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A Domain Decomposition Method for Micropolar Fluids

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1 Introduction

In this paper, we consider a mixed finite element discretization of the following system of partial differential equations with Dirichlet boundary conditions:

$$\begin{cases} -(\nu+\nu_1)\Delta u + (u\cdot\nabla)u + \nabla p = 2\nu_1 \operatorname{curl}\omega + f & \text{in } \Omega, \\ \operatorname{div} u = 0 & \text{in } \Omega, \\ -c_1\Delta\omega + (u\cdot\nabla)\omega - c_2\nabla\operatorname{div}\omega + 4\nu_1\omega = 2\nu_1\operatorname{curl} u + g & \text{in } \Omega, \\ u = u_0 & \text{on } \partial\Omega, \\ \omega = \omega_0 & \text{on } \partial\Omega \end{cases} \tag{1}$$

This system is a mathematical model of stationary flow of a viscous micropolar fluid, which describes the motion of solid particle suspension in a liquid. Such model is also a basis for more complicated ones used in applied sciences, for example in the theory of lubrication [BL95], [Kho90] or in the theory of blood flow [P$^+$74].

The unknowns are the velocity vector u, the pressure p and the internal microrotation vector ω. We denote the external force and the angular momentum force by f and g, respectively. The usual (constant) kinematic viscosity is denoted by $\nu > 0$, while other positive constants ν_1, c_1 and c_2 are additional viscosities introduced by the field of internal rotation ω.

Existence and uniqueness theorems for (1) are proved in [Luk88]. Here, we provide their discrete counterparts for the mixed finite element discretization of (1). The nonlinear discrete problem is then solved using the Newton's method. Each iteration step requires solution of a linear system with a nonsymmetric indefinite matrix, which is ill conditioned with respect to the mesh size h.

We propose and analyse a preconditioning method for the linear system, based on a block diagonal preconditioner. Our goal is to make it possible to reuse the methods already existing for simpler problems, like for the Poisson equation. Since the theory and methods for preconditioning the discrete Laplacian are well developed, our preconditioner can be easily constructed and implemented, using, for example, an efficient domain decomposition preconditioner.

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The preconditioned system *is* symmetric and positive definite with respect to some auxiliary scalar product, so standard iterative methods, like conjugate gradient method, can be used for this system. Each step of CG method requires solution of three smaller, independent problems of small computational cost.

Block diagonal preconditioners for Stokes-like problems have been considered by many authors before, see, for example, [D'y87], [BP88], [BP90], [RW92], [ES94], [SW94] or [Kla96]. However, our analysis of the preconditioned system relies neither on the symmetry nor the positive definiteness of the matrix.

Notation

Throughout the paper we assume that Ω is an open, bounded polyhedron in R^3 with Lipschitz continuous boundary. The differential operators $\Delta, \nabla, \mathrm{curl}, \mathrm{div}$, appearing in (1), are defined in a standard way, see [GR86].

We use several function spaces whose properties are described, for example, in [Ada75]. By $H^k(\Omega)$ we denote the usual Sobolev spaces, identifying $H^0(\Omega)$ with the $L^2(\Omega)$ space of square integrable functions. The standard norm in $H^k(\Omega)$ is denoted by $||\cdot||_k$, while the seminorm by $|\cdot|_k$. $H^1_0(\Omega)$ denotes the subspace of $H^1(\Omega)$ of functions whose traces on $\partial\Omega$ are equal to zero, while $L^2_0(\Omega)$ is the subspace of $L^2(\Omega)$, defined as $L^2_0(\Omega) = \{w \in L^2(\Omega) : \int_\Omega w = 0\}$.

For a positive integer N, we denote the inner product in $[L^2(\Omega)]^N = L^2(\Omega) \times L^2(\Omega) \ldots \times L^2(\Omega)$ by

$$(u, v) := \sum_{i=1}^{N} \int_\Omega u_i\, v_i\, dx.$$

For the inner product in $[H^1_0(\Omega)]^N$ we use

$$((u, v)) := \sum_{j=1}^{N} (\nabla u_j, \nabla v_j) = \sum_{j=1}^{N} \sum_{i=1}^{N} \int_\Omega \frac{\partial u_j}{\partial x_i} \frac{\partial v_j}{\partial x_i}.$$

By "C" we denote a generic positive constant which, if necessary, we shall distinguish by subscripts. Where there is no risk of confusion, we shall write H^k, H^1_0, L^2_0 instead of $H^k(\Omega)$, $H^1_0(\Omega)$, $L^2_0(\Omega)$ and use the same symbols for N-fold products of such spaces.

The Discrete Problem

We pose our original problem (1) in a variational form, using the following function spaces:

$$V := [H^1_0(\Omega)]^3, \qquad W := L^2_0(\Omega).$$

In the rest of the paper we shall assume that the data for problem (1) satisfy $f, g \in [L^2(\Omega)]^3$ and the boundary conditions on u, ω are homogeneous.

We cover $\bar{\Omega}$ with a quasi-uniform triangulation [Cia91] $\mathcal{T}_h$, dividing $\bar{\Omega}$ into tetrahedra K:

$$\bigcup_{K \in \mathcal{T}_h} K = \bar{\Omega},$$

with the mesh parameter h. We make standard assumption that at least one vertex of each $K \in \mathcal{T}_h$ lies inside Ω.

For approximation of the velocity u and pressure p we shall use the Taylor – Hood finite spaces V_h, W_h (see, e.g. [BF91]),

$$V_h = \{v \in V \cap C(\bar{\Omega}) : v\big|_K \in P_2(K) \quad \forall K \in \mathcal{T}_h\},$$

and

$$W_h = \{w \in W \cap C(\bar{\Omega}) : w\big|_K \in P_1(K) \quad \forall K \in \mathcal{T}_h\}.$$

The microrotation field ω is approximated in the same space V_h as the velocity. It is well known that V_h and W_h satisfy the *inf-sup* condition.

The mixed variational formulation of the approximate problem (1) in the finite element spaces $V_h \subset V$, $W_h \subset W$ is as follows:

Problem 1.1 *Find $(u_h, p_h, \omega_h) \in V_h \times W_h \times V_h$, such that*

$$\begin{cases} (\nu+\nu_1)(\nabla u_h, \nabla v) + d(u_h,\, u_h,\, v) - (p_h, \operatorname{div} v) = 2\nu_1(\operatorname{curl}\omega_h, v) + (f, v), \\ (\operatorname{div} u_h, q) = 0, \\ c_1(\nabla\omega_h, \nabla\xi) + d(u_h,\, \omega_h,\, \xi) + c_2(\operatorname{div}\omega_h, \operatorname{div}\xi) + 4\nu_1(\omega_h, \xi) \\ \qquad\qquad = 2\nu_1(\operatorname{curl} u_h, \xi) + (g, \xi), \end{cases} \tag{2}$$

for all (v, q, ξ) in $V_h \times W_h \times V_h$.

Here, $d(\cdot,\cdot,\cdot)$ denotes the convective term, defined either as

$$d_1(u,\, v,\, w) := ((u\cdot\nabla)v, w) = \sum_{i,j=1}^{3} \int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j,$$

or, following [Tem79],

$$d_2(u,\, v,\, w) := \frac{1}{2}\Big(((u\cdot\nabla)v, w) - ((u\cdot\nabla)w, v) \Big)$$

for any $u, v, w \in [H^1(\Omega)]^3$. Note that if $\operatorname{div} u = 0$ then $d_1(u,\,\cdot,\,\cdot) \equiv d_2(u,\,\cdot,\,\cdot)$. The form $d_2(\cdot,\,\cdot,\,\cdot)$ is by the definition skew-symmetric with respect to the last two arguments (which reflects the skew-symmetry of $((u\cdot\nabla)v, w)$ on the solution u of (1)).

2 Existence and Uniqueness Results

We begin with a general existence result for the case when the form $d(\cdot,\,\cdot,\,\cdot)$ is equal to $d_2(\cdot,\,\cdot,\,\cdot)$.

Theorem 2.1 *For any $f, g \in L^2$ and any positive ν, ν_1, c_1, c_2 there exists at least one triple $(u_h, p_h, \omega_h) \in V_h \times W_h \times V_h$ which solves the discrete nonlinear system (2). Moreover, the solution is unique, provided that the data f, g are sufficiently small with respect to ν, ν_1, c_1, c_2.*

Remark 2.1 *The "small data" assumption in Theorem 2.1 reflects similar requirements of the uniqueness statement for the continuous case, see [Luk88].*

The next theorem is valid for $d(\cdot,\cdot,\cdot) \equiv d_1(\cdot,\cdot,\cdot)$ or $d_2(\cdot,\cdot,\cdot)$ and provides a generalization of the discrete Navier–Stokes local uniqueness and approximation result of [GR86] for the discrete micropolar equations.

Theorem 2.2 *Let us set* $\lambda = (\nu + \nu_1)^{-1}$. *Let* Λ *be a compact interval in* R_+ *and assume that* $\{(\lambda, (u(\lambda), \lambda p(\lambda), \omega(\lambda)) : \lambda \in \Lambda\}$ *is a branch of nonsingular solutions of (1) such that* $u(\lambda) \in H^{l+1}$, $p(\lambda) \in H^l$, $\omega(\lambda) \in H^{l+1}$ *for* $l = 1$ *or* $l = 2$ *and for all* $\lambda \in \Lambda$. *Then there exists* h_0 *(small enough), such that for* $h \leq h_0$ *there exists a unique smooth function* $\lambda \in \Lambda \to (u_h(\lambda), \lambda p_h(\lambda), \omega_h(\lambda)) \in V_h \times W_h \times V_h$ *such that:*

(i) $\{(\lambda, (u_h(\lambda), \lambda p_h(\lambda), \omega_h(\lambda)) : \lambda \in \Lambda\}$ *is a branch of nonsingular solutions of Problem 1.1,*

(ii) there exists $C > 0$, *independent of* h *and* λ, *such that for all* $\lambda \in \Lambda$

$$|u_h(\lambda) - u(\lambda)|_1 + |\lambda| ||p_h(\lambda) - p(\lambda)||_0 + |\omega_h(\lambda) - \omega(\lambda)|_1 \\ \leq Ch^l(||u(\lambda)||_{l+1} + |\lambda| ||p(\lambda)||_l + ||\omega(\lambda)||_{l+1})$$

For the proofs of Theorem 2.1 and Theorem 2.2 we refer the reader to [Krz96]. Existence and local uniqueness results stated in Theorem 2.1 and in Theorem 2.2 can be easily extended to other conforming finite elements satisfying the inf-sup condition.

3 A Preconditioning Method for Newton's Iteration Step

In this section we propose and analyse a preconditioning method for one step of Newton's method for Problem 1.1.

Newton's algorithm. Given $(u_h^n, p_h^n, \omega_h^n) \in V_h \times W_h \times V_h$, find $(u_h^{n+1}, p_h^{n+1}, \omega_h^{n+1}) \in V_h \times W_h \times V_h$, which satisfies

$$\begin{cases} (\nu + \nu_1)(\nabla u_h^{n+1}, \nabla v) + d(u_h^n, u_h^{n+1}, v) + d(u_h^{n+1}, u_h^n, v) - (p_h, \operatorname{div} v) \\ \qquad = 2\nu_1(\operatorname{curl} \omega_h^{n+1}, v) + d(u_h^n, u_h^n, v) + (f, v), \\ (\operatorname{div} u_h^{n+1}, q) = 0, \\ c_1(\nabla \omega_h^{n+1}, \nabla \xi) + d(u_h^n, \omega_h^{n+1}, \xi) + d(u_h^{n+1}, \omega_h^n, \xi) + c_2(\operatorname{div} \omega_h^{n+1}, \operatorname{div} \xi) \\ \qquad + 4\nu_1(\omega_h^{n+1}, \xi) = 2\nu_1(\operatorname{curl} u_h^{n+1}, \xi) + d(u_h^n, \omega_h^n, \xi) + (g, \xi) \end{cases} \tag{3}$$

for all $(v, q, \xi) \in V_h \times W_h \times V_h$.

Actually, we are dealing with a family of such problems, indexed by the mesh parameter h. Under assumptions as in Theorem 2.2 the Newton's method is locally quadratically convergent to the solution of the discrete system. The rate of convergence is affected by the parameter $\lambda = (\nu + \nu_r)^{-1}$, but is independent (see [Krz96]) of the mesh parameter h.

We are going to analyse a preconditioning method for these problems so that the resulting problem is given by a symmetric positive definite operator whose condition number is independent of h. Let us denote for short $(u_h^{n+1}, p_h^{n+1}, \omega_h^{n+1})$ by (u, p, ω). Define linear operators A, B, C, T_A, T_B by variational identities for all $u, v \in V_h$ and

$w \in W_h$:

$$\begin{array}{ll} A : V_h \to V_h', & \langle\langle Au, v\rangle\rangle = (\nu+\nu_1)(\nabla u, \nabla v) + d(u_h^n,\, u,\, v) + d(u,\, u_h^n,\, v), \\ B : V_h \to W_h, & \langle\langle Bu, w\rangle\rangle = -(\operatorname{div} u, w), \quad B' : W_h \to V_h', \langle\langle B'w, u\rangle\rangle = -(\operatorname{div} u, w), \\ C : V_h \to V_h', & \langle\langle Cu, v\rangle\rangle = c_1(\nabla\omega, \nabla\xi) + d(u_h^n,\, \omega,\, \xi) + c_2(\operatorname{div}\omega, \operatorname{div}\xi) + 4\nu_1(\omega,\xi), \\ T_A : V_h \to V_h', & \langle\langle T_A u, v\rangle\rangle = -2\nu_1(\operatorname{curl}\omega, v), \\ T_C : V_h \to V_h', & \langle\langle T_C u, v\rangle\rangle = -2\nu_1(\operatorname{curl} u, \xi) + d(u,\, \omega_h^n,\, \xi). \end{array}$$

The dual pairing between V_h and V_h', or W_h and W_h', respectively, is denoted by $\langle\langle\cdot,\cdot\rangle\rangle$. We use the same symbol for these two different pairings, since its meaning will be always clear form the context. Then we can express (3) in an operator form:

Problem 3.1 *For $\mathcal{F} = (\phi,\psi,\varphi) \in V_h' \times W_h' \times V_h'$, find $(u,p,\omega) \in V_h \times W_h \times V_h$ such that*

$$\mathcal{M}\begin{pmatrix} u \\ p \\ \omega \end{pmatrix} \equiv \begin{pmatrix} A & B' & T_A \\ B & 0 & 0 \\ T_C & 0 & C \end{pmatrix}\begin{pmatrix} u \\ p \\ \omega \end{pmatrix} = \mathcal{F}.$$

The following lemma is a consequence of Theorem 2.2, see [Krz96].

Lemma 3.1 *Suppose that the assumptions of Theorem 2.2 hold with sufficiently small h_0. In addition, let us assume that $(u_h^n, p_h^n, \omega_h^n)$ is close enough to the solution of Problem 1.1. Then for any $\mathcal{F} \in V_h' \times W_h' \times V_h'$ there exists a unique solution (u,p,ω) of Problem 3.1 and*

$$||u||_1 + ||p||_0 + ||\omega||_1 \le C(||\phi||_{V_h'} + ||\psi||_{W_h'} + ||\varphi||_{V_h'})$$

with C independent of h.

With the inner product $((\cdot,\cdot))$ in V_h we associate the discrete Laplace operator $-\Delta_h : V_h \to V_h'$, defined by $\langle\langle -\Delta_h u, v\rangle\rangle \equiv ((u,v))$. We also define canonical mapping $J : W_h \to W_h'$ (the "mass matrix" operator), $\langle\langle Jp, q\rangle\rangle \equiv (p,q)$.

Let $A_0 : V_h \to V_h'$ be a good preconditioner for the discrete Laplace operator $-\Delta_h$, so that

(i) $\langle\langle A_0 u, v\rangle\rangle = \langle\langle A_0 v, u\rangle\rangle$ for all $u, v \in V_h$,

(ii) there exist constants $\alpha_1, \alpha_2 > 0$, independent of h, such that

$$\alpha_1\langle\langle -\Delta_h u, u\rangle\rangle \le \langle\langle A_0 u, u\rangle\rangle \le \alpha_2\langle\langle -\Delta_h u, u\rangle\rangle \tag{4}$$

for all $u \in V_h$,

(iii) A_0^{-1} is easy to apply.

Likewise, we introduce (cf. [Kla96]) a good preconditioner for the "mass matrix" operator, $J_0 : W_h \to W_h'$, i.e.

(iv) $\langle\langle J_0 p, q\rangle\rangle = \langle\langle J_0 q, p\rangle\rangle$ for all $p, q \in W_h$,

(v) there exist constants $\beta_1, \beta_2 > 0$, independent of h, such that

$$\beta_1\langle\langle Jp, p\rangle\rangle \le \langle\langle J_0 p, p\rangle\rangle \le \beta_2\langle\langle Jp, p\rangle\rangle \tag{5}$$

for all $p \in W_h$,

(vi) J_0^{-1} is easy to apply.

In the implementation, efficient preconditioners A_0 and J_0 may be obtained using domain decomposition methods.

Introducing a block diagonal operator matrix

$$\mathcal{M}_0 = \begin{pmatrix} A_0 & 0 & 0 \\ 0 & J_0 & 0 \\ 0 & 0 & A_0 \end{pmatrix}, \tag{6}$$

we define a preconditioned version of problem (3).

Problem 3.2 *Find $(u,p,\omega) \in V_h \times W_h \times V_h$ such that*

$$\mathcal{M}_0^{-1}\mathcal{M}'\mathcal{M}_0^{-1}\mathcal{M}\begin{pmatrix} u \\ p \\ \omega \end{pmatrix} = \mathcal{M}_0^{-1}\mathcal{M}'\mathcal{M}_0^{-1}\mathcal{F}.$$

Lemma 3.2 *The operator $\mathcal{P} = \mathcal{M}_0^{-1}\mathcal{M}'\mathcal{M}_0^{-1}\mathcal{M}$ is self-adjoint with respect to the auxiliary scalar product $[\cdot,\cdot]$ defined as*

$$\left[\begin{pmatrix} u \\ p \\ \omega \end{pmatrix}, \begin{pmatrix} v \\ q \\ z \end{pmatrix}\right] \equiv \langle\langle A_0 u, v\rangle\rangle + \langle\langle J_0 p, q\rangle\rangle + \langle\langle A_0 \omega, z\rangle\rangle. \tag{7}$$

The main result of this section is an estimate of the condition number of the operator $\mathcal{P}$ in the norm induced by $[\cdot,\cdot]$.

Theorem 3.3 *Let $\mathcal{P}$ be defined as in Lemma 3.2. Suppose Lemma 3.1 holds and A_0, J_0 satisfy assumptions (i) – (vi) of this section. Then there exist positive constants m_1, m_2, independent of h, such that*

$$m_1\left[\begin{pmatrix} u \\ p \\ \omega \end{pmatrix}, \begin{pmatrix} u \\ p \\ \omega \end{pmatrix}\right] \le \left[\mathcal{P}\begin{pmatrix} u \\ p \\ \omega \end{pmatrix}, \begin{pmatrix} u \\ p \\ \omega \end{pmatrix}\right] \le m_2\left[\begin{pmatrix} u \\ p \\ \omega \end{pmatrix}, \begin{pmatrix} u \\ p \\ \omega \end{pmatrix}\right]$$

for any $(u,p,\omega) \in V_h \times W_h \times V_h$.

Proof. We have

$$\begin{aligned}\left[\mathcal{P}\begin{pmatrix} u \\ p \\ \omega \end{pmatrix}, \begin{pmatrix} u \\ p \\ \omega \end{pmatrix}\right] &= \langle\langle Au + B'p + T_A\omega, A_0^{-1}(Au + B'p + T_A\omega)\rangle\rangle + \langle\langle Bu, J_0^{-1}Bu\rangle\rangle \\ &\quad + \langle\langle T_C u + C\omega, A_0^{-1}(T_C u + C\omega)\rangle\rangle.\end{aligned} \tag{8}$$

Since there exist constants $\gamma_1, \gamma_2 > 0$, and $\delta_1, \delta_2 > 0$, independent of h, such that

$$\begin{aligned} \gamma_1||\phi||_{V_h'}^2 &\le \langle\langle \phi, A_0^{-1}\phi\rangle\rangle \le \gamma_2||\phi||_{V_h'}^2 \qquad \forall\phi \in V_h', \\ \delta_1||q||_0^2 &\le \langle\langle q, J_0^{-1}q\rangle\rangle \le \delta_2||q||_0^2 \qquad \forall q \in W_h', \end{aligned} \tag{9}$$

we obtain

$$\left[\mathcal{P}\begin{pmatrix} u \\ p \\ \omega \end{pmatrix}, \begin{pmatrix} u \\ p \\ \omega \end{pmatrix}\right] \geq \gamma_1 ||Au + B'p + T_A\omega||^2_{V_h'} + \delta_1 ||Bu||_0^2 + \gamma_1 ||T_C u + C\omega||^2_{V_h'},$$

which by Lemma 3.1 together with (4) and (5) yields the lower bound

$$\left[\mathcal{P}\begin{pmatrix} u \\ p \\ \omega \end{pmatrix}, \begin{pmatrix} u \\ p \\ \omega \end{pmatrix}\right] \geq C \left[\begin{pmatrix} u \\ p \\ \omega \end{pmatrix}, \begin{pmatrix} u \\ p \\ \omega \end{pmatrix}\right].$$

Similarly we can establish the upper bound. Indeed, from (9) together with (8) we have

$$\begin{aligned}\left[\mathcal{P}\begin{pmatrix} u \\ p \\ \omega \end{pmatrix}, \begin{pmatrix} u \\ p \\ \omega \end{pmatrix}\right] &\leq \gamma_2 ||Au + B'p + T_A\omega||^2_{V_h'} + \delta_2 ||Bu||_0^2 + \gamma_2 ||T_C u + C\omega||^2_{V_h'} \\ &\leq C(||Au||^2_{V_h'} + ||B'p||^2_{V_h'} + ||T_A\omega||^2_{V_h'} + ||Bu||_0^2 + ||T_C u||^2_{V_h'} + ||C\omega||^2_{V_h'}).\end{aligned}$$

Obviously, each of the operators A, B, C, T_A, T_B is bounded from above independently of h. Estimating each term in the sum, we get

$$\left[\mathcal{P}\begin{pmatrix} u \\ p \\ \omega \end{pmatrix}, \begin{pmatrix} u \\ p \\ \omega \end{pmatrix}\right] \leq C \left[\begin{pmatrix} u \\ p \\ \omega \end{pmatrix}, \begin{pmatrix} u \\ p \\ \omega \end{pmatrix}\right],$$

which completes the proof.

4 Remarks

The resulting system can be solved by the conjugate gradient method since its matrix is symmetric and positive definite. By Theorem 3.3, the number of iterations required for reducing the residual by a given factor is independent of h. As it has been pointed out in [BP88], computing the inner product during the CG step requires only one solution of a system with the operator $\mathcal{M}_0$.

Many authors contributed to the area of numerical solution of saddle point problems, see for example [D'y87], [BP88], [BP90], [RW92], [ES94], [SW94], [Kla96], addressing mostly (if not exclusively) the symmetric operator case. The idea of symmetrizing the saddle point system with the aid of a preconditioner for the Laplacian has been considered previously in, for example, [D'y87] and [BP90]. However, our analysis remains also valid for nonsymmetric operators.

In the case of $v_r = c_1 = c_2 = 0$ our system reduces to Newton's linearization of the Navier-Stokes equations, therefore our preconditioner applies also to this particular case. Moreover, this preconditioning method generalizes to the case of abstract saddle point equations with nonsymmetric, indefinite diagonal part. Different preconditioning methods for these problems will be analysed in a forthcoming paper.

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