ON HOLOMORPHICALLY PROJECTIVE FLAT PARABOLICALLY-KÄHLERIAN SPACES *

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We consider holomorphically projective mappings of parabolically-Kählerian spaces and define holomorphically projective flat parabolically-Kählerian spaces. We found the tensor characteristic of these spaces and obtained their metric tensors.

1. Introduction

Many authors studied holomorphically projective mappings of Kählerian spaces and their generalizations [1, 17]. Some facts from the theory of holomorphically projective mappings of parabolically-Kählerian spaces $K_n^{o(m)}$ were published in [2, 9]−[15].

A (pseudo-) Riemannian space $K_n^{o(m)}$ is said to be *parabolically-Kählerian space* if together with a metric tensor $g_{ij}(x)$ it possesses an affinor structure $F_i^h(x)$ of rank $m \geq 2$ satisfying the following relations

$$a)\quad F_\alpha^h F_i^\alpha = 0, \qquad b)\quad g_{i\alpha}F_j^\alpha + g_{j\alpha}F_i^\alpha = 0, \qquad c)\quad F_{i,j}^h = 0, \tag{1}$$

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where the comma denotes the covariant derivation.

2. Holomorphically projective mappings of parabolically-Kählerian spaces

The following criteria from the papers [10, 13] hold for holomorphically projective mappings from a parabolically-Kählerian space $K_n^{o(m)}$ onto a parabolically-Kählerian space $\bar{K}_n^{o(m)}$.

An *analytically planar curve* of the parabolically-Kählerian space $K_n^{o(m)}$ is a curve defined by the equations $x^h = x^h(t)$ which tangent vector $\lambda^h = dx^h/dt$, being translated, remains in the area element formed by the tangent vector λ^h and its conjugate $\lambda^\alpha F_\alpha^h$, i.e., the conditions

$$\frac{d\lambda^h}{dt} + \Gamma_{\alpha\beta}^h \lambda^\alpha \lambda^\beta = \rho_1(t)\lambda^h + \rho_2(t)\lambda^\alpha F_\alpha^h,$$

are fulfilled. Here Γ_{ij}^h is the Christoffel symbol and ρ_1, ρ_2 are functions of the argument t.

The diffeomorphism f of $K_n^{o(m)}$ onto $\bar{K}_n^{o(m)}$ is a *holomorphically projective mapping*, if it transform all analytically planar curves of $K_n^{o(m)}$ into analytically planar curves of $\bar{K}_n^{o(m)}$.

Consider a concrete mapping f: $K_n^{o(m)} \longrightarrow \bar{K}_n^{o(m)}$, both spaces being referred to the general coordinate system x with respect to this mapping. This is a coordinate system where two corresponding points $M \in K_n^{o(m)}$ and $f(M) \in \bar{K}_n^{o(m)}$ have equal coordinates $x = (x^1, x^2, \dots, x^n)$; the corresponding geometric objects in $\bar{K}_n^{o(m)}$ will be marked with a bar. For example, Γ_{ij}^h and $\bar{\Gamma}_{ij}^h$ are components of the Christoffel symbols on $K_n^{o(m)}$ and $\bar{K}_n^{o(m)}$, respectively.

Structures of $K_n^{o(m)}$ and $\bar{K}_n^{o(m)}$ are preserved under f, i.e. $\bar{F}_i^h(x) = F_i^h(x)$. Among others, the structure F_i^h is covariantly constant, and $\bar{g}_{i\alpha}F_j^\alpha + \bar{g}_{j\alpha}F_i^\alpha = 0$ holds.

It is proved in [10, 13] that a parabolically-Kählerian space $K_n^{o(m)}$ admits a *holomorphically projective mapping* f onto a parabolically-Kählerian space $\bar{K}_n^{o(m)}$ if and only if the following conditions (in the common coordinate system x) hold:

$$\bar{\Gamma}_{ij}^h(x) = \Gamma_{ij}^h(x) + \psi_i\delta_j^h + \psi_j\delta_i^h + \varphi_i F_j^h + \varphi_j F_i^h, \tag{2}$$

where φ_i is a covector, $\psi_i = \varphi_\alpha F_i^\alpha$, and $\psi_i(x)$ is a gradient, i.e. there is a function $\psi(x)$, such that $\psi_i(x) = \partial\psi(x)/\partial x^i$.

If $\varphi_i \not\equiv 0$ then a holomorphically projective mapping is called *nontrivial*; otherwise it is said to be *trivial* or *affine*.

Condition (2) is equivalent to

$$\bar{g}_{ij,k} = 2\psi_k \bar{g}_{ij} + \psi_i \bar{g}_{jk} + \psi_j \bar{g}_{ik} + \varphi_i \bar{g}_{j\alpha} F_k^\alpha + \varphi_j \bar{g}_{i\alpha} F_k^\alpha. \tag{3}$$

Under a holomorphically projective mapping f: $K_n^{o(m)} \longrightarrow \bar{K}_n^{o(m)}$, the following conditions hold:

$$\bar{R}^h_{ijk} = R^h_{ijk} + \psi_{ij}\delta^h_k - \psi_{ik}\delta^h_j + \varphi_{ij}F^h_k - \varphi_{ik}F^h_j - (\varphi_{jk} - \varphi_{kj})F^h_i, \tag{4}$$

where R^h_{ijk} and $\bar{R}^h_{ijk}$ are Riemannian tensors of $K_n^{o(m)}$ and $\bar{K}_n^{o(m)}$,

$$\varphi_{ij} = \varphi_{i,j} - \psi_i\varphi_j - \varphi_i\psi_j, \quad \psi_{ij} = \varphi_{\alpha j}F^\alpha_i \quad (= \psi_{ji} = \psi_{i,j} - \psi_i\psi_j). \tag{5}$$

3. Holomorphically projective flat parabolically-Kählerian space

A parabolically-Kählerian space $K_n^{o(m)}$ is said to be *holomorphically projective flat*, if it admits a holomorphically projective mapping onto a flat space, i.e. the space with the vanishing Riemannian tensor.

We have the following theorem.

Theorem 3.1 *The parabolically-Kählerian space $K_n^{o(m)}$ is holomorphically projective flat if and only if the following conditions are true for the Riemannian tensor*

$$R_{hijk} = c\,(2\,F_{hi}F_{jk} + F_{hj}F_{ik} - F_{hk}F_{ij}) \tag{6}$$

where c = const, $F_{ij} = g_{i\alpha}F^\alpha_j$.

Proof. Let a parabolically-Kählerian space $K_n^{o(m)}$ admit a holomorphically projective mapping onto a flat space $\bar{V}_n$ ($\bar{R}^h_{ijk} = 0$), which should be a parabolically-Kählerian space $\bar{K}_n^{o(m)}$ same.

If $\bar{R}^h_{ijk} = 0$ then after omitting the index h (4) takes the form

$$R_{hijk} = -\psi_{ij}g_{kh} + \psi_{ik}g_{jh} - \varphi_{ij}F_{hk} + \varphi_{ik}F_{hj} + (\varphi_{jk} - \varphi_{kj})F_{hi}. \tag{7}$$

Let us symmetrize (7) at indices h and i. Then, using the properties of the Riemannian tensor we get:

$$0 = -\psi_{ij}g_{kh} + \psi_{ik}g_{jh} - \varphi_{ij}F_{hk} + \varphi_{ik}F_{hj} - \psi_{hj}g_{ki} + \psi_{hk}g_{ji} - \varphi_{hj}F_{ik} + \varphi_{hk}F_{ij}.$$

Analyzing of this formula, we obtain $\psi_{ij} = 0$ and

$$\varphi_{ij} = c\,F_{ij}, \tag{8}$$

where c is a certain function. Thus (8) takes the form (6).

On the basis (5), formula (8) takes the form

$$\varphi_{i,j} = \psi_i\varphi_j + \varphi_i\psi_j + c\,F_{ij}. \tag{9}$$

The condition of integrability takes the form: $c_{,k}F_{ij} - c_{,j}F_{ik} = 0$. From foregoing one it is implied, that $c_{,i} = 0$ and $c = \text{const}$.

So, we have shown that the Riemannian tensor at all holomorphically projective flat parabolically-Kählerian spaces $K_n^{o(m)}$ satisfies (6).

It is easy to check that any parabolically-Kählerian space $K_n^{o(m)}$, in which the Riemannian tensor satisfies (6), admits holomorphically projective mapping onto a flat space $\bar{K}_n^{o(m)}$.

Make sure that the system of equations (3) and (9) is completely integrable in this $K_n^{o(m)}$ and has the solution $\bar{g}_{ij}(x)$, $\varphi_i(x)$ for any initial conditions

$$\bar{g}_{ij}(x_o) = \overset{o}{\bar{g}}_{ij} \quad \text{and} \quad \varphi_i(x_o) = \overset{o}{\varphi}_i \tag{10}$$

for which $\det\| \overset{o}{\bar{g}}_{ij} \| \not\equiv 0, \quad \overset{o}{\bar{g}}_{ij} = \overset{o}{\bar{g}}_{ji} \quad \text{and} \quad \overset{o}{\bar{g}}_{i\alpha}F_j^\alpha(x_o) + \overset{o}{\bar{g}}_{j\alpha}F_i^\alpha(x_o) = 0.$

Consequently, the space $K_n^{o(m)}$ admits a holomorphically projective mapping onto a space $\bar{K}_n^{o(m)}$ with the metric tensor $\bar{g}_{ij}(x)$ and the structure $F_i^h(x)$. Using (4) we can see, that $\bar{R}^h_{ijk} = 0$, hence $\bar{K}_n^{o(m)}$ is a flat space. This completes the proof.

The direct analysis of (6) leads us to the following

Lemma 3.1 *A holomorphically projective flat parabolically-Kählerian space $K_n^{o(m)}$ is a Ricci flat symmetric space, i.e. a Ricci tensor is vanishing and the Riemannian tensor is covariantly constant in this $K_n^{o(m)}$.*

4. On isometries between holomorphically projective flat parabolically-Kählerian spaces

We denote $K_n^{o(m,c)}$ a holomorphically projective flat parabolically-Kählerian space, which determined by (6), and prove the following theorem.

Theorem 4.1 *Two holomorphically projective flat parabolically-Kählerian spaces $K_n^{o(m,c)}$ and $\bar{K}_n^{o(\bar{m},\bar{c})}$ are locally isometric if and only if $\bar{m} = m$, the metric signatures are coincident, and the constants c and $\bar{c}$ have the same sign.*

Proof. Let us consider the given spaces $K_n^{o(m,c)}$ and $\bar{K}_n^{o(\bar{m},\bar{c})}$ which are related to the coordinate systems x and $\bar{x}$ respectively. It is natural to consider the case, when the constants c and $\bar{c}$ are not equal to zero.

We will search an isometric mapping $f\colon\ K_n^{o(m,c)} \longrightarrow \bar{K}_n^{o(\bar{m},\bar{c})}$. As it is known, the mapping $f\colon\ \bar{x}^h = \bar{x}^h(x^1, x^2, \ldots, x^n)$ is an isometric mapping if and only if

$$g_{ij}(x) = \bar{g}_{\alpha\beta}(\bar{x}(x))\partial_i\bar{x}^\alpha\partial_j\bar{x}^\beta. \tag{11}$$

Denote $\bar{x}_i^h \equiv \partial_i\bar{x}^h$. From (11) it follows that

$$\partial_i\bar{x}^h = \bar{x}_i^h, \quad \partial_j\bar{x}_i^h = \bar{\Gamma}^h_{\alpha\beta}\bar{x}_i^\alpha\bar{x}_j^\beta - \Gamma^\alpha_{ij}\bar{x}^h_\alpha, \tag{12}$$

where Γ^h_{ij} and $\bar{\Gamma}^h_{ij}$ are the Christoffel symbols of $K_n^{o(m,c)}$ and $\bar{K}_n^{o(\bar{m},\bar{c})}$.

The system (12) for the unknown functions $\bar{x}^h(x)$, $\bar{x}_i^h(x)$ has a solution for initial conditions $\bar{x}^h(x_o) = \bar{x}_o^h$ and $\bar{x}_i^h(x_o) = y_i^h$, where the following properties are satisfied

$$\bar{g}_{\alpha\beta}(\bar{x}_o)y_i^\alpha y_j^\beta = g_{ij}(x_o), \quad F_i^\alpha(x_o)\, y_\alpha^h = \sqrt{\bar{c}/c}\, \bar{F}_\alpha^h(\bar{x}_o)\, y_i^\alpha, \tag{13}$$

where F_i^h and $\bar{F}_i^h$ are the structures of $K_n^{o(m,c)}$ and $\bar{K}_n^{o(m,c)}$, respectively.

Initial conditions y_i^h from (13) exist if only if $\bar{m} = m$, the signatures of the metric g and $\bar{g}$ are coincident, and the constants c and $\bar{c}$ have the same sign. Conditions (13) follow from (11) and from an integrability condition of system (12): $R_{hijk} = \bar{R}_{\alpha\beta\gamma\delta}\bar{x}_h^\alpha\bar{x}_i^\beta\bar{x}_j^\gamma\bar{x}_k^\delta$.

5. Holomorphically projective mappings of holomorphically projective flat parabolically-Kählerian spaces

We can prove the next theorem in the similar way as Theorem 3.1.

Theorem 5.1 *If the holomorphically projective flat parabolically-Kählerian space $K_n^{o(m,c)}$ admits a holomorphically projective mapping onto some parabolically-Kählerian space $\bar{K}_n^{o(m)}$, then $\bar{K}_n^{o(m)}$ is a holomorphically projective flat parabolically-Kählerian space $\bar{K}_n^{o(m,\bar{c})}$ too.*

In addition the next theorem holds

Theorem 5.2 *Any holomorphically projective flat parabolically-Kählerian space $K_n^{o(m,c)}$ admits a nontrivial holomorphically projective mapping onto some holomorphically projective flat parabolically-Kählerian space $\bar{K}_n^{o(m,\bar{c})}$ with a given constant $\bar{c}$ and a given signature of the metric $\bar{g}_{ij}$.*

Proof. The availability of this theorem follows from the existence of the solutions $\bar{g}_{ij}(x)$ and $\varphi_i(x)$ of equations (3) and

$$\varphi_{i,j} = \psi_i\varphi_j + \varphi_i\psi_j + c\,F_{ij} - \bar{c}\,\bar{F}_{ij},$$

where $\bar{F}_{ij} = \bar{g}_{i\alpha}F^{\alpha}_{j}$, for any initial conditions (10) for which det $\| \overset{o}{\bar{g}}_{ij} \| \neq 0$, $\overset{o}{\bar{g}}_{ij} = \overset{o}{\bar{g}}_{ji}$ and $\overset{o}{\bar{g}}_{i\alpha}F^{\alpha}_{j}(x_o) + \overset{o}{\bar{g}}_{j\alpha}F^{\alpha}_{i}(x_o) = 0$, in the space $K_n^{o(m,c)}$.

Theorem 5.3 *Between any holomorphically projective flat parabolically-Kählerian spaces it is possible to establish a nontrivial holomorphically projective mapping.*

Proof. Let us have two arbitrary holomorphically projective flat parabolically-Kählerian spaces $K_n^{o(m,c)}$ and $\bar{K}_n^{o(m,\bar{c})}$. By Theorem 5.2, there exists some space $\tilde{K}_n^{o(m,c)}$ with a signature of a metric of $\bar{K}_n^{o(m,\bar{c})}$, on which $K_n^{o(m,c)}$ admits nontrivial holomorphically projective mapping. By Theorem 4.1, the spaces $\bar{K}_n^{o(m,\bar{c})}$ and $\tilde{K}_n^{o(m,c)}$ are isometric, which prove the theorem.

6. Metric of holomorphically projective flat parabolically-Kählerian spaces

In a symmetric space its a metric tensor may be rebuilt in some Riemannian coordinate system $(y^1, y^2, \ldots y^n)$ at a point x_o by the known formulas [5]

$$g_{ij} = \overset{o}{g}_{ij} + \frac{1}{2}\sum_{k=1}^{\infty}\frac{(-1)^k\,2^{2k+2}}{(2k+2)!}m_i^{\sigma_1}m_{\sigma_1}^{\sigma_2}\cdots m_{\sigma_{k-1}j}, \tag{14}$$

where $m_{ij} = \overset{o}{R}_{i\alpha j\beta}y^\alpha y^\beta$, $m^i_j = m_{i\alpha}\,\overset{o}{g}^{\alpha i}$ and $\overset{o}{g}_{ij}$, $\overset{o}{g}^{ij}$ and $\overset{o}{R}_{hijk}$ are the components of the metric, its inverse and Riemannian tensors at the point x_o.

Taking into account the representation of Riemannian tensor (6) and properties of structures F^h_i the formulas (14) take the form:

$$g_{ij} = \overset{o}{g}_{ij} - c\,F_iF_j, \tag{15}$$

where $F_i = \overset{o}{F}_{i\alpha}y^\alpha$, $\overset{o}{F}_{ij}$ are the components of tensor F_{ij} at x_o.

Note, that for a given point x_o of holomorphically projective flat parabolically-Kählerian space $K_n^{o(m,c)}$ the metric and structure tensors may be simultaneously reduced to the form:

$$\overset{o}{g}_{ij} = \begin{pmatrix} 0 & 0 & b_{ab} \\ 0 & \overset{*}{e} & 0 \\ b^T_{ab} & 0 & 0 \end{pmatrix} \quad \text{and} \quad \overset{o}{F}{}^h_i = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ E_m & 0 & 0 \end{pmatrix},$$

where b^T_{ab} is a transposed matrix b_{ab}, $a, b = \overline{1, m}$, E_m is the identity matrix,

$$b_{ab} = \begin{pmatrix} 0 & 1 & & & & & & 0 \\ -1 & 0 & & & & & & \\ & & 0 & 1 & & & & \\ & & -1 & 0 & & & & \\ & & & & \ddots & & & \\ & & & & & & 0 & 1 \\ 0 & & & & & & -1 & 0 \end{pmatrix} \quad \text{and} \quad \overset{*}{e} = \begin{pmatrix} e_1 & & & 0 \\ & e_2 & & \\ & & \ddots & \\ 0 & & & e_{n-2m} \end{pmatrix}, e_a = \pm 1.$$

Thus, we proved the following theorem.

Theorem 6.1 *In the holomorphically projective flat parabolically-Kählerian space* $K_n^{o(m,c)}$ *there exists a coordinate system* y *in which the metric tensor has the form (15).*

Not neglecting generality of reasons, on basic of theorem 4.1 we can consider $c = 0, \pm 1$ that is, spaces $K_n^{o(m,0)}$, $K_n^{o(m,+1)}$ and $K_n^{o(m,-1)}$.

References

1. D. V. Beklemishev, *Differential geometry of spaces with an almost complex structure*, (Russian), In: Itogi Nauki, Geometria, 1963, All-Union Institute for Scietific and Technical Information, Moscow, (1965), 165-212.
2. J. Mikeš, *Holomorphically projective mappings and their generalizations*, J. Math. Sci., New York, **89**, **3** (1998), 1334-1353.
3. J. Mikeš and N. S. Sinyukov, *On quasi planar mappings of affine-connected spaces*, Sov. Math. **27**, **1** (1983), 63-70.
4. T. Otsuki and Y. Tashiro, *On curves in Kaehlerian spaces*, Math. J. Okayama Univ. **4** (1954), 57-78.
5. A. Z. Petrov, *New method in general relativity theory*, Nauka, Moscow, 1966.
6. A. Z. Petrov, *Simulation of physical fields*, In: Gravitation and the Theory of Relativity, Vol. **4-5**, Kazan' State Univ., Kazan, (1968), 7-21.
7. M. Prvanovic, *Holomorphically projective transformations in a locally product space*, Math. Balk. **1** (1971), 195-213.
8. M. Prvanovic, *A note on holomorphically projective transformations of the Kähler spaces*, Tensor, New Ser. **35** (1981), 99-104.
9. Zh. Radulovich, *Holomorphically-projective mappings of parabolically-Kählerian spaces*, Math. Montisnigri, Vol. **VIII** (1997), 159-184.
10. M. Shiha, *On the theory of holomorphically-projective mappings parabolically-Kählerian spaces*, Diff. Geometry and Its Appl. Proc. Conf. Opava. Silesian Univ., Opava, (1993), 157-160.
11. M. Shiha, *Geodesic and holomorphically projective mappings of parabolically-Kählerian spaces*, (Russian), PhD. Thesis, Moscow, (1994).

12. M. Shiha, *Geodesic and holomorphically projective mappings of parabolically-Kählerian spaces*, (Russian), Abstract of PhD. Thesis, Moscow, (1994).
13. M. Shiha and J. Mikeš, *The holomorphically projective mappings of parabolically-Kählerian spaces*, (Russian), Dep. in UkrNIINTI, Kiev, No 1128-Uk91, 19p., (1991).
14. M. Shiha and J. Mikeš, *On equidistant parabolically-Kählerian spaces*, (Russian), Trudy Geom. Sem., **22** (1994), 97-107.
15. M. Shiha and J. Mikeš, *On parabolically Sasakian and equidistant parabolically-Kählerian spaces*, (Russian), Dvizh. v obobshch. prostranstvach. Inter. Sci. Sb. Nauchn. Trudov, Penza (Russia), (1999), 190-198.
16. V. V. Vishnevsky, A. P. Shirokov, V. V. Shurigin, *Spaces over Algebras*, Kazan Univ. Press, Kazan, 1985.
17. K. Yano, *Differential Geometry on Complex and Almost Complex Spaces*, Pergamon Press, Oxford, 1965.