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Conformal Geometry of Surfaces in S^4 and Quaternions

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Quaternionic methods recently proved to be a powerful tool in conformal surface theory. A first account of this was presented in the ICM 98 article [5], and the theory was developed further by the authors of [1] in an informal seminar at Technische Universität Berlin since then [1]. I presented this research to the Summer School on Differential Geometry at Coimbra 1999 in a series of lectures that gave an elementary introduction to the new concepts, but also included several recent results with complete proofs. The present article contains an extended abstract of my lectures, while the details will appear in [1].

Contents

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1. Linear Algebra over the Quaternions

We consider the skew-field $\mathbb{H}$ of the Hamiltonian quaternions in the usual representation

$$a = a_0 + a_1 i + a_2 j + a_3 k, \quad a_l \in \mathbb{R}, \tag{1.1}$$

with imaginary units i, j, k. We define

$$\bar{a} := a_0 - a_1 i - a_2 j - a_3 k,$$
$$\operatorname{Re} a := a_0, \quad \operatorname{Im} a := a_1 i + a_2 j + a_3 k,$$
$$< a, b >_{\mathbb{R}} = \operatorname{Re}(\bar{a}b) = \operatorname{Re}(a\bar{b}) = \frac{1}{2}(\bar{a}b + \bar{b}a).$$

We shall identify the real vector space $\mathbb{H}$ in the obvious way with $\mathbb{R}^4$, and the subspace of purely imaginary quaternions with $\mathbb{R}^3$:

$$\mathbb{R}^3 = \operatorname{Im} \mathbb{H}.$$

The quaternion multiplication incorporates both the usual vector and scalar products on $\mathbb{R}^3$. For $a, b \in \operatorname{Im}\mathbb{H} = \mathbb{R}^3$ we have

$$ab = a \times b - < a, b >_{\mathbb{R}} . \tag{1.2}$$

In particular, the usual two-sphere is given by

$$S^2 \subset \mathbb{R}^3 = \operatorname{Im}\mathbb{H} = \{a \in \mathbb{H} \,|\, a^2 = -1\}.$$

We choose quaternion vector spaces to be *right* vector spaces, i.e. vectors are multiplied by quaternions from the right. We shall often have an *additional* complex structure on V, acting from the left, and hence commuting with the quaternionic structure. In other words, we consider a fixed $J \in \operatorname{End}(V)$ such that $J^2 = -I$. In this case we call (V, J) a *complex quaternionic (bi-)vector space.* If (V, J) and (W, J) are such spaces, then the quaternionic linear maps from V to W split as a direct sum of the real vector spaces of complex linear and anti-linear homomorphisms:

$$\operatorname{Hom}(V, W) = \operatorname{Hom}_+(V, W) \oplus \operatorname{Hom}_-(V, W).$$

Lemma 1. (Fundamental lemma) *If $N, R \in \mathbb{H}$ with $N^2 = -1 = R^2$, then $\sigma(x) := NxR$ defines a self-adjoint endomorphism of $\mathbb{H} = \mathbb{R}^4$, and its ± 1-eigenspaces $U, U^\perp$ are real 2-planes. Conversely, each real 2-plane in $\mathbb{H}$ determines a pair (N, R) of imaginary units unique up to sign. An orientation of U can be used to fix that sign.*

If $U \subset \operatorname{Im}\mathbb{H}$, then $N = R$, and this is perpendicular to U. We, therefore, call N and R the *left* and *right normal* of U though, in general, they are *not at all* orthogonal to U.
Conformal maps of Euclidean spaces are maps with differential preserving the metric up to a factor. In the case of maps from $\mathbb{R}^2 = \mathbb{C}$ into itself, this can be characterized in terms of the complex structure $J : z \mapsto iz$ alone, not referring to the metric. In fact, f is conformal, if and only if it satisfies the Cauchy-Riemann equation

$$*df := df \circ J = \pm J \circ df,$$

where the sign depends on the orientation behaviour of df. We generalize this to maps into the quaternions in the following fundamental

Definition. *Let M be a Riemann surface, i.e. a 2-dimensional manifold endowed with a complex structure $J : TM \to TM, J^2 = -I$. A map $f : M \to \mathbb{H} = \mathbb{R}^4$ is called conformal, if there exist $N, R : M \to \mathbb{H}$ such that, with $*df := df \circ J$,*

$$N^2 = -1 = R^2 \tag{1.3}$$

$$*df = Ndf = -dfR. \tag{1.4}$$

If f is an immersion then (1.3) follows from (1.4), and N and R are unique, called the left and right normal vector of f.

Equation (1.4) is a Cauchy-Riemann equation with a "variable complex structure" on the target space side. In this sense conformal maps into $\mathbb{H}$ are a generalization of complex holomorphic maps.
For an immersion f the existence of $N : M \to \mathbb{H}$ such that $*df = Ndf$ already implies that the immersion $f : M \to \mathbb{H}$ is conformal. Similarly for R.
If $f : M \to \operatorname{Im}\mathbb{H} = \mathbb{R}^3$ is an immersion then $N = R$ is "the classical" unit normal vector of f. But for general $f : M \to \mathbb{H}$, the vectors N and R are *not* orthogonal to $df(TM)$.

2. Projective Spaces

In complex function theory the Riemann sphere $\mathbb{C}P^1$ is more convenient as a target space for holomorphic functions than the complex plane. Similarly, the natural target space for conformal immersions is $\mathbb{H}P^1$, rather than $\mathbb{H}$. We therefore give a description of the quaternionic projective space.
The quaternionic projective space $\mathbb{H}P^n$ is defined, similar to its real and complex cousins, as the set of quaternionic lines in $\mathbb{H}^{n+1}$. We have the (continuous) canonical projection

$$\pi : \mathbb{H}^{n+1}\backslash\{0\} \to \mathbb{H}P^n, x \mapsto \pi(x) = [x] = x\mathbb{H}.$$

The manifold structure of $\mathbb{H}P^n$ is defined as follows: For any linear form $\beta \neq 0$ on $\mathbb{H}^{n+1}$

$$u : \pi(x) \mapsto x < \beta, x >^{-1}$$

is well-defined and maps the open set $\{\pi(x) \mid < \beta, x > \neq 0\}$ onto the affine hyperplane $\beta = 1$, which is isomorphic to $\mathbb{H}^n$. Coordinates of this type are called *affine coordinates* for $\mathbb{H}P^n$. They define a (real-analytic) atlas for $\mathbb{H}P^n$.
The set

$$\{\pi(x) \mid < \beta, x >= 0\}$$

is called the *hyperplane at infinity.*

Example. In the special case $n = 1$, the hyperplane at infinity is a single point: $\mathbb{H}P^1$ is the one-point compactification of $\mathbb{R}^4$, hence "the" 4-sphere:

$$\mathbb{H}P^1 = S^4.$$

The projective line is a perfect model for the conformal structure of S^4. On the other hand, for instance *the antipodal map* is natural on the usual 4-sphere, but not on $\mathbb{H}P^1$ – unless we introduce additional structure, like a metric.

The tangent space of $\mathbb{H}P^n$ admits a well-defined isomorphism

$$T_l\mathbb{H}P^n \cong \mathrm{Hom}(l, \mathbb{H}^{n+1}/l) \tag{2.1}$$

using the following

Proposition 1. *Let* $\tilde{f} : M \to \mathbb{H}^{n+1}\backslash\{0\}$ *and* $f = \pi\tilde{f} : M \to \mathbb{H}P^n$. *Let* $p \in M, l := f(p), v \in T_pM$. *Then*

$$d_pf : T_pM \to T_{f(p)}\mathbb{H}P^n = \mathrm{Hom}(f(p), \mathbb{H}^{n+1}/f(p))$$

is given by $d_pf(v)(\tilde{f}(p)\lambda) = \pi_l(d_p\tilde{f}(v)\lambda)$. *We denote the differential in this interpretation by* δf*:*

$$\delta f(v)(\tilde{f}) = d\tilde{f}(v) \mod f. \tag{2.2}$$

Given a non-degenerate quaternionic hermitian inner product $< .,. >$ on $\mathbb{H}^{n+1}$, we define a (possibly degenerate Pseudo-) Riemannian metric on $\mathbb{H}P^n$ as follows: For $x \in \mathbb{H}^{n+1}$ with $< x, x > \neq 0$ and $v, w \in (x\mathbb{H})^\perp$, we put

$$< d_x\pi(v), d_x\pi(w) >= \frac{1}{< x, x >} \mathrm{Re} < v, w > .$$

Example 1. For $< v, w >= \sum \bar{v}_k w_k$ we obtain the standard Riemannian metric on $\mathbb{H}P^n$. (In the complex case, this is the so-called Fubini-Study metric.) The corresponding conformal structure is in the background of all of the following considerations. Comparison with the stereographic projection shows that on $\mathbb{H}P^1$ it induces a metric of constant curvature 4.

Example 2. If we consider an *indefinite* hermitian metric on $\mathbb{H}^{n+1}$, then the above construction of a metric on $\mathbb{H}P^n$ fails for isotropic lines ($< l, l >= 0$), but these points are scarce. We consider the case $n = 1$, and the hermitian inner product $< v, w >= \bar{v}_1 w_2 + \bar{v}_2 w_1$. Isotropic lines are characterized in affine coordinates $h : x \mapsto \begin{pmatrix} x \\ 1 \end{pmatrix}$ by $\bar{x} + x = 0$, i.e. by $x \in \mathrm{Im}\,\mathbb{H} = \mathbb{R}^3$. Therefore, the set of isotropic points is a 3-sphere $S^3 \subset S^4$, and its complement consists of two open discs or – in affine coordinates – two open half-spaces. The induced metric is – up to a constant factor – the standard hyperbolic metric on these half-spaces.

Let

$$\mathcal{Z} = \{S \in \mathrm{End}(\mathbb{H}^2) \mid S^2 = -I\}$$

and, for $S \in \mathcal{Z}$, define $S' := \{l \in \mathbb{H}P^1 \mid Sl = l\}$. Using affine coordinates with $\infty \in S'$ and Lemma 1 one shows that

$$\mathcal{Z} \text{ is the set of oriented 2-spheres in } S^4 = \mathbb{H}P^1.$$

3. Vector Bundles

A quaternionic vector bundle $\pi : V \to M$ of rank n over a smooth manifold M is a real vector bundle of rank $4n$ together with a smooth fibre-preserving action of $\mathbb{H}$ on V from the right such that the fibres become quaternionic vector spaces.

Example 3. The points of the projective space $\mathbb{H}P^n$ are the 1-dimensional subspaces of $\mathbb{H}^{n+1}$. The *tautological bundle*

$$\pi : \Sigma \to \mathbb{H}P^n$$

is the line bundle with $\Sigma_l = l$. More precisely

$$\Sigma := \{(l, v) \in \mathbb{H}P^n \times \mathbb{H}^{n+1} \mid v \in l\}, \quad \pi(l, v) = l.$$

We shall be concerned with maps $f : M \to \mathbb{H}P^n$ from a surface into the projective space. To f we associate the bundle $L := f^*\Sigma$, whose fibre over x is $f(x) \subset \mathbb{H}^{n+1} = \{x\} \times \mathbb{H}^{n+1}$. The bundle L is a line subbundle of the product bundle

$$H := M \times \mathbb{H}^{n+1}.$$

Conversely, every line subbundle L of H over M determines a map $f : M \to \mathbb{H}P^n$ by $f(x) := L_x$. We obtain an identification

Maps $f : M \to \mathbb{H}P^n$	$\leftrightarrow$	Line subbundles $L \subset H = M \times \mathbb{H}^{n+1}$

All natural constructions for quaternionic vector spaces extend, fibre-wise, to operations in the category of quaternionic vector bundles. For example, a subbundle L of a vector bundle H induces a quotient bundle H/L with fibres H_x/L_x. However, given two quaternionic vector bundles V_1, V_2, the homomorphism bundle $\text{Hom}(V_1, V_2)$ is merely a *real* vector bundle.

Example 4. Over $\mathbb{H}P^n$ we have the product bundle $H = \mathbb{H}P^n \times \mathbb{H}^{n+1}$ and, inside it, the tautological subbundle Σ. Then

$$T\mathbb{H}P^n \cong \text{Hom}(\Sigma, H/\Sigma),$$

see (2.1).

Example 5 (and Definition). Let L be a line subbundle of $H = M \times \mathbb{H}^{n+1}$ corresponding to $f : M \to \mathbb{H}P^n$. Then, by Proposition 1,

$$f^*T\mathbb{H}P^n \cong \text{Hom}(L, H/L),$$

and the differential of $f = L$ should be viewed as a 1-form on M with values in $\text{Hom}(L, H/L)$:

$$\delta \in \Omega^1(\text{Hom}(L, H/L)). \tag{3.1}$$

We now endow quaternionic vector bundles with additional structure. A *complex quaternionic vector bundle* is a pair (V, J) consisting of a quaternionic vector bundle V and a section $J \in \Gamma(\mathrm{End}(V))$ with

$$J^2 = -I,$$

see Section 1.

Example 6. Given $f : M \to \mathbb{H}, *df = Ndf$, the quaternionic line bundle $L = M \times \mathbb{H}$ has a complex structure given by

$$Jv := Nv.$$

With the use of Lemma 1 we obtain

Proposition 2. *Let $L \subset H = M \times \mathbb{H}^2$ be an immersed oriented surface in $\mathbb{H}P^1$ with derivative $\delta \in \Omega^1(\mathrm{Hom}(L, H/L))$. Then there exist unique complex structures on L and H/L, denoted by $J, \tilde{J}$, such that for all $x \in M$*

$$\tilde{J}\delta(T_xM) = \delta(T_xM) = \delta(T_xM)J,$$
$$\tilde{J}\delta = \delta J,$$

and J is compatible with the orientation induced by $\delta : T_xM \to \delta(T_xM)$.

Definition. *A line subbundle $L \subset H = M \times \mathbb{H}^{n+1}$ over a Riemann surface M is called conformal or a holomorphic curve in $\mathbb{H}P^n$, if there exists a complex structure J on L such that*

$$*\delta = \delta J.$$

From the proposition we see: If L is an *immersed* holomorphic curve in $\mathbb{H}P^1$, i.e. if δ is in addition injective, such that $\delta(TM) \subset \mathrm{Hom}(L, H/L)$ is a real subbundle of rank 2, then there is also a complex structure $\tilde{J} \in \Gamma(\mathrm{End}(H/L))$ such that

$$*\delta = \delta J = \tilde{J}\delta. \tag{3.2}$$

A Riemann surface immersed into $\mathbb{H}P^1$ is a holomorphic curve if and only if the complex structures given by the proposition are compatible with the complex structure given on M in the sense of (3.2).

Example 7. Let $f : M \to \mathbb{H}$ be a conformally immersed Riemann surface with right normal vector R, and let L be the line bundle corresponding to

$$\begin{bmatrix} f \\ 1 \end{bmatrix} : M \to \mathbb{H}P^1.$$

Then $J\begin{pmatrix} f \\ 1 \end{pmatrix} = -\begin{pmatrix} f \\ 1 \end{pmatrix}R$ defines a complex structure with

$$\delta J = *\delta.$$

Hence (L, J) is a holomorphic curve. Conversely, if (L, J) is a holomorphic curve, then $J\begin{pmatrix} f \\ 1 \end{pmatrix} = -\begin{pmatrix} f \\ 1 \end{pmatrix}R$ for some $R : M \to \mathbb{H}$, and f is conformal with right normal vector R.

Let (V, J) be a complex quaternionic vector bundle over the Riemann surface M. We decompose

$$\mathrm{Hom}_{\mathbb{R}}(TM, V) = KV \oplus \bar{K}V,$$

where

$$KV := \{\omega : TM \to V \mid *\,\omega = J\omega\},$$
$$\bar{K}V := \{\omega : TM \to V \mid *\,\omega = -J\omega\}.$$

Definition. *A holomorphic structure on (V, J) is a quaternionic linear map*

$$D : \Gamma(V) \to \Gamma(\bar{K}V)$$

such that for all $\psi \in \Gamma(V)$ and $\lambda : M \to \mathbb{H}$

$$D(\psi\lambda) = (D\psi)\lambda + \frac{1}{2}(\psi d\lambda + J\psi * d\lambda). \tag{3.3}$$

A section $\psi \in \Gamma(V)$ is called holomorphic if $D\psi = 0$, and we put

$$H^0(V) = \ker D \subset \Gamma(V).$$

Remark 1. For a better understanding of this, note that for complex-valued λ the anti-$\mathbb{C}$-linear part (the $\bar{K}$-part) of $d\lambda$ is given by $\bar{\partial}\lambda = \frac{1}{2}(d\lambda + i * d\lambda)$. In fact,

$$(d\lambda + i * d\lambda)(JX) = *d\lambda(X) - i\,d\lambda(X) = -i(d\lambda + i * d\lambda)(X).$$

A holomorphic structure is a generalized $\bar{\partial}$-operator. Equation (3.3) is the only natural way to make sense of a product rule of the form $D(\psi\lambda) = (D\psi)\lambda + \psi\bar{\partial}\lambda$.

Example 8. Any given $J \in \mathrm{End}(\mathbb{H}^n)$, $J^2 = -1$, turns $H = M \times \mathbb{H}^n$ into a complex quaternionic vector bundle. Then $\Gamma(H) = \{\psi : M \to \mathbb{H}^n\}$, and

$$D\psi := \frac{1}{2}(d\psi + J * d\psi)$$

is a holomorphic structure.

Example 9. If L is a complex quaternionic line bundle and $\phi \in \Gamma(L)$ has no zeros, then there exists exactly one holomorphic structure D on (L, J) such that ϕ becomes holomorphic. In fact, any $\psi \in \Gamma(L)$ can be written as $\psi = \phi\mu$ with $\mu : M \to \mathbb{H}$, and our only chance is

$$D\psi := \frac{1}{2}(\phi d\mu + J\phi * d\mu). \tag{3.4}$$

This, indeed, satisfies the definition of a holomorphic structure.

4. The Mean Curvature Sphere

Let M be a Riemann surface. Let $H := M \times \mathbb{H}^2$ denote the product bundle over M, and let $S : M \to \mathrm{End}(\mathbb{H}^2) \in \Gamma(\mathrm{End}(H))$ with $S^2 = -I$ be a complex structure on H. We split the differential according to type:

$$d\psi = d'\psi + d''\psi,$$

where d' and d'' denote the $\mathbb{C}$-linear and anti-linear components, respectively. In general $d(S\psi) \neq Sd\psi$, and we decompose further:

$$d' = \partial + A, \quad d'' = \bar{\partial} + Q,$$

where

$$\partial(S\psi) = S\partial\psi, \quad \bar{\partial}(S\psi) = S\bar{\partial}\psi,$$
$$AS = -SA, \quad QS = -SQ.$$

Then $\bar{\partial}$ defines a holomorphic structure and ∂ an anti-holomorphic structure on H, while A and Q are tensorial:

$$A \in \Gamma(K\,\mathrm{End}_-(H)), \quad Q \in \Gamma(\bar{K}\,\mathrm{End}_-(H)). \tag{4.1}$$

We find

$$SdS = 2(Q + A), \tag{4.2}$$

$$Q = \frac{1}{4}(SdS - *dS), \quad A = \frac{1}{4}(SdS + *dS). \tag{4.3}$$

Remark 2. Since A and Q are of different type, $dS = 0$ if and only if $A = 0$ and $Q = 0$. If $dS = 0$, then the $\pm i$-eigenspaces of the complex endomorphism S decompose $H = (M \times \mathbb{C}) \oplus (M \times \mathbb{C})$. Therefore A and Q measure the deviation from the "complex case".

We now consider an immersed holomorphic curve $L \subset H$ in $\mathbb{H}P^1$ with derivative $\delta = \delta_L \in \Omega^1(\mathrm{Hom}(L, H/L))$. Then there exist complex structures J on L and $\tilde{J}$ on H/L such that

$$*\delta = \delta J = \tilde{J}\delta.$$

We want to extend J and $\tilde{J}$ to a complex structure of H, i.e. to find an

$$S \in \Gamma(\mathrm{End}(H))$$

such that

$$SL = L, \quad S|_L = J, \quad \pi S = \tilde{J}\pi.$$

Note that this implies $\pi dS(\psi) = \pi(d(S\psi) - Sd\psi) = \delta J\psi - \tilde{J}\delta\psi = 0$, and therefore

$$dSL \subset L. \tag{4.4}$$

The existence of S is clear: Write $H = L \oplus L'$ for some complementary bundle L'. Identify L' with H/L using π, and define $S|_L := J$, $S|_{L'} := \tilde{J}$. Since L' is not unique, S is not unique. It is a very important feature of the 4-dimensional case that we can add a natural condition to single out a unique extension:

Theorem 1. *Let $L \subset H = M \times \mathbb{H}^2$ be a holomorphic curve immersed into $\mathbb{H}P^1$. Then there exists a unique complex structure S on H such that*

$$SL = L, \quad dSL \subset L, \tag{4.5}$$
$$*\delta = \delta \circ S = S \circ \delta, \tag{4.6}$$
$$Q|_L = 0. \tag{4.7}$$

S is a family of 2-spheres, a *sphere congruence* in classical terms. Because $S_p L_p = L_p$ the sphere S_p goes through $L_p \in \mathbb{H}P^1$, while $dSL \subset L$ (or, equivalently, $\delta S = S\delta$) implies it is tangent to L in p, see Example 5. In an affine coordinate system $\begin{bmatrix} f \\ 1 \end{bmatrix} = L$ the sphere S_p has the same mean curvature vector as $f : M \to \mathbb{R}^4 = \mathbb{H}$ at p. This motivates the

Definition. *S is called the mean curvature sphere (congruence) of L. The differential forms $A, Q \in \Omega^1(\mathrm{End}(H))$ are called the Hopf fields of L.*

Remark 3. Equations (4.5), (4.6) imply $d\psi + S * d\psi \in \Gamma(L)$ for $\psi \in \Gamma(L)$, whence $d'' = \bar{\partial} + Q = \frac{1}{2}(d + S * d)$ leaves L invariant. Hence an immersed holomorphic curve in $\mathbb{H}P^1$ is a holomorphic subbundle of (H, S, d'') and, in particular, is a holomorphic quaternionic vector bundle itself.

We now collect some information about the Hopf fields and the mean curvature sphere congruence $S : M \to \mathcal{Z}$.

Lemma 2.

$$d(A + Q) = 2(Q \wedge Q + A \wedge A).$$

Lemma 3. *Let $L \subset H$ be an immersed surface and S a complex structure on H stabilizing L such that $dSL \subset L$. Then $Q_{|L} = 0$ is equivalent to $AH \subset L$.*

Definition. *For a quaternionic vector space or bundle V of rank n and an endomorphism $A \in \mathrm{End}(V)$ we define*

$$< A > := \frac{1}{4n} \operatorname{trace}_{\mathbb{R}} A,$$

where the trace is taken of the real endomorphism A. In particular $< I > = 1$. We obtain an indefinite scalar product $< A, B > := < AB >$.

Proposition 3. *The mean curvature sphere S of an immersed Riemann surface L satisfies*

$$< dS, dS > = < *dS, *dS >, \quad < dS, *dS > = 0,$$

i.e. $S : M \to \mathcal{Z}$ is conformal.

Because of this proposition, S is also called *the conformal Gauss map*, see Bryant [2].

5. Willmore Surfaces

Throughout this section M denotes a *compact* surface.
The set

$$\mathcal{Z} = \{S \in \mathrm{End}(\mathbb{H}^2) \,|\, S^2 = -I\}$$

of oriented 2-spheres in $\mathbb{H}P^1$ is a submanifold of $\mathrm{End}(\mathbb{H}^2)$ with

$$T_S\mathcal{Z} = \{X \in \mathrm{End}(\mathbb{H}^2) \,|\, XS = -SX\}, \quad \perp_S \mathcal{Z} = \{Y \in \mathrm{End}(\mathbb{H}^2) \,|\, YS = SY\}.$$

Here we use the (indefinite) inner product $< A, B > := < AB > = \frac{1}{8}\,\mathrm{trace}_{\mathbb{R}}(AB)$.

Definition. *The energy functional of a map $S : M \to \mathcal{Z}$ of a Riemann surface M is defined by*

$$E(S) := \int_M < dS \wedge *dS > .$$

Critical points S of this functional with respect to variations of S are called harmonic maps from M to $\mathcal{Z}$.

Proposition 4. *S is harmonic if and only if the $\mathcal{Z}$-tangential component of $d * dS$ vanishes:*

$$(d * dS)^T = 0. \tag{5.1}$$

This condition is equivalent to any of the following:

$$d(S * dS) = 0, \tag{5.2}$$

$$d * A = 0, \tag{5.3}$$

$$d * Q = 0. \tag{5.4}$$

We now consider the case where S is the mean curvature sphere of an immersed holomorphic curve. We decompose dS into the Hopf fields.

Lemma 4.

$$< dS \wedge *dS > = 4(< A \wedge *A > + < Q \wedge *Q >), \tag{5.5}$$

$$< dS \wedge SdS > = 4(< A \wedge *A > - < Q \wedge *Q >). \tag{5.6}$$

Proposition 5. *The integral*

$$\deg S := \frac{1}{\pi} \int_M < A \wedge *A > - < Q \wedge *Q >$$

is a topological invariant of S.

Therefore,

$$E(S) = 4\int_M < A \wedge *A > + < Q \wedge *Q > = 8\int_M < A \wedge *A > -4\pi \deg S,$$

and for variational problems the energy functional can be replaced by the integral of $< A \wedge *A >$.

Definition. *Let L be a compact immersed holomorphic curve in $\mathbb{H}P^1$ with Hopf field A. The Willmore functional of L is defined as*

$$W(L) := \frac{1}{\pi}\int_M < A \wedge *A > .$$

If we vary the immersion $L : M \to \mathbb{H}P^1$, it will in general not remain a holomorphic curve. On the other hand, any immersion induces a complex structure J on M such that with respect to this it is a holomorphic curve, see Proposition 2. Critical points of W with respect to such variations are called Willmore surfaces.

Computation of the first variation of W yields

Theorem 2. (Ejiri [3], Rigoli [8]) *An immersed holomorphic curve L is Willmore if and only if its mean curvature sphere S is harmonic.*

6. Metric and Affine Conformal Geometry

We consider the metric extrinsic geometry of $f : M \to \mathbb{R}^4$ in relation to the quantities associated to

$$L := \begin{bmatrix} f \\ 1 \end{bmatrix} : M \to \mathbb{H}P^1.$$

For brevity we write $< ., . >$ instead of $< ., . >_{\mathbb{R}}$.
Let N, R denote the left and right normal vector of $f : M \to \mathbb{H}$, i.e.

$$*df = Ndf = -df R.$$

Then

Proposition 6. (i) *The second fundamental form $II(X,Y) = (X \cdot df(Y))^{\perp}$ of f is given by*

$$II(X,Y) = \frac{1}{2}(*df(Y)dR(X) - dN(X) * df(Y)). \tag{6.1}$$

(ii) *The mean curvature vector $\mathcal{H} = \frac{1}{2}\operatorname{trace} II$, the Gaussian curvature K and the normal curvature $K^{\perp}$ are given by*

$$\bar{\mathcal{H}} df = \frac{1}{2}(*dR + RdR), \quad df\bar{\mathcal{H}} = -\frac{1}{2}(*dN + NdN) \tag{6.2}$$

$$K|df|^2 = \frac{1}{2}(< *dR, RdR > + < *dN, NdN >) \tag{6.3}$$

$$K^{\perp}|df|^2 = \frac{1}{2}(< *dR, RdR > - < *dN, NdN >) \tag{6.4}$$

(iii) *We obtain*

$$(|\mathcal{H}|^2 - K - K^\perp)|df|^2 = \frac{1}{4}| * dR - RdR|^2$$

In particular, if $f : M \to \operatorname{Im}\mathbb{H} = \mathbb{R}^3$ *then* $K^\perp = 0$, *and the classical Willmore integrand is given by*

$$(|\mathcal{H}|^2 - K)|df|^2 = \frac{1}{4}| * dR - RdR|^2. \tag{6.5}$$

The Hopf fields are given by

Proposition 7.

$$4 * Q = G \begin{pmatrix} dN + N * dN & 0 \\ -2dH + w & 0 \end{pmatrix} G^{-1}, \qquad 4 * A = G \begin{pmatrix} 0 & 0 \\ w & dR + R * dR \end{pmatrix} G^{-1},$$

where $G = \begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix}$, *and* $w = dH + H * df H + R * dH - H * dN$.

Proposition 8. *The Willmore integrand is given by*

$$< A \wedge *A > = \frac{1}{16}|RdR - *dR|^2 = \frac{1}{4}(|\mathcal{H}|^2 - K - K^\perp)|df|^2.$$

For $f : M \to \mathbb{R}^3$, *this is the classical integrand* $\frac{1}{4}(|\mathcal{H}|^2 - K)|df|^2$.

We finally express the Euler-Lagrange equation $d * A = 0$ for Willmore surfaces in affine coordinates.

Proposition 9.

$$d * A = \frac{1}{4} G \begin{pmatrix} 0 & 0 \\ dw & 0 \end{pmatrix} G^{-1} = \begin{pmatrix} -f dw & -f dw f \\ dw & dw f \end{pmatrix}.$$

with $w = dH + R * dH + \frac{1}{2}H(NdN - *dN)$. *Therefore,* f *is Willmore if and only if* $dw = 0$.

Example 10. (Willmore Cylinder) Let $\gamma : \mathbb{R} \to \operatorname{Im}\mathbb{H}$ be a unit-speed curve, and $f : \mathbb{R}^2 \to \mathbb{H}$ the cylinder defined by

$$f(s,t) = \gamma(s) + t$$

with the conformal structure $J\frac{\partial}{\partial s} = \frac{\partial}{\partial t}$. Then using Proposition 9, we obtain, after some computation, that f is (non-compact) Willmore if and only if

$$\frac{1}{2}\kappa^3 + \kappa'' - \kappa\tau^2 = 0, \quad (\kappa^2\tau)' = 0.$$

This is exactly the condition that γ be a free elastic curve.

7. Twistor Projections

Let $E \subset H := M \times \mathbb{H}^2 = M \times \mathbb{C}^4$ be a *complex* (not a quaternionic) line subbundle over a Riemann surface M with complex structure J_E induced from right multiplication by i on $\mathbb{H}^2$. We define $\delta_E \in \Omega^1(\text{Hom}(E, H/E))$ by $\delta_E \phi := \pi_E d\phi$, and call E a *holomorphic curve* in $\mathbb{C}P^3$, if

$$*\delta_E = \delta_E J_E.$$

This is equivalent to the fact that the holomorphic structure

$$d''\psi = \frac{1}{2}(d\psi + i * d\psi) \tag{7.1}$$

of H maps $\Gamma(E)$ into itself, and hence induces a holomorphic structure on the complex line bundle E.
A complex line bundle $E \subset H$ induces a quaternionic line bundle

$$L = E\mathbb{H} = E \oplus Ej \subset H.$$

The complex structure J_E admits a unique extension to the structure of a complex quaternionic bundle (L, J), namely right-multiplication by $(-i)$ on Ej. Conversely, a complex quaternionic line bundle $(L, J) \subset H$ induces a complex line bundle

$$E := \{\phi \in L \mid J\phi = \phi i\}.$$

Definition. *We call (L, J) the twistor projection of E, and E the twistor lift of (L, J).*

Remark 4. As in the quaternionic case, any map $f : M \to \mathbb{C}P^3$ induces a complex line bundle E, where the fibre over p is $f(p)$, and vice versa. Holomorphic curves as defined above correspond to holomorphic curves in the sense of complex analysis. The correspondence between E and (L, J) is mediated by the Penrose twistor projection $\mathbb{C}P^3 \to \mathbb{H}P^1$.

Theorem 3. *Let $E \subset H$ be a a complex line subbundle over a Riemann surface M, and (L, J) its twistor projection. Assume L to be immersed. Then E is a holomorphic curve in $\mathbb{C}P^3$ if and only if (L, J) is a holomorphic curve in $\mathbb{H}P^1$ with $A|_L = 0$.*

Given a surface conformally immersed into $\mathbb{R}^4$, the image of a tangential circle under the quadratic second fundamental form is (a double cover of) an ellipse in the normal space, centered at the mean curvature vector, the so-called *curvature ellipse*. The surface is called *super-conformal* if this ellipse is a circle. Using results of Section 6 in connection with Proposition 7, one can show that the condition $A|_L = 0$ is essentially the super-conformality:

Theorem 4. *A conformally immersed Riemann surface $f : M \to \mathbb{H} = \mathbb{R}^4$ is super-conformal if and only if $\begin{bmatrix} f \\ 1 \end{bmatrix} : M \to \mathbb{H}P^1$ or $\begin{bmatrix} \bar{f} \\ 1 \end{bmatrix} : M \to \mathbb{H}P^1$ is the twistor projection of a holomorphic curve in $\mathbb{C}P^3$.*

8. Bäcklund Transforms of Willmore Surfaces

In this section we shall describe a method to construct new Willmore surfaces from a given one. The construction depends on the choice of a point ∞, and therefore generously offers a 4-parameter family of such transformations. On the other hand, the necessary computations are not invariant, and therefore ought to be done in affine coordinates.
Let $f : M \to \mathbb{H}$ be a Willmore surface with N, R, H, and

$$w = dH + H * df H + R * dH - H * dN.$$

Then $dw = 0$, and hence we can integrate it (on the universal covering of M). Assume that $g : M \to \mathbb{H}$ is an immersion with

$$dg = \frac{1}{2}w. \tag{8.1}$$

We want to show that g is again a Willmore surface called a *Bäcklund transform* of f. Using this name, we refer to the fact that in a given category of surfaces we construct new examples from old ones by solving an ODE (8.1), similar to the classical Bäcklund transforms of K-surfaces, see Tenenblat [9].
We denote the associated to g by a subscript $(.)_g$, and want to prove $dw_g = 0$. The computation of w_g can be done under the weaker assumption (8.2), which holds in the case above.

Proposition 10. *Let $f, g : M \to \mathbb{H}$ be immersions such that*

$$df \wedge dg = 0. \tag{8.2}$$

Then f and g induce the same conformal structure on M, and

$$N_g = -R, \tag{8.3}$$

$$dg(2dH_g - w_g) = -w df. \tag{8.4}$$

If f is Willmore, and g is defined by (8.1), then

$$dg(2df + 2dH_g - w_g) = 2dgdf + dg(2dH_g - w_g) = (2dg - w)df = 0.$$

Hence

$$w_g = 2d(f + H_g), \tag{8.5}$$

and g is Willmore, too.
Now assume that $h := g - H$ is again an immersion. Then,

$$2dh \wedge df = (2dg - 2dH) \wedge df = (w - 2dH) \wedge df = 0.$$

Proposition 10 applied to (h, f) instead of (f, g) then says

$$-w_h dh = df(2dH - w) = df(2dH - 2dg) = -2df dh.$$

We find $w_h = 2df$, whence h is again a Willmore surface. We call g a *forward*, and h a *backward Bäcklund transform* of f. h can be obtained without reference to g by integrating $d(g - H) = \frac{1}{2}w - dH$.
Note that f is a forward Bäcklund transform of h because $df = \frac{1}{2}w_h$, and is also a backward transform of g because $df = \frac{1}{2}w_g - dH_g$, see (8.5).
The concept of Bäcklund transformations depends on the choice of affine coordinates. The following theorem clarifies this situation.

Theorem 5. *Let L be a Willmore surface in $\mathbb{H}P^1$. Choose non-zero $\beta \in (\mathbb{H}^2)^*, a \in \mathbb{H}^2$ such that $<\beta, a> = 0$. Then $d <\beta, *Aa> = 0 = d <\beta, *Qa>$. If $g, h : M \to \mathbb{H} \subset \mathbb{H}P^1$ are immersions that satisfy*

$$dg = 2 <\beta, *Aa>, \quad dh = 2 <\beta, *Qa>,$$

they are again Willmore surfaces, called forward respectively backward Bäcklund transforms of L. The free choice of β implies that there is a whole S^4 of such pairs of Bäcklund transforms. (Different choices of a result in Moebius transforms $g \to g\lambda$, or $h \to h\lambda$, for a constant λ.)

We can now proceed from g with another forward Bäcklund transform. To do so, we must integrate $\frac{1}{2}w_g = d(f + H_g)$. But, up to a translational constant, this yields

$$\tilde{f} := f + H_g. \tag{8.6}$$

Using Proposition 7, we now get for $\tilde{f}$, and similarly for the twofold *backward* Bäcklund transform $\hat{f}$

$$\begin{pmatrix} \tilde{f} \\ 1 \end{pmatrix} \in \ker A, \qquad \begin{pmatrix} \hat{f} \\ 1 \end{pmatrix} \mathbb{H} \supset \operatorname{image} Q.$$

But this means that away from the zeros of A or Q the 2-step Bäcklund transforms of a Willmore surface L in $\mathbb{H}P^1$ can be described simply as $\tilde{L} = \ker A$ or $\hat{L} = \operatorname{image} Q$. In particular there are no periods arising. It can be shown that (as a consequence of holomorphicity) the kernel of A defines a non-singular line bundle $\tilde{L}$ even at the zeros of A:

Proposition 11. *Let L be a Willmore surface in $\mathbb{H}P^1$, and $A \not\equiv 0$ on each component of M. Then there exists a unique line bundle $\tilde{L} \subset H$ such that on an open dense subset of M we have:*

$$\tilde{L} = \ker A, \text{ and } H = L \oplus \tilde{L}.$$

A similar assertion holds for $\operatorname{image} Q$.

Assuming that $\tilde{L}$ is immersed, its invariants can be obtained directly from L. E.g. for the Q-Hopf field we get

Theorem 6. *For the 2-step Bäcklund transform $\tilde{L}$ of L we have*

$$\tilde{Q} = A. \tag{8.7}$$

Hence $\tilde{L}$ is again a Willmore surface.

We obtain a chain of Bäcklund transforms

$$\begin{array}{ccccccccccccc} \cdots & \to & \hat{f} & \to & h & \to & f & \to & g & \to & \tilde{f} & \to & \cdots \\ & & \| & & & & \| & & & & \| & & \\ \to & & \hat{L} & & \to & & L & & \to & & \tilde{L} & & \to \end{array}$$

Of course, the chain may break down if we arrive at non-immersed surfaces, or it may close up.
Taking the two-step backward transform of $\tilde{L}$, we get $\operatorname{image}\tilde{Q} = \operatorname{image} A = L$. Hence $\hat{\tilde{L}} = L$. We remark that the above results similarly apply to the backward two-step Bäcklund transformation $L \to \hat{L} = \operatorname{image} Q$. As a corollary of (8.7) and its analog $\hat{A} = Q$ we obtain

Theorem 7.

$$\hat{\tilde{L}} = L = \tilde{\hat{L}}.$$

9. Willmore Surfaces in S^3

Let $< .,. >$ be an indefinite hermitian inner product on $\mathbb{H}^2$. Then the set of isotropic lines $< l, l >= 0$ defines an $S^3 \subset \mathbb{H}P^1$, while the complementary 4-discs are hyperbolic 4-spaces. Let L be a line bundle with mean curvature sphere S. We look at the adjoint map $M \to \mathcal{Z}, p \mapsto S_p^*$ with respect to $< .,. >$. It is easily seen that $S = S^*$ implies that L is isotropic. Conversely, if L is isotropic, then S^* satisfies the characterization of S given in Theorem 1.

Proposition 12. *An immersed holomorphic curve L in $\mathbb{H}P^1$ is isotropic, i.e. a surface in S^3, if and only if $S = S^*$.*

On the other hand, a 2-sphere $S \in \mathcal{Z}$ intersects the isotropic S^3 orthogonally, if and only if it represents totally geodesic 2-planes in the complementary hyperbolic 4-spaces. The orthogonality can be described using affine coordinates with isotropic point ∞ and the reflexion $\mathbb{H} \to \mathbb{H}, x \mapsto -\bar{x}$ at $\operatorname{Im}\mathbb{H} = S^3$. We get

Proposition 13. *A 2-sphere $S \in \mathcal{Z}$ intersects the hyperbolic 4-spaces determined by an indefinite inner product in hyperbolic 2-planes if and only if $S^* = -S$.*

Now let L be a connected Willmore surface in $S^3 \subset \mathbb{H}P^1$, where S^3 is the isotropic set of an indefinite hermitian form on $\mathbb{H}^2$. Then its mean curvature sphere satisfies

$$S^* = S.$$

Let us assume that $A \not\equiv 0$, and let $\tilde{L} = \ker A$ and $\hat{L} = \operatorname{image} Q$ be the 2-step Bäcklund transforms of L. Then $S^* = S$ implies $\hat{L} = \tilde{L}$. Using this one shows that $-S$ satisfies the requirements for $\tilde{S}$, and hence $\tilde{S} = -S$. We now turn to the 1-step Bäcklund transform of L. If $dF = 2 * A$, then

$$d(F + F^*) = 2 * A + 2 * A^* = 2 * A - 2 * Q = -dS.$$

Because $S^* = S$, we can choose suitable initial conditions for F such that

$$F + F^* = -S. \tag{9.1}$$

We now use affine coordinates with $L = \begin{bmatrix} f \\ 1 \end{bmatrix}$. Then the lower left entry g of F is a Bäcklund transform of f, and from $S^* = S$ we get $g + \bar{g} = H$. Using the properties of Bäcklund transforms we compute the mean curvature sphere S_g and find $S_g^* = -S_g$. Hence the mean curvature spheres of g intersect S^3 orthogonally, and therefore are hyperbolic planes. We know that, using affine coordinates and a Euclidean metric, the mean curvature spheres are tangent to g and have the same mean curvature vector as g. This property remains under conformal changes of the ambient metric. Therefore, in the hyperbolic metric, g has mean curvature 0, and hence is minimal. If $A \equiv 0$, then $w = 0$, and the "Bäcklund transform" is constant, which may be considered as a degenerate minimal surface. In general g will be singular in the (isolated) zeros of $dg = \frac{1}{2}w$, but minimal elsewhere.
The converse is also true:

Theorem 8. (Richter [7]) *Let $< .,. >$ be an indefinite hermitian product on $\mathbb{H}^2$. Then the isotropic lines form an $S^3 \subset \mathbb{H}P^1$, while the two complementary discs inherit complete hyperbolic metrics. Let L be a Willmore surface in $S^3 \subset \mathbb{H}P^1$. Then a suitable forward Bäcklund transform of L is hyperbolic minimal. Conversely, an immersed holomorphic curve that is hyperbolic minimal is Willmore, and a suitable backward Bäcklund transformation is a Willmore surface in S^3.*

10. Spherical Willmore Surfaces in $\mathbb{H}P^1$

In this section we sketch a proof of the following theorem of Montiel, which generalizes an earlier result of Bryant [2] for Willmore spheres in S^3.

Theorem 9. (Montiel [4]) *A Willmore* sphere *in $\mathbb{H}P^1$ is a twistor projection of a holomorphic or anti-holomorphic curve in $\mathbb{C}P^3$, or, in suitable affine coordinates, corresponds to a minimal surface in $\mathbb{R}^4$.*

The material differs from what we have treated so far: The theorem is global, and therefore requires global methods of proof,imported from complex function theory.
Let E be a complex vector bundles. We keep the symbol $J \in \text{End}(H)$ for the endomorphism given by multiplication with the imaginary unit i.
We denote by $\bar{E}$ the bundle where J is replaced by $-J$. If $< .,. >$ is a hermitian metric on E, then

$$\bar{E} \to E^* = E^{-1}, \psi \to < \psi, . >$$

is an isomorphism of complex vector bundles. Also note that for complex *line* bundles E_1, E_2 the bundle $\text{Hom}(E_1, E_2)$ is again a complex line bundle.
There is a powerful integer invariant for complex line bundles E over a compact Riemann surface, the *degree*, classifying these bundles up to isomorphism. It can be defined as the

integrated curvature form ω of a complex connection, or, equivalently, as the algebraic number of zeros of a generic section $\phi \in \Gamma(E)$:

$$\deg(E) := \frac{1}{2\pi}\int_M \omega, = \operatorname{ord}\phi := \sum_{\phi(p)=0} \operatorname{ind}_p \phi.$$

The degree satisfies

$$\deg(\bar{E}) = \deg E^{-1} = -\deg E,$$
$$\deg(E_1 \otimes E_2) = \deg E_1 + \deg E_2.$$

Example 11. Let M be a compact Riemann surface of genus g, and E its tangent bundle, viewed as a complex line bundle. Using a complex connection, we compute its degree to find $\deg(E) = \chi(M) = (2 - 2g)$. For the canonical bundle $K := E^{-1}$ we therefore get

$$\deg(K) = 2g - 2.$$

Now, let $(E, \bar{\partial})$ be a holomorphic complex vector bundle. We denote by $H^0(E|_U)$ the vector space of holomorphic sections over U.
If E is a complex *line* bundle with holomorphic structure, and $\psi \in H^0(E)\backslash\{0\}$, then the zeros of ψ are isolated and of positive index because holomorphic maps preserve orientation. In particular, if M is compact and $\deg E < 0$, then any global holomorphic section in E vanishes identically.
In the proof of the Montiel theorem we shall apply the concepts of degree and holomorphicity to several complex bundles obtained from quaternionic ones. We relate these concepts.

Definition. *If (L, J) is a complex quaternionic line bundle, then $E_L := \{\psi \in L \mid J\psi = \psi i\}$ is a complex line bundle. We define* $\deg L := \deg E_L$.

If L_1, L_2 are complex quaternionic line bundles, and $E_i := E_{L_i}$, then $\operatorname{Hom}_+(L_1, L_2)$ is isomorphic to $\operatorname{Hom}_{\mathbb{C}}(E_1, E_2)$ as a complex vector bundle. In particular

$$\deg \operatorname{Hom}_+(L_1, L_2) = -\deg L_1 + \deg L_2.$$

We now consider an immersed holomorphic curve $L \subset H = M \times \mathbb{H}^2$ in $\mathbb{H}P^1$ with mean curvature sphere S. The bundle $K\operatorname{End}_-(H)$ is a complex vector bundle, the complex structure being given by post-composition with S. The complex structure $\bar{\partial}$ on TM, and the (quaternionic) holomorphic structures $\bar{\partial}$ on H and ∂ on $\bar{H}$ induce a holomorphic structure on

$$K\operatorname{End}_-(H) = K\operatorname{Hom}_+(\bar{H}, H) = K\operatorname{Hom}_{\mathbb{C}}(\bar{H}, H).$$

If L is Willmore, then $d * A = 0$, and this implies $\bar{\partial}A = 0$, and A is holomorphic:

$$A \in H^0(K\operatorname{End}_-(H)) = H^0(K\operatorname{Hom}_+(\bar{H}, H)).$$

As a consequence, either $A \equiv 0$, or the zeros of A are isolated, and there exists a line bundle $\tilde{L} \subset H$ such that $\tilde{L} = \ker A$ away from the zeros of A. For local $\psi \in \Gamma(\tilde{L})$ and holomorphic $Y \in H^0(TM)$ we have

$$\underbrace{\bar{\partial} A}_{=0}(Y)\psi = \bar{\partial}(\underbrace{A(Y)\psi}_{=0}) - A(Y)\partial\psi.$$

Therefore $\tilde{L}$ is invariant under ∂, like L is invariant under $\bar{\partial}$, see Remark 3. As above, we get a holomorphic structure on the complex *line* bundle $K\,\mathrm{Hom}_+(\bar{H}/\tilde{L}, L)$ and A defines a holomorphic section of this bundle:

$$A \in H^0(K\,\mathrm{Hom}_+(\bar{H}/\tilde{L}, L)).$$

We turn to the

Proof of Theorem 9. If $A \equiv 0$ or $Q \equiv 0$, then L is a twistor projection by Theorem 4. Otherwise we have the line bundle $\tilde{L}$, and similarly a line bundle $\hat{L}$ that coincides with the image of Q almost everywhere.

Proposition 14. *We have the following holomorphic sections of complex holomorphic line bundles:*

$$\begin{aligned} A &\in H^0(K\,\mathrm{Hom}_+(\bar{H}/\tilde{L}, L)), & Q &\in H^0(K\,\mathrm{Hom}_+(H/L, \bar{\hat{L}})),\\ \delta_L &\in H^0(K\,\mathrm{Hom}_+(L, H/L)), & AQ &\in H^0(K^2\,\mathrm{Hom}_+(H/L, L)),\\ &\textit{and if } AQ = 0 \textit{ then} & \delta_{\tilde{L}} &\in H^0(K\,\mathrm{Hom}_+(\bar{\tilde{L}}, \bar{H}/\tilde{L})). \end{aligned}$$

We proved the statement about A, the other proofs are similar.
The degree formula then yields

$$\begin{aligned} \mathrm{ord}\,\delta_L &= \deg K - \deg L + \deg H/L\\ \mathrm{ord}(AQ) &= 2\deg K - \deg H/L + \deg L\\ &= 3\deg K - \mathrm{ord}\,\delta_L\\ &= 6(g-1) - \mathrm{ord}\,\delta_L. \end{aligned}$$

For $M = S^2$, i.e. $g = 0$, we get $\mathrm{ord}(AQ) < 0$, whence $AQ = 0$. Then $\tilde{L} = \hat{L}$, and

$$\begin{aligned} \mathrm{ord}\,A &= \deg K + \deg H/\tilde{L} + \deg L\\ \mathrm{ord}\,Q &= \deg K - \deg H/L - \deg \tilde{L}\\ \mathrm{ord}\,\delta_{\tilde{L}} &= \deg K + \deg \tilde{L} - \deg H/\tilde{L}. \end{aligned}$$

Addition yields

$$\begin{aligned} \mathrm{ord}\,\delta_{\tilde{L}} + \mathrm{ord}\,Q + \mathrm{ord}\,A &= 3\deg K - \deg H/L + \deg L\\ &= 4\deg K - \mathrm{ord}\,\delta_L = -8 - \mathrm{ord}\,\delta_L. \end{aligned}$$

It follows that $\mathrm{ord}\,\delta_{\tilde{L}} < 0$, i.e. $\delta_{\tilde{L}} = 0$, and $\tilde{L}$ is d-stable, hence constant in $H = M \times \mathbb{H}^2$. From $AS = -SA = 0$ we conclude $S\tilde{L} = \tilde{L}$. Therefore all mean curvature spheres of L pass through the fixed point $\tilde{L}$. Choosing affine coordinates with $\tilde{L} = \infty$, all mean curvature spheres are affine planes, and L corresponds to a minimal surface in $\mathbb{R}^4$. □

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