NONSINGULAR BILINEAR MAPS, SPACES OF MATRICES, IMMERSIONS AND EMBEDDINGS *

ZORAN PETROVIĆ [†]

Faculty of Mathematics, Studentski trg 16,
11000 Belgrade, Serbia

E-mail: zoranp@matf.bg.ac.yu

We present in this short paper relations between vector spaces of matrices which satisfy some rank conditions, problem of the existence of nonsingular bilinear maps and the problem of the existence of immersions and embeddings of real projective spaces into Euclidean spaces.

1. Introduction

In every differential topology textbook one finds Whitney's immersion and embedding theorems. But, the real story of immersions started with the results of Hirsch ([15]) and Smale ([30]). We do not intend to follow closely all these developments. Rather, we restrict ourselves mainly to the problem of the existence of immersions of real projective spaces into Euclidean spaces and the problems related to this one.

We show in the next section why the real projective spaces are especially interesting as we discuss immersions of manifolds and how one attempts to prove the existence of particular immersions. These methods turn out to be related to the question of the dimension of the vector spaces of matrices on which some condition on the rank is imposed and which we discuss in the section dedicated to the spaces of matrices. These problems are also interesting per se, since these are such basic objects and the questions we ask are so natural, that one would certainly like to know the answer to them.

Finally, we point out to some recent works which show how the question of nonsingular bilinear maps is related to special kinds of embeddings into Euclidean spaces.

2. Immersions

One of the long-standing problems of differential topology had certainly been the resolution of the so-called *immersion conjecture*: Every compact n-dimensional manifold M^n admits an immersion in the Euclidean space $\mathbb{R}^{2n-\alpha(n)}$, where $\alpha(n)$ stands for the number of 1's in the binary expansion of n. Certainly, the most natural question one can ask after hearing about this conjecture is: How this strange number $\alpha(n)$ comes into play? But, surprisingly, this number is rather natural for this problem. Namely, one can show, using only Stiefel–Whithey characteristic classes, that the following product of real projective spaces $\mathbb{R}P^{2^{t_1}} \times \mathbb{R}P^{2^{t_2}} \times \cdots \times \mathbb{R}P^{2^{t_k}}$ *does not* immerse into $\mathbb{R}^{2n-k-1}$, if $n = 2^{t_1} + \cdots + 2^{t_k}$ is the representation of n as the sum od different powers of 2. From this example we can conclude two things. First, one cannot do better than what the immersion conjecture claims. Second, the real projective spaces are rather important for the immersion problem.

Actually, one can say that the work on the immersion conjecture has started with the paper [22] where the author had shown, using the Steenrod squares, that $\bar{w}_i(M^n) = 0$ for $i > n - \alpha(n)$ for any n-dimensional compact manifold M^n. Of course, $\bar{w}_i(M^n)$ stand for the Stiefel-Whitney classes of the normal bundle of M^n. In the papers [6],[7] and [8], the authors first expanded this algebraic result by computing the ideal of all relations among the Stiefel-Whitney classes of normal bundles of all compact, n-dimensional manifolds and then proceeded to find the geometric realization of the algebraic structures which appeared in this investigation, namely to construct certain commutative diagram of spaces and maps realizing corresponding diagram od algebras. This program was finally completed by Ralph Cohen in 1985 (see [10]). Since this paper is rather long and involved and the result is so important, some topologists think that it would be nice to find another proof, perhaps using some other methods.

The following question still remains unresolved:

Q: What is the smallest k such that $\mathbb{R}P^n \looparrowright \mathbb{R}^{n+k}$?

We use $\looparrowright$ to denote an immersion.

One can approach this problem from two directions. First, one tries to

actually construct immersions and second, to apply various methods from algebraic topology in order to show that some immersions cannot exist. We will be mainly interested in the first approach. As far as proving the nonexistence of immersions, let us just mention that various methods have been used, mainly the obstruction theory, but also some methods specially suited for this problem.

Let us go back to the question of constructing immersions. Suppose $t \geq n$. An *axial map* $g : \mathbb{R}P^n \times \mathbb{R}P^n \longrightarrow \mathbb{R}P^t$ is a map g such that $g(x, *) = x = g(*, x)$ for $x \in \mathbb{R}P^n$, where $*$ stands for the base point and we regard real projective space $\mathbb{R}P^n$ as embedded into $\mathbb{R}P^t$. The existence of such a map is equivalent to the existence of an immersion of $\mathbb{R}P^n$ into $\mathbb{R}P^t$ (see [19],[29]). It is interesting to note that Hopf in 1940 (see [17]) used cohomology to prove that certain axial maps do not exist. Much later, some generalized cohomology theories were used to prove nonimmersion results (see [2], [9], [5]).

One can construct axial maps using *nonsingular bilinear maps.*

Definition 2.1 We define a bilinear map

$$f : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^{n+k+1}$$

to be nonsingular if $f(x, y) = 0$ occurs only when $x = 0$ or $y = 0$.

We see that this notion is a kind of generalization of the product without zero divisors. For example, one can take $k = 0$ only if $n = 0, 1, 3$ or 7 in which case we get the usual products for real numbers, complex numbers, quaternions and octonions. The relevance of this notion to our immersion problem stems from the following theorem (see e. g. [11]).

Theorem 2.1 *Existence of a bilinear map* $f : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^{n+k+1}$ *which is nonsingular, implies* $\mathbb{R}P^n \looparrowright \mathbb{R}^{n+k}$.

Actually, one can show that the existence of nonsingular bilinear map implies existence of a map which is homotopic to the corresponding axial map. But, the previous theorem can be proved without the help of axial maps. This is related to the spaces of matrices and will be dealt with in the following section.

Various methods were used to construct such maps. For example, we have the following general result:

$$\mathbb{R}P^n \looparrowright \mathbb{R}^{2n-\alpha(n)-k(n)},$$

where $k(n) = 0, 1$ or 4, depending on n (see [23], [20]). As one can see, this result is pretty close to the general result which holds for all manifolds. For concrete n one has better results, but we do not intend to discuss these results here.

Let us look at this problem from a slightly different perspective. Any bilinear map $f : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^{n+k+1}$ naturally induces a linear map $L : \mathbb{R}^{n+1} \otimes \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^{n+k+1}$. Since $\mathbb{R}^{n+1} \otimes \mathbb{R}^{n+1} \cong M(n+1, n+1, \mathbb{R})$ (we denote by $M(m, n, \mathbb{R})$ the space of all $m \times n$ matrices over $\mathbb{R}$), we can reformulate our question as follows:

Q: What is the highest dimension of the vector subspace of $M(n, n, \mathbb{R})$ which does not contain matrices of rank 1?

Namely, if V is any subspace not containing matrices of rank 1, the projection $L : \mathbb{R}^{n+1} \otimes \mathbb{R}^{n+1} \longrightarrow M(n+1, n+1, \mathbb{R})/V$ gives us a nonsingular bilinear map. This brings us to the discussion of spaces of matrices.

3. Spaces of matrices

Let us first introduce some notation. We denote by $L_{\mathbb{F}}(m, n; k)$ the maximal dimension of a subspace of $M(m, n, \mathbb{F})$, seen as a vector space over the real numbers, consisting, in addition to the zero matrix, only of matrices of rank at least equal to k ($\mathbb{F}$ is either $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, where $\mathbb{H}$ denotes the skew-field of quaternions). Similarly, $l_{\mathbb{F}}(m, n; k)$ stands for the maximal dimension of the subspace of the above mentioned space consisting only of matrices of rank equal to k (and the zero matrix of course).

If we look at the problem of finding subspaces consisting of matrices of rank at least equal to k we mention two results. Handel ([14]) showed that if $k = 2$, one can find such a subspace which has a basis consisting only of rank 2 matrices. Petrović ([25]) has generalized this result and has proved that it is true for all k.

How about matrices of fixed rank? Let us first look at matrices of high rank. In the paper [3] the authors have shown that the following is true:

$$l_{\mathbb{R}}(n, n; n) = \rho(n);$$
$$l_{\mathbb{C}}(n, n; n) = 2\nu_2(n) + 2;$$
$$l_{\mathbb{H}}(n, n; n) = \rho(\frac{n}{2}) + 4.$$

Before we proceed, let us explain this notation. The *Radon-Hurwitz* number

$\rho(n)$ (see [3], [27], [18]) is defined as follows:

$$\rho((2m+1)2^{4a+b}) = 8a + 2^b, a \geq 0, 0 \leq b \leq 3.$$

Of course, if n is odd, we take $\rho(n/2) = 0$. The notation $\nu_2(n)$ stands for the highest power of 2 dividing n. The paper [3] refers to the famous Adams paper [1]. In this paper Adams has proved that the maximum number of linearly independent vector fields on S^{n-1} is $\rho(n) - 1$. In this paper Adams has used the full machinery of multiplicative structure as well as operations in K-theory to prove this result. At the end of the correction to their paper [3](see [4]) the authors pose a question to relate their results to some topological phenomena.

In my thesis [25] and the paper [26], I have used only the additive structure of real K-theory as well as symplectic K-theory to prove these results. Since the Bott periodicity is inherent in the relation between real and symplectic K-theory, this shows that it is also responsible for this algebraic phenomena.

Various people have worked on the problem of determining the values of $l_{\mathbb{F}}(m, n; k)$ and $L_{\mathbb{F}}(m, n; k)$. Let us present a general method for dealing with this kind of problem. Suppose we have matrices $A_1, \ldots, A_r$ in $M(m, n, \mathbb{R})$. We define a bundle map

$$n\xi_{r-1} \xrightarrow{f} \varepsilon^m,$$

where by ξ_{r-1} we denote the canonical line bundle over $\mathbb{R}P^{r-1}$ and by ε^m the trivial m-bundle over the same space, as follows

$$f([x]; \lambda_1 x, \ldots, \lambda_n x) = ([x]; (x_1 A_1 + \cdots + x_r A_r) \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix}),$$

where $[x]$ is the class of $x = (x_1, \ldots, x_r) \in S^{r-1}$ in $\mathbb{R}P^{r-1}$. This construction was introduced for the first time in [31] for the complex case, and later used in [32], [24], [21], [25] for the complex and the real case (we concentrate in our presentation to the real case).

If our matrices $A_1, \ldots, A_r$ all belong to a vector subspace of matrices of constant rank k and are linearly independent, then $\text{Im}(f)$ becomes a vector bundle of dimension k we have the following splittings

$$\text{Im}(f) \oplus \nu^{m-k} = \varepsilon^m; \qquad \text{Im}(f) \oplus \eta^{m-k} = n\xi_{r-1}.$$

To extract further information from these splittings, Stiefel-Whitney classes and real K-theory have been used in the previously mentioned papers. Of

course, in order to find subspaces of matrices of constant rank, one does not use algebraic topology or algebraic geometry — one constructs such spaces directly. These methods can only show which subspaces may not exist.

Meshulam (see [24]) has shown that $l_{\mathbb{R}}(n, n; k) \leq n$. In [26] one may find that $l_{\mathbb{R}}(n, n; 2) = 2[n/2]$ if $n \neq 3$ while $l_{\mathbb{R}}(3, 3; 2) = 3$. Lam and Yiu have used topological K-theory to determine $l_{\mathbb{R}}(n, n; n - 1)$, $l_{\mathbb{R}}(n, n - 1; n - 2)$ and $l_{\mathbb{R}}(n, n; n - 2)$ (for details, see [21]).

Sylvester has shown that $L_{\mathbb{C}}(n, n; n - 1) = 4$ (see [31]), while in [28] one finds that $L_{\mathbb{R}}(n, n; n - 1) = 4$ if $n = 4k + 2$. It is shown in [26] that for all n one has $L_{\mathbb{R}}(n, n; n - 1) \leq 2^{\nu_2(n)} + 1$. For the results concerning $L(n, n; k)$ the authors have used some rudimentary algebraic geometry. While the previously mentioned inequality can certainly be improved, one cannot hope for the nice result for the real case as the one we have for the complex one. Namely, it is clear that $L_{\mathbb{R}}(4k+1, 4k+1; 4k) \geq l_{\mathbb{R}}(4k+1, 4k+1; 4k) = \rho(4k)$ (see [21]). Therefore, $L_{\mathbb{R}}(n, n; n - 1)$ can be arbitrarily large for n of the form $4k + 1$. The reader should consult papers mentioned in the references below in order to get further information.

Let us, at the end of this section, show how one can prove the existence of an immersion of $\mathbb{R}P^n$ into $\mathbb{R}^{n+k}$ from the existence of a nonsingular bilinear map $f : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^{n+k+1}$. We follow [25].

Any nonsingular bilinear map $f : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^{n+k+1}$ induces a linear map $L : \mathbb{R}^{n+1} \longrightarrow \mathrm{Hom}(\mathbb{R}^{n+1}, \mathbb{R}^{n+k+1})$ defined by $L(x)(y) := f(x, y)$. This map is mono and for any $x \neq 0$ $L(x)$ is also mono (these facts follow from the definition of a nonsingular bilinear map). We conclude from this the following inequality $l_{\mathbb{R}}(n + 1, n + k + 1; n + 1) \geq n + 1$. Using the previous results, one sees that there exist the following splitting

$$\varepsilon^{n+k} = (n + 1)\xi_n \oplus \nu^k.$$

Since $(n + 1)\xi_n \cong \tau(\mathbb{R}P^n) \oplus \varepsilon^1$. it follows that $\tau(\mathbb{R}P^n)$ has a stable normal k-bundle and so by a theorem of Hirsch (see [15]), $\mathbb{R}P^n$ can be immersed into $\mathbb{R}^{n+k}$.

4. Recent developments

Suppose that we would like to be more stringent and instead of looking for arbitrary nonsingular bilinear maps, we look for the *symmetric* ones. Of course, by a symmetric bilinear map f we understand a map f such that $f(x, y) = f(y, x)$. Actually, people have used these maps (see [16]) to construct *embeddings* of real projective spaces into Euclidean ones. But

some recent developments shows that they can be used to get some more interesting results.

In [12] and [13], the notion of *totally skew embeddings* have been introduced. Namely, if we have an embedding $M \hookrightarrow \mathbb{R}^N$, then this embedding is *totally skew* if for all $x \neq y$ in M and lines l in $T_x M$ and l' in $T_y M$ one has: l and l' are *skew* (as lines in $\mathbb{R}^N$). For example, the embedding of $\mathbb{R}$ into $\mathbb{R}^3$ given by the map $x \mapsto (x, x^2, x^3)$ is totally skew, while one has the result that there are no totally skew embeddings of S^1 into $\mathbb{R}^3$. In [12] one finds the following

Theorem 4.1 *If we denote by $N(M^n)$ the smallest dimension N of the Euclidean space for which there exists a totally skew embedding of M^n into $\mathbb{R}^N$, then $2n+1 \leq N(M^n) \leq 4n+1$. If M^n is closed, then $N(M^n) \geq 2n+2$.*

The reader may ask for the relevance of these results to our general theme concerning nonsingular bilinear maps. Here it is. In the same paper, the authors have proved that if $B : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^m$ is a *symmetric* nonsingular bilinear map, then one totally skew embedding $S^n \hookrightarrow \mathbb{R}^{n+1} \times \mathbb{R}^m$ is given by $x \mapsto (x, B(x,x))$.

So, it is interesting to ask what do we know about the existence of symmetric nonsingular bilinear maps. Let us give a simple example. We know that if we multiply two polynomials with real coefficients we get a zero polynomial only if one of the factors is zero. But this multiplication gives a symmetric bilinear map (look at the coefficients of these polynomials) and this result shows that this map is nonsingular. So, we have a symmetric nonsingular bilinear map $m : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^{2n+1}$. If we use polynomials with complex coefficients and see $\mathbb{C}$ as $\mathbb{R}^2$ we can show that there exist a symmetric nonsingular bilinear map $m : \mathbb{R}^{2k} \times \mathbb{R}^{2k} \longrightarrow \mathbb{R}^{4k-2}$. This example looks rather simple. The reader might wonder why we discuss such a simple example at all. Well, for the very simple reason: it is not known whether this result can be improved — it is not known whether there exist a symmetric nonsingular bilinear map $g : \mathbb{R}^{2k} \times \mathbb{R}^{2k} \longrightarrow \mathbb{R}^{4k-3}$. So, here is certainly something to think about.

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