \cdots |x_{n-1}|^{\alpha_{r,n-1}}$$

$$\forall\, t \in [0, \infty),\ x \in [-\alpha, \alpha],\ x_1, \ldots, x_{n-1} \in \mathbb{R}$$

for some $\alpha_{i,j} \geq 0$ where $\int_{t_1}^{t_2} g_i(t)\, dt = o(|t_2 - t_1|^{\gamma_i})$ as $t_1, t_2 \in [0, \infty)$ with $0 < |t_2 - t_1| \to 0$ and $\gamma_i \in [0, 1]$ $(i = 1, \ldots, p)$. Then

$$\Phi(b_0, \ldots, b_{n-1}; \delta, t) = o\left(\sum_{r=1}^{p} \delta^{\gamma_r} b_1^{\alpha_{r,1}} \cdots b_{n-1}^{\alpha_{r,n-1}}\right) \qquad (\delta \to 0).$$

Hence we obtain, arguing as in Example 3.1, that if

$$\sum_{k=1}^{n-1} k\alpha_{r,k} - \gamma_r \leq n - 1 \qquad (r = 1, \ldots, p),$$

then the function $t \mapsto M(t)$ is bounded for all $t \geq 0$ and any $\alpha > 0$, i.e. Theorem 1 is applicable in its 1-st variant.

3.3. Consider the system of scalar equations with bounded functions g_i

$$\left.\begin{aligned} x_1''(t) &= g_1\big(t, x(t), x'(t)\big)\, |x_1'(t)|^{\beta_{11}} |x_2'(t)|^{\beta_{12}} \\ x_2''(t) &= g_2\big(t, x(t), x'(t)\big)\, |x_1'(t)|^{\beta_{21}} |x_2'(t)|^{\beta_{22}} \end{aligned}\right\} \qquad (0 \leq t < \infty) \tag{12}$$

for some $\beta_{ij} > 0$ $(1 \leq i, j \leq 2)$ and $x = (x_1, x_2)$. Here

$$\Phi_i(b_{11}, b_{21}; \delta, t) \leq G\, b_{11}^{\beta_{i1}} b_{21}^{\beta_{i2}} \delta \qquad (i = 1, 2;\ G > 0),$$

therefore the system of inequalities (10) has the form

$$M_i(t) \leq \max\left\{C_1\alpha_i, C_2 M_1(t)^{\beta_{i1}} M_2(t)^{\beta_{i2}} \alpha_i M_i(t)^{-1}\right\} \qquad (i = 1, 2) \tag{13}$$

for $t \geq t_0 > 0$ where $C_1 > 0$ and $C_2 > 0$ are certain constants. After taking the logarithm of both sides of (13) and denoting $y_i = \ln M_i(t)$ we obtain the inequality system

$$\left.\begin{aligned} y_1 &\leq \max\left\{ \ln(C_1\alpha_1), \ln(C_2\alpha_1) + (\beta_{11}-1)y_1 + \beta_{12}y_2 \right\} \\ y_2 &\leq \max\left\{ \ln(C_1\alpha_2), \ln(C_2\alpha_2) + \beta_{21}y_1 + (\beta_{22}-1)y_2 \right\} \end{aligned}\right\}. \tag{14}$$

Inequality $(14)_1$ defines an angle in the (y_1, y_2)-plane which is larger than π and bounded by two rays given as

$$\left.\begin{aligned} y_1 &= \ln(C_1\alpha_1) \\ y_2 &= \frac{2-\beta_{11}}{\beta_{12}}y_1 - \frac{\ln(C_2\alpha_1)}{\beta_{12}} \end{aligned}\right\}$$

where the first goes downwards and the second one to the right. Further, inequality $(14)_2$ defines an angle which is larger than π and bounded by two rays given as

$$\left.\begin{aligned} y_2 &= \ln(C_1\alpha_2) \\ y_1 &= \frac{2-\beta_{22}}{\beta_{21}}y_2 - \frac{\ln(C_2\alpha_2)}{\beta_{21}} \end{aligned}\right\}$$

where the first goes to the left and the second one upwards. The direct consideration of the intersection of these angles shows that the conditions

$$\beta_{11} < 2,\ \frac{\beta_{12}}{2-\beta_{11}} < \frac{2-\beta_{22}}{\beta_{21}} \qquad \text{or} \qquad \beta_{11} < 2,\ \frac{\beta_{12}}{2-\beta_{11}} \leq \frac{2-\beta_{22}}{\beta_{21}} \tag{15}$$

are necessary and sufficient for the boundedness from above both coordinates of its points for any, or respectively any sufficiently small values of α_1 and α_2.

Thus Theorem 1 is applicable to system (12) in the 1-st or 2-nd variant, if inequalities $(15)_1$ or equality in $(15)_2$ hold.

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