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STRUCTURE OF GEODESICS IN A 13-DIMENSIONAL GROUP OF HEISENBERG TYPE

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Abstract. A *g.o. space* is a homogeneous Riemannian manifold $(G/H, g)$ on which every geodesic is an orbit of a one-parameter subgroup of the group G. Each g.o. space can be characterized by the degree of certain rational map called "geodesic graph", which is linear in the naturally reductive case and non-linear otherwise. For a g.o. space which is not naturally reductive we try to find a geodesic graph of "minimal" degree.

Up to now only geodesic graphs of degree zero or two were observed. Here we study the generalized Heisenberg group (in the sense of A. Kaplan) of dimension 13 and with 5-dimensional center, which gives a new interesting example of a g.o. space for which the degree of the canonical geodesic graph is equal to six.

1. Introduction

Let (M, g) be a connected Riemannian manifold and let G be a connected group of isometries which acts transitively on M. Then M can be viewed as a homogeneous space $(G/H, g)$, where H is the isotropy subgroup of a fixed point p. On the Lie algebra $\mathfrak{g}$ of the group G there exists an $\mathrm{Ad}(H)$-invariant decomposition (reductive decomposition) $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$, where $\mathfrak{h}$ is the Lie algebra of the group H and $\mathfrak{m}$ is a vector space $\mathfrak{m} \subset \mathfrak{g}$. (Such a decomposition is not unique.)

Definition 1. A homogeneous space $(G/H, g)$ is called a *(Riemannian) g.o. space*, if each geodesic of $(G/H, g)$ (with respect to the Riemannian connection) is an orbit of a one-parameter subgroup $\{\exp(tZ)\}, Z \in \mathfrak{g}$ of the group of isometries G.

Definition 2. A homogeneous space $(G/H, g)$ is *naturally reductive* with respect to a reductive decomposition $\mathfrak{g} = \mathfrak{m}+\mathfrak{h}$ if, for any vector $X \in \mathfrak{m}\backslash\{0\}$, the curve $\gamma(t) = (\exp(tX))(p)$ is a geodesic (with respect to the Riemannian connection).

It is known that naturally reductive spaces form a proper subclass of the class of g.o. spaces. The difference can be described in terms of "geodesic vectors" and special maps called "geodesic graphs".

A geodesic graph (see the next section) is an $\mathrm{Ad}(H)$-equivariant map $\xi : \mathfrak{m} \mapsto \mathfrak{h}$ such that for any $X \in \mathfrak{m}\backslash\{0\}$ the curve $\exp t(X + \xi(X))(p)$ is a geodesic. This map is either linear (and the space is naturally reductive with respect to some reductive decomposition $\mathfrak{g} = \mathfrak{m}' + \mathfrak{h}$) or it is non-differentiable at the origin. Then the geodesic graph is a rational map and we are interested in the degree of this rational map.

We briefly recall some facts about the construction of the geodesis graphs and then we proceed to the H-type groups, which provide examples of g.o. spaces which are in no way naturally reductive (so each geodesic graph is non-linear).

2. Geodesic graph

Definition 3. Let $(G/H, g)$ be a Riemannian g.o. space. A vector $X \in \mathfrak{g}\backslash\{0\}$ is called a *geodesic vector* if the curve $\exp(tX)(p)$ is a geodesic.

If we consider a reductive decomposition $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ of a g.o. space, there is a natural $\mathrm{Ad}(H)$-invariant scalar product on the vector space $\mathfrak{m}$. It comes from the identification of $\mathfrak{m} \subset T_eG$ with the tangent space T_pM via the projection $\pi : G \mapsto M$.

We have the simple criterion for the property of the vector to be a geodesic vector:

Proposition 1 (cf. [3], Corollary 2.2)**.** A vector $Z \in \mathfrak{g}\backslash\{0\}$ is geodesic if and only if

$$\langle [Z, Y]_{\mathfrak{m}}, Z_{\mathfrak{m}} \rangle = 0 \quad \text{for all} \quad Y \in \mathfrak{m}. \tag{1}$$

Here the subscript $\mathfrak{m}$ indicates the projection into $\mathfrak{m}$.

Now we will head to the construction of the *canonical* geodesic graph, which is an $\mathrm{Ad}(H)$-equivariant map assigning to each vector $X \in \mathfrak{m}$ the vector $A \in \mathfrak{h}$, so that the vector $X + A$ is geodesic. Let $X \in \mathfrak{m}\backslash\{0\}$, denote

$$\mathfrak{q}_X = \{A \in \mathfrak{h} \mid [A, X] = 0\},$$

$$N_X = \{B \in \mathfrak{h} \mid [B, A] \in \mathfrak{q}_X \ \forall A \in \mathfrak{q}_X\}.$$

Obviously $\mathfrak{q}_X$ is a subalgebra of $\mathfrak{h}$ and $\mathfrak{q}_X \subset N_X$.

Let's briefly recall some facts from [1]:

• Let $X \in \mathfrak{m}\backslash\{0\}$. There is at least one element $A \in \mathfrak{h}$ such that $X + A$ is a geodesic vector and it holds $A \in N_X$.

• Let $Z \in \mathfrak{g}\backslash\{0\}$ be a geodesic vector and $A \in \mathfrak{h}$. Then the vector $Z + A$ is geodesic if and only if $A \in \mathfrak{q}_{Z_{\mathfrak{m}}}$.

Now choose an $\mathrm{Ad}(H)$-invariant scalar product on $\mathfrak{h}$ and let $N_X = \mathfrak{q}_X + \mathfrak{c}_X$ be the orthogonal decomposition with respect to this scalar product. Let $X \in \mathfrak{m}\backslash\{0\}$ and $A \in \mathfrak{h}$ so that $X + A$ is a geodesic vector. Denote A' the orthogonal projection of A into $\mathfrak{c}_X$. Then the vector $X + A'$ is also geodesic.

Definition 4. Let $(G/H, g)$ be a Riemannian g.o. space and $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ an $\mathrm{Ad}(H)$-invariant decomposition. For any fixed $X \in \mathfrak{m}\backslash\{0\}$ let $N_X = \mathfrak{q}_X + \mathfrak{c}_X$ be the decomposition as above and let $\pi_X : N_X \mapsto \mathfrak{c}_X$ be the orthogonal projection. If $A \in \mathfrak{h}$ is any vector such that $X + A$ is a geodesic vector, we define the canonical geodesic graph ξ as follows:

$$\begin{aligned} &\xi(X) = \pi_X(A) \quad \text{for each} \quad X \in \mathfrak{m}\backslash\{0\}, \\ &\xi(0) = 0. \end{aligned}$$

Remark. The canonical geodesic graph from Definition 4 is uniquely determined and $\mathrm{Ad}(H)$-equivariant.

A practical way of finding the geodesic graph of a g.o. space is solving the equation (1) with respect to some bases. For a given vector $X \in \mathfrak{m}\backslash\{0\}$ we put $X = \sum_{i=1}^{m} x_i E_i$ and $\xi(X) = \sum_{j=1}^{r} d_j F_j$, where $\{E_i\}_{i=1}^{m}$ is a basis of $\mathfrak{m}$ and $\{F_j\}_{j=1}^{r}$ is a basis of $\mathfrak{h}$. From the equation (1) we obtain a system of m (nonhomogeneous) linear equations for the parameters d_j, by puting $Z = X + \xi(X)$ and $Y = E_1, \dots, E_m$, respectively. These equations are compatible but not necessarily linearly independent.

The corresponding system of homogeneous linear equations characterizes the property $\xi(X) = \sum_{j=1}^{r} b_j F_j \in \mathfrak{q}_X$. Let us restrict ourselves to the open dense subset of $\mathfrak{m}\backslash\{0\}$ on which this system has the maximal rank, so the subalgebra $\mathfrak{q}_X$ has the minimal dimension. This minimal dimension is called *anihilating dimension* and denoted by q. A vector $X \in \mathfrak{m}\backslash\{0\}$ which fulfills the condition $\dim\mathfrak{q}_X = q$ is called a *generic vector*. The number q is independent on the decomposition $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$.

If X is generic, then the vector $\xi(X)$ must satisfy a system of exactly $r-q$ linearly independent nonhomogeneous linear equations and it must be also orthogonal to the subspace $\mathfrak{q}_X$ (of dimension q) described by corresponding homogeneous system. Hence we get additional q homogeneous linear equations. Altogether we obtain r independent equations for r independent parameters (r is the dimension of $\mathfrak{h}$).

Such a system of linear equations is solvable by using the Cramer's rule. We obtain *formally* a solution in the form $d_i = \tilde{P}_i/\tilde{P}$, where $\tilde{P}$ is polynomial of degree $\tilde{p} = (q+1)(r-q)$ and $\tilde{P}_i$ are polynomials of degree $\tilde{p}+1$. It may happen that the polynomials $\tilde{P}$ and $\tilde{P}_i$ have a nontrivial greatest common divizor. After cancelling out we obtain finally $d_i = P_i/P$, where the degree p of P is less then $\tilde{p}$. This "minimal" degree is called *the degree of the geodesic graph* ξ.

3. H-type groups

Let's briefly recall basic facts about generalized Heisenberg groups (H-type groups). These are very important, because the first examples of g.o. spaces which are in no way naturally reductive were found among H-type groups (see[4]).

Definition 5. Let $\mathfrak{n}$ be a 2-step nilpotent Lie algebra with an inner product $\langle , \rangle$. Let $\mathfrak{z}$ be the center of $\mathfrak{n}$ and let $\mathfrak{v}$ be it's orthogonal complement. For each vector $Z \in \mathfrak{z}$ define the operator $J_Z : \mathfrak{v} \mapsto \mathfrak{v}$ by the relation

$$\langle J_Z X, Y \rangle = \langle Z, [X, Y] \rangle \quad \text{for all} \quad X, Y \in \mathfrak{v}. \tag{2}$$

The algebra $\mathfrak{n}$ is called a *generalized Heisenberg algebra (H-type algebra)* if, for each $Z \in \mathfrak{z}$, the operator J_Z satisfies the identity

$$J_Z{}^2 = -\langle Z, Z \rangle \mathrm{id}_{\mathfrak{v}}. \tag{3}$$

A connected, simply connected Lie group whose Lie algebra is an H-type algebra is called an *H-type group.* It is endowed with a left-invariant metric.

The classification of H-type algebras is based on the classification of representations of real Clifford algebras, which is already known. Particularly, the H-type groups which are g.o. spaces but in no way naturally reductive are also classified. There are just the following cases:

(i) $\dim\mathfrak{z} = 2$, or
(ii) $\dim\mathfrak{z} = 3$ and $\mathfrak{v}$ is not an isotypic module, or
(iii) $\dim\mathfrak{z} = 5, 6, 7$ and $\dim\mathfrak{v} = 8$, or
(iv) $\dim\mathfrak{z} = 7,\ \dim\mathfrak{v} = 16, 24$ and $\mathfrak{v}$ is an isotypic module.

(A comprehensive information can be found in [2] but there the case (ii) was incorrectly described. The same error appeared in [1].)

The example of an H-type group with $\dim\mathfrak{z} = 2$ and $\dim\mathfrak{v} = 4$ was already treated in [4], [3] and [1] and the computation was made by hand. The general cases (i) and (ii) were treated by the present author (to be published elsewhere). Here the degree of a geodesic graph is always equal to 2. Now we shall continue with the next example (of dimension 13), where the computer aid is necessary.

4. H-type group of dimension 13

Let $\mathfrak{m}$ be a vector space of dimension 13 equipped with a scalar product and let $\mathfrak{m} = \mathrm{span}(E_i, Z_j), i = 1, \ldots, 8, j = 1, \ldots, 5$, where $\{E_i, Z_j\}_{i=1}^{8}{}_{j=1}^{5}$ form an orthonormal basis. We define the structure of a Lie algebra by the following relations:

$$
\begin{array}{llll}
[E_1,E_2]=-Z_1, & & & \\
[E_1,E_3]=-Z_2, & [E_2,E_3]=-Z_3, & & \\
[E_1,E_4]=Z_3, & [E_2,E_4]=-Z_2, & [E_3,E_4]=Z_1, & \\
[E_1,E_5]=Z_4, & [E_2,E_5]=Z_5, & [E_3,E_5]=0, & [E_4,E_5]=0,\\
[E_1,E_6]=-Z_5, & [E_2,E_6]=Z_4, & [E_3,E_6]=0, & [E_4,E_6]=0,\\
[E_1,E_7]=0, & [E_2,E_7]=0, & [E_3,E_7]=-Z_4, & [E_4,E_7]=-Z_5,\\
[E_1,E_8]=0, & [E_2,E_8]=0, & [E_3,E_8]=Z_5, & [E_4,E_8]=-Z_4,\\
[E_5,E_6]=Z_1, & & & \\
[E_5,E_7]=-Z_2, & [E_6,E_7]=-Z_3, & & [Z_i,Z_j]=0,\\
[E_5,E_8]=Z_3, & [E_6,E_8]=-Z_2, & [E_7,E_8]=-Z_1, & [E_i,Z_j]=0.
\end{array}
$$

It is obvious that $\mathfrak{z}=\operatorname{span}(Z_1,\dots,Z_5)$ is the center of the algebra $\mathfrak{m}$. One can easily verify, for $\mathfrak{v}=\operatorname{span}(E_1,\dots,E_8)$, that the operators J_Z defined by the equation (2) satisfy the condition (3), so the algebra $\mathfrak{m}$ is a generalized Heisenberg algebra.

Now, let us express the H-type group M corresponding to the Lie algebra $\mathfrak{m}$ as a homogeneous space G/H. On the Lie algebra level we have a decomposition $\mathfrak{g}=\mathfrak{h}+\mathfrak{m}$. Put $\mathfrak{h}=\operatorname{Der}(\mathfrak{m})\cap\mathfrak{so}(\mathfrak{m})$ - the algebra of skew-symetric derivations on $\mathfrak{m}$. Then $\mathfrak{g}=\mathfrak{h}+\mathfrak{m}$ is the Lie algebra of the maximal group of isometries $G=I_0(M)$ of the space M. We are going to make explicit computations here. The generators of $\mathfrak{h}$ can be derived by solving the equation

$$D[X,Y]=[DX,Y]+[X,DY],$$

$$\text{where } X,Y\in\mathfrak{m},\ D=\sum_{1\le i<j\le 8}a_{ij}A_{ij}+\sum_{1\le k<l\le 5}b_{kl}B_{kl}\in\mathfrak{h}.$$

A_{ij} is the basis of $\mathfrak{so}(\mathfrak{v})$ acting on $\mathfrak{v}$ by $A_{ij}(E_k)=\delta_{ik}E_j-\delta_{jk}E_i$ and, similarly, B_{ij} is the basis of $\mathfrak{so}(\mathfrak{z})$ acting on $\mathfrak{z}$ by $B_{ij}(Z_k)=\delta_{ik}Z_j-\delta_{jk}Z_i$. As a solution we obtain the following eleven operators:

$$
\begin{array}{ll}
D_1=2\,B_{12}-A_{14}+A_{23}+A_{58}-A_{67}, & D_7=2\,B_{25}+A_{18}-A_{27}+A_{36}-A_{45},\\
D_2=2\,B_{13}-A_{13}-A_{24}+A_{57}+A_{68}, & D_8=2\,B_{34}+A_{18}-A_{27}-A_{36}+A_{45},\\
D_3=2\,B_{14}+A_{16}-A_{25}+A_{38}-A_{47}, & D_9=2\,B_{35}+A_{17}+A_{28}-A_{35}-A_{46},\\
D_4=2\,B_{15}+A_{15}+A_{26}+A_{37}+A_{48}, & D_{10}=2\,B_{45}+A_{12}+A_{34}-A_{56}-A_{78},\\
D_5=2\,B_{23}+A_{12}-A_{34}+A_{56}-A_{78}, & D_{11}=A_{12}+A_{34}+A_{56}+A_{78}.\\
D_6=2\,B_{24}-A_{17}-A_{28}-A_{35}-A_{46}, &
\end{array}
$$

Thus $\mathfrak{h} = \mathrm{span}(D_1, \dots, D_{11})$ and we can write each $D \in \mathfrak{h}$ in the form $D = \sum_{i=1}^{11} d_i D_i$. Now the canonical geodesic graph $\xi : \mathfrak{m} \mapsto \mathfrak{h}$ satisfies the equation

$$\langle [X + D, Y]_m, X \rangle = 0,$$

where $X \in \mathfrak{m} \backslash \{0\}$ is given, Y runs over all $\mathfrak{m}$ and $D = \xi(X) \in \mathfrak{h}$ is to be determined. Here for $Y \in \mathfrak{m}$ we substitute, step by step, all 13 elements $\{E_i, Z_j\}_{i=1}^{8}{}_{j=1}^{5}$ of the given orthonormal basis. We obtain a system of 13 linear equations for the parameters $d_1, \dots, d_{11}$, but the rank of this system is 10. We select, in a convenient way, a subsystem of 10 linearly independent equations. The matrix $\mathbf{A}$ of the coefficients of the corresponding homogeneous system and the vector $\mathbf{b}$ of the right-hand sides are given by

$$\mathbf{A} = \begin{bmatrix}
-x_4 & -x_3 & x_6 & x_5 & x_2 & -x_7 & x_8 & x_8 & x_7 & x_2 & x_2 \\
x_3 & -x_4 & -x_5 & x_6 & -x_1 & -x_8 & -x_7 & -x_7 & x_8 & -x_1 & -x_1 \\
-x_2 & x_1 & x_8 & x_7 & -x_4 & -x_5 & x_6 & -x_6 & -x_5 & x_4 & x_4 \\
x_1 & x_2 & -x_7 & x_8 & x_3 & -x_6 & -x_5 & x_5 & -x_6 & -x_3 & -x_3 \\
x_8 & x_7 & x_2 & -x_1 & x_6 & x_3 & x_4 & -x_4 & x_3 & -x_6 & x_6 \\
-x_7 & x_8 & -x_1 & -x_2 & -x_5 & x_4 & -x_3 & x_3 & x_4 & x_5 & -x_5 \\
-2\,z_1 & 0 & 0 & 0 & 2\,z_3 & 2\,z_4 & 2\,z_5 & 0 & 0 & 0 & 0 \\
0 & -2\,z_1 & 0 & 0 & -2\,z_2 & 0 & 0 & 2\,z_4 & 2\,z_5 & 0 & 0 \\
0 & 0 & -2\,z_1 & 0 & 0 & -2\,z_2 & 0 & -2\,z_3 & 0 & 2\,z_5 & 0 \\
0 & 0 & 0 & -2\,z_1 & 0 & 0 & -2\,z_2 & 0 & -2\,z_3 & -2\,z_4 & 0
\end{bmatrix},$$

$$\mathbf{b} = \begin{bmatrix}
x_2 z_1 + x_3 z_2 - x_4 z_3 - x_5 z_4 + x_6 z_5 \\
-x_1 z_1 + x_3 z_3 + x_4 z_2 - x_5 z_5 - x_6 z_4 \\
-x_1 z_2 - x_2 z_3 - x_4 z_1 + x_7 z_4 - x_8 z_5 \\
x_1 z_3 - x_2 z_2 + x_3 z_1 + x_7 z_5 + x_8 z_4 \\
x_1 z_4 + x_2 z_5 - x_6 z_1 + x_7 z_2 - x_8 z_3 \\
-x_1 z_5 + x_2 z_4 + x_5 z_1 + x_7 z_3 + x_8 z_2 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}.$$

Next, there are just two cases to study, for which one obtains a g.o. space.

Case 1) $M = G'/H'$, where $H' = \mathrm{Spin}(5)$ and $G' = M \rtimes H'$.

We consider the 10-dimensional subalgebra $\mathfrak{h}' = \mathrm{span}(D_1, \dots, D_{10})$ and we put $\mathfrak{g}' = \mathfrak{m} + \mathfrak{h}'$. ($D_{11} \in \mathfrak{h}$ is an element of the center of $\mathfrak{h}$.) It is known from the theory (see [5]) that the corresponding homogeneous space G'/H' is a g.o. space. We get a restricted matrix equation $\mathbf{A}'\mathbf{d} = \mathbf{b}$, where $\mathbf{A}'$ is the matrix consisting of the first ten columns of the matrix $\mathbf{A}$. (The last, omitted column of the matrix $\mathbf{A}$ corresponds to the omitted operator D_{11}.) This equation has obviously the unique solution. Solving it by the Cramer's rule we obtain a vector $\mathbf{d}$ with 10 components, which are of the form $d_i = \tilde{P}_i/\tilde{P}, i = 1, \dots, 10$. Polynomials $\tilde{P}_i, i = 1, \dots, 10$ are of degree 11 and $\tilde{P}$ is of degree 10. They can be shown to have a common factor

$$\begin{aligned}\alpha_1 = 16\, z_1\, ((x_7x_5 + x_6x_8)\, z_2 + (-x_8x_5 + x_6x_7)\, z_3 + \\ + (x_7x_3 + x_4x_8)\, z_4 + (-x_8x_3 + x_4x_7)\, z_5)\end{aligned} \tag{4}$$

of degree 4, and hence we can simplify

$$d_i = \frac{\alpha_1 P_i}{\alpha_1 P} = \frac{P_i}{P},\ i = 1, \ldots, 10.$$

The polynomials P_i are of degree 7 and the polynomial P is of degree 6. Using the substitutions

$$\begin{aligned}K_1 &= {x_3}^2 + {x_4}^2 + {x_5}^2 + {x_6}^2 - {x_1}^2 - {x_2}^2 - {x_7}^2 - {x_8}^2,\\ K_2 &= 2\, x_1x_4 - 2\, x_2x_3 - 2\, x_7x_6 + 2\, x_8x_5,\\ K_3 &= 2\, x_1x_3 + 2\, x_2x_4 + 2\, x_7x_5 + 2\, x_8x_6,\\ K_4 &= -2\, x_1x_6 + 2\, x_2x_5 - 2\, x_7x_4 + 2\, x_8x_3,\\ K_5 &= -2\, x_1x_5 - 2\, x_2x_6 + 2\, x_7x_3 + 2\, x_8x_4,\\ I_1 = \|x\|^2 &= \sum_{j=1}^{8} {x_j}^2,\ I_2 = \|z\|^2 = \sum_{j=1}^{5} {z_j}^2\ I_3 = \sum_{j=1}^{5} z_jK_j.\end{aligned}$$

we can express the denominator in the form

$$P = -{I_1}^2 I_2 + {I_3}^2.$$

All the terms I_1, I_2 and I_3 are algebraic invariants with respect to the action of the group G' on M and so is P. Moreover, I_1, I_2, I_3 form a *basis* of all algebraic invariants. But none of P_i is an invariant and it can't be expressed in such a nice form. Also α_1 is not an invariant. Because of the excessive length of the terms P_i we prefer to write down only the terms which arise after the substitution

$$x_1 = x_3 = x_5 = x_6 = x_7 = z_2 = z_3 = z_5 = 0.$$

We obtain for example

$$\begin{aligned}\alpha_1 &= z_1z_4x_4x_8,\\ P_7 &= -4\, x_2{x_4}^2x_8z_1\left({z_1}^2 + {z_4}^2\right),\\ P &= -4\, {x_4}^2{z_1}^2\left({x_2}^2 + {x_8}^2\right) - {z_4}^2\left({x_2}^2 + {x_4}^2 + {x_8}^2\right)^2.\end{aligned}$$

In this way one easily verifies that P_7 and P have no longer a common factor. So the degree of the geodesic graph in the space G'/H' is equal to six. The full expressions for the polynomials $P, P_1, \ldots, P_{10}$ can be viewed on the Internet address http://www.karlin.mff.cuni.cz/~dusek/h13/h13.php.

Case 2) $M = G/H$, where $H = \mathrm{Spin}(5) \times \mathrm{So}(2)$ and $G = M \rtimes H$.

Here we consider the geodesic graph in the full isometry group $G = I_0(M)$. (Let us remark that the algebraic invariants with respect to G are the same as those with respect to G'.) In this case we need to find the "canonical" solution of the matrix equation $\mathbf{Ac} = \mathbf{b}$, where $\mathbf{c}$ denotes the vector $(c_1, \ldots, c_{11})^t$. For this purpose we have to express the additional property $\xi(X) \perp \mathfrak{q}_X$. This gives a new linear equation whose coefficients form an additional row $\mathbf{e} = (e_1, \ldots, e_{11})$ extending the matrix $\mathbf{A}$ (of type (10,11)) to a square matrix $\mathbf{A}^S$. The last row $\mathbf{e}$ will be fixed as the most natural solution of the homogeneous linear matrix equation $\mathbf{A}\mathbf{e} = 0$, i.e., e_i are chosen as the corresponding subdeterminants (up to the sign) of the matrix $\mathbf{A}$. But then we obtain $e_i = \alpha_1 T_i, i = 1, \ldots, 11$, where α_1 is the polynomial of degree 4 as in (4) and T_i are polynomials of degree 6. We put $\mathbf{T} = (T_1, \ldots, T_{11})$.

Now, solving the extended matrix equation $\mathbf{A}^S\mathbf{c} = \mathbf{b}$ by the Cramer's rule we obtain the eleven components of the vector $\mathbf{c}$, which are (after factorization) of the form

$$c_i = \frac{{\alpha_1}^2 \alpha_2 R_i}{{\alpha_1}^2 \alpha_2 R}\ , \qquad i = 1, \ldots, 11. \tag{5}$$

Here α_1 is the same polynomial of degree 4 as in (4), α_2 and R are polynomials of degree 6 and $R_i, i = 1, \ldots, 10$, are polynomials of degree 7. The degree of the denominator in the expressions for $c_i, i = 1, \ldots, 10$, is 20. (This is the formal upper bound $\tilde{p}$ mentioned in section **2**.) After canceling out the common factors we see that the geodesic graph is in fact of degree 6. Even more, we see that $R = -P, R_i = -P_i$ for $i = 1, \ldots, 10$ and $R_{11} = 0$. So the final solution $\mathbf{c} = (c_i)_{i=1}^{11}$ is "the same" as the solution $\mathbf{d} = (d_i)_{i=1}^{10}$ in the smaler group G'. It is also worth mentioning that the second factor α_2 and the square of the norm of the vector $\mathbf{T}$ are algebraic invariants and it holds $\alpha_2 = 2\left({I_1}^2 I_2 - {I_3}^2\right) = -2\,P$, $\|\mathbf{T}\|^2 = 2\,({I_1}^2 I_2 - {I_3}^2)^2$.

5. Conclusion

For the computation using the computer was essential. We used the software "Maple V", ©Waterloo Maple Inc., on the computer equipped with the processor Alpha EV6/500MHz and 1GB of memory. As an example - the computation of the factorization (5) allocated 170 MB of memory (used 55 GB) and took approximately 26 hours of time.

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