PURPOSEFUL TASK DESIGN AND THE EMERGENCE OF TRANSPARENCY

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In the Purposeful Algebraic Activity project[1] we have produced a teaching programme of spreadsheet-based tasks, using purpose *and* utility *as the framework for task design. Here we look in detail at the design of one of the tasks, using the notions of visibility and invisibility to examine examples of pupils' activity when working on this task and the role which perceptions of purpose play in the way in which transparency emerges.*

INTRODUCTION

This paper focuses on one aspect of the *Purposeful Algebraic Activity* project. The overall aim of the project has been to study pupils' construction of meaning for algebra in the early part of secondary education. The project takes up the challenge set by Sutherland (1991) to create '*a school algebra culture in which pupils find a need for algebraic symbolism*'. Central to the project is a programme of six tasks, based on the use of spreadsheets. These tasks have been designed to offer purposeful contexts for algebraic activity. In this paper we discuss in detail the design of one task, and use the notion of *transparency* (Lave & Wenger, 1991) to examine potential trajectories through the task, and some specific activity by pupils in response to it.

DESIGNING PURPOSEFUL TASKS

The relative lack of relevance in much of school mathematics, compared to the high levels of engagement with mathematical ideas in out-of-school settings, has been recognised by a number of researchers and curriculum developers. Schliemann (1995) identifies the need for '*school situations that are as challenging and relevant for school children as getting the correct amount of change is for the street seller and his customers*'. However, setting school tasks in the context of 'real world' situations does not provide a simple solution: there is considerable evidence of the problematic nature of pedagogic materials which contextualise mathematics in supposedly real-world settings, but fail to provide purpose (see for example Cooper and Dunne, 2000). Ainley and Pratt (2002) identify the purposeful nature of activity as a key feature which contributes to the challenge and relevance of mathematics in everyday settings, and propose a framework for pedagogic task design in which *purpose for the learner, within the classroom environment* is a key construct.

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This use of '*purpose'* is quite specifically related to the perceptions of the learner. It may be quite distinct from any objectives identified by the teacher, and does not depend on any apparent connection to a 'real world' context. It may, of course, be true in a trivial sense that learners construct the purpose of any task in ways other than those intended by the teacher. In using purpose as a design principle, we have tried to provide purposeful outcomes through the creation of actual or virtual products, solutions to intriguing questions or explanation and justification of results.

We have also used the notion of the *utility* of mathematical ideas: that is knowing how, when and why such ideas are useful (Ainley and Pratt, 2002). Within a purposeful task, opportunities can be provided for learners to use and learn about particular mathematical ideas in ways that allow them to appreciate their *utility*. In contrast, within much of school mathematics, ideas are learnt in contexts which are divorced from any sense of how or why such mathematical ideas may be useful.

In addition to these two general design principles, we have been concerned to include within the design of our tasks three other features: opportunities to exploit the algebraic potential of the spreadsheet (Ainley, Bills & Wilson, 2004), opportunities for pupils to engage in a balance of *generational*, *transformational* and *meta-level* algebraic activities (Kieran, 1996) and opportunities to build on pupils' fluency with arithmetic to make links to both the spreadsheet notation and standard algebra.

AN EXAMPLE: THE FAIRGROUND GAME TASK

We now describe the design of the sixth and final task in our teaching programme. The task was based on an idea which appears fairly frequently (in the UK at least) in resources for teaching algebra in the early years of secondary school. The example in Figure 1 is taken from the *Framework for Teaching Mathematics* for ages 11-14, which forms the basis of the curriculum which schools in England have to follow (DfES, 2001). It is from the section headed '*Equations, formulae, identities*', for pupils in the first year of secondary school (age 11-12).

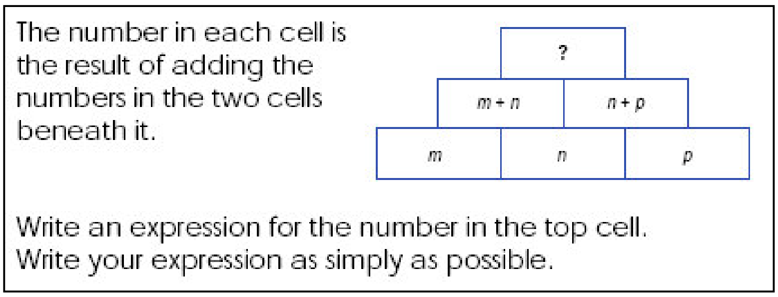

Figure 1: The original example task

The example task given here seems to us to be limited in a number of ways. It is set in a purely algebraic context. Although the text refers to 'numbers' no numbers are given in this example (although further examples based on the same idea appear in the sections for subsequent age groups which ask pupils to find the value of a missing number from a pyramid array). It may be that teachers and pupils would already be familiar with the pyramid array from previous numerical activities, but there is no attempt made in this example to make explicit links to arithmetic experience.

The choice of letters to represent numbers in the array is likely to suggest to some pupils that the numbers in the bottom row are ordered, and indeed consecutive. This task does not give any sense of the letters as variables, representing *any* number, and indeed subsequent tasks based on the pyramid array are concerned with finding the value of particular unknowns.

There is no purpose offered for the task. What is the outcome of adding numbers in this way? And what is the benefit of writing the final expression 'as simply as possible'? For many pupils it would be difficult to see why $m+2n+p$ is a simpler or more usable expression than $m+n+n+p$, because they are offered no context in which the usefulness of simplification might be apparent.

Producing a spreadsheet-based task

Despite these limitations, the pyramid array does seem to offer rich possibilities for algebraic activity, and its cell structure lends itself well to use with a spreadsheet. The spatial arrangement of the cells provides a visual metaphor for the repeating additive structure of the mathematical problem, and thus offers the potential for the array to be *transparent* for users: allowing them to look *at* the visible physical structure so that the content of cells can be manipulated, and to look *through* this (transparent) structure to get a sense of the mathematical structure which underlies it (Lave & Wenger, 1991). However, Meira's study of instructional devices suggests that transparency emerges in the use of tools and symbols, rather than being an inherent characteristic of them (Meira, 1998). Thus the design of tasks may be as significant in the emergence of transparency as the design of tools themselves (Ainley, 2000).

In order to create a task which would offer purposeful activities, we explored the questions which might be asked about the pyramid array, and the challenges which might be set. If the pyramid is used for numerical activities, then one obvious group of questions concerns the effect of changing the numbers used in the bottom row on the subsequent rows, and the final total. Does changing the order of these numbers alter the total? How can the highest or lowest total be achieved from any given set of numbers? Recreating the array on a spreadsheet offers an environment in which it is easy to explore such questions.

In our task we used the structure of the pyramid array as the basis for a game which might be used at a school fair. The game uses a version of the array on a spreadsheet as shown below. The player is given five numbers, which they can enter into the left hand column in any order they like. To win, the player has to make a total (which will appear in the cell on the far right) which is as high as, or higher than, a target set by the stallholder. Pupils are presented with the example shown in Figure 2 on a worksheet, with a description of the game.

	A	B	C	D	E
1	4				
2	2	6			
3	1	3	9		
4	3	4	7	16	
5	5	8	12	19	35

There is a game at the school fair. Players are given five numbers to enter into column A in any order they wish. The stallholder sets a target number. If the number that appears on the right (column E) is the same as or higher than the target number then the player wins!

Figure 2: Extract from the pupils' worksheet

The decisions which we made in transferring the pyramid array to the spreadsheet, and creating the game context, had a number of effects on the potential activity of pupils working on the task.

Because of the 'row and column' structure of the spreadsheet, it was necessary to change the spatial relationship of the cells from that in the original pyramid array[2]. This arrangement may make it less clear which cells were added to produce the next column. In the pupils' worksheet for this task, the whole array of numbers is presented, but how the array is constructed is not made explicit. We chose to rotate the image, partly to make the spreadsheet operations more comfortable, and partly so that pupils would not immediately associate this task with previous experiences they may have had of working with the pyramid array. The array was enlarged to use five starting numbers rather than three to make the challenge more realistic.

The first stage of the task is to recreate the game array on a spreadsheet, and to explore the effects of changing the positions of the numbers in the first column, and in particular to try to make the highest possible total which will become the target number of the game. The next stage of the task concerns what happens when a player wins by making the first target number. The stall holder must then offer a new set of starting numbers, so pupils need to find a method of getting the highest total for any set of five numbers. The final challenge is to find a way for the stallholder to calculate what the target number should be for any set of starting numbers.

TRAJECTORIES THROUGH THIS TASK

We now discuss features of this task and the learning trajectories which we had anticipated in relation to these, and compare these to examples of data from pupils working on this task within our teaching programme. The teaching programme was carried out in five classes in the first year of two secondary schools (i.e. pupils aged 11-12, representing a range of achievement). Four teachers who had been involved in the development of the tasks used them as part of their regular teaching during the year. For each task, pupils' worksheets and detailed teachers' notes were prepared. The teachers were encouraged to introduce the tasks through whole class discussion before pupils began work in pairs, and to bring the class together for further plenary sessions as they felt appropriate. The pupils' worksheets were designed to support the pupils' activity, but not to 'stand alone' in presenting the tasks. The six tasks were

[2] Although a closer approximation to the pyramid structure could have been produced by using alternate cells, this would have added an unnecessary complication.

used as three pairs during the year, with the *Fairground Game* task being the last in the sequence. Classes spent two lessons of about an hour each on each of the tasks. Most of these lessons took place in computer rooms rather than in the normal classrooms. This had the advantage of providing enough access for all pupils to work singly or in pairs at the machines, but had the disadvantages that pupils were generally not very familiar with this environment, and that the layout of the rooms was not well designed for the teacher to be able to circulate and monitor the progress of all pairs of pupils.

During the teaching programme data was collected through fieldnotes and audio recording of the teacher, to give an overall picture of the progress of the lesson, and video and screen recording of one pair of pupils in each class working on each task.

Setting up the game and finding the highest total

In the first stage of the task the intended purpose was to produce a version of the game on the spreadsheet, and then to use this to find how to get the highest total. We anticipated that having to spot the pattern in the array of numbers and generate the formulae to create the array on the spreadsheet would encourage pupils to attend closely to the arithmetic structure of the game. By the time they undertook this task, pupils were reasonably familiar with using the spreadsheet and most could enter formulae confidently. The formulae that are required in the spreadsheet, as shown in Figure 3, make the iterative, column to column structure of the array very clear. However, this view of the spreadsheet was not available to pupils as they worked on the task. The formulae have become (literally) invisible to pupils, and what they see are the numbers in each cell changing.

	A	B	C	D	E
1	1				
2	4	=A1+A2			
3	2	=A2+A3	=B2+B3		
4	3	=A3+A4	=B3+B4	=C3+C4	
5	5	=A4+A5	=B4+B5	=C4+C5	=D4+D5

Figure 3: the completed spreadsheet formulae

Searching for the highest total involves repeatedly changing the values entered in the cells in column A, and seeing the effect of this on the remaining cells in the array. Our intention for pupils' learning was that this would reinforce the notion of the cell reference in a formula representing a variable: any number which may be entered into a particular cell.

Once they had created their own version of the game, most pupils were able to engage with exploring the effects of changing the order of the starting numbers, and many worked systematically to identify a winning strategy. Kayleigh and Christopher, in a low attaining set, did not immediately understand that they needed to produce a spreadsheet made with formulae on which the game could be played. At first they simply reproduced the array they had been shown by typing in the numbers. After an intervention, they were able to put in the formulae, and use their game to explore the effects of changing the order of the starting numbers. However, they seemed to see the purpose at this stage as *getting* the highest total, rather than as *finding how to get* the highest total. Their attention was only on the final total, and they did not see any reason to record how they had used the numbers to get each result. There was no evidence that they were engaging with the notion of variable.

Finding how to get the highest total for any set of numbers

In the next stage of the task the purpose shifts to finding a strategy that will always give the highest total. This not only reinforces the variable nature of the cell reference by increasing the range of possible numbers, but focuses attention on the structure of the array, and how the total is formed. Many pupils had already made a conjecture about a method for placing the numbers to give the highest total, and using a different set of numbers was a way of confirming their ideas.

Pupils' offered a variety explanations for the method they had chosen. In some cases their explanations suggest that the array of numbers on the spreadsheet became a transparent tool which they were able to look through to see features of the underlying arithmetic structure. We conjecture that their experience of entering the formulae supported this as they explored the effects of changing the starting numbers. For example, Hugo, in a middle attaining set, wrote '*you get the highest overall number when the two highest starting numbers are in the middle because they get included in every sum until the overall answer*'. Rupinda, in the same set, said, '*You have to put the largest number in the middle because when you travel through the columns the big number will make a higher total*'. In a high attaining set, a pupil said in a class discussion *' the three middle numbers like carry them on and the other two just get lost somewhere'*.

Kayleigh and Christopher, in a lower attaining set, were initially motivated by a competition to find the highest total with a new set of numbers, but still focussed on the total rather than on a method for getting it. After some further intervention, however, Christopher began to focus on the arrangement of the numbers, and talked about why some gave higher totals in terms of how numbers '*travelled*' across the grid. For him it seemed that opportunities to articulate his exploration were important in allowing him to begin to look through the numbers to gain a sense of the structure and the use of variable inputs.

Explaining the method and calculating the target number

The final stage of the task is designed to introduce purposeful use of standard algebraic notation. The purpose is to give the stallholder a way to quickly calculate the appropriate target number for any new set of starting numbers. Obviously this could be done very easily using the spreadsheet array, or more laboriously by working through the calculations by hand. However, in the context of the Fairground game story the stallholder needs to do this calculation quickly and without his customers seeing the outcome, and so another method is needed. To find such a method, it is necessary to look at the structure of the array in a different way. Using spreadsheet formulae, the cumulative effect of the arithmetic structure is invisible since each formula only refers to the previous column. The teachers' notes for this part of the task suggest that pupils should move away from the computer to find ways of showing why their method will produce the highest total using standard notation.

Many of the higher attaining pupils were able to use letters in place of numbers and work through the array simplifying their answers to give a expression for the total in standard notation. Of these, some made comments in their written work that suggested that they had appreciated the utility of the notation for showing structure. Robin wrote, '*Algebra helped me to find the strategy because it made it easier to see how many letters were used and how often*', and Mandy commented '*Algebra did make it easier because it showed you how the numbers were added up*'.

Other pupils found ways of showing the structure by working through (generic) numerical examples. Amanpreet worked on a paper grid, using the starting numbers 3, 5, 4, 6, 10 (in that order). In each cell he showed the calculation that was to be done, but did not work out any of the results. In the final cell he recorded

$$3+5+5+4+5+4+4+6+5+4+4+6+4+6+6+10$$

He did not feel the need to collect like terms to simplify this result, but was happy that it showed that the number on the middle position (4) was used most, and that 3 and 10 (in the first and last positions) were used least. Other pupils used the grid in similar ways, but showing how to actually get the highest total, and some did simplify their final calculation. Whilst these pupils seemed able to some extent to treat the array as transparent, they had not yet fully appreciated the utility of standard notation to express the generalised structure, or engaged with the purpose of finding a way to calculate the total. In practice the time which teachers allowed for working on this task was not long enough for most pupils to really address this final stage of the task.

Faith also preferred to work with numerical examples to illustrate her general method for finding the highest total for five numbers. However, when challenged to find a strategy for six numbers, she went immediately to algebra, setting out a, b, c, d, e, f in the first column and completing the grid without error to finish with $10c + 10d + a + 5e + 5b + f$ in the final cell. She wrote "***c*** *and* ***d*** *appear most often so that is where the largest numbers should be placed*". The additional challenge of working with six numbers may have helped her to appreciate the utility of using standard notation.

THE ROLE OF PURPOSE

In analysing these examples of pupils' activity we see the role of purpose as significant in shaping the focus of their attention, and thus the ways in which they work with, and look at and through the tools involved, that is, the game array, the spreadsheet formulae and the standard notation. For Kayleigh and Christopher, the challenge of getting the highest total was engaging. Initially the effect of changing the order of the starting numbers on the total was highly visible, but their lack of appreciation of the purpose of the exploration (i.e. finding *how* to get the highest total) prevented them from also attending to the underlying structure of the array. When his attention had been focussed on the purpose of finding a general strategy through the teacher's intervention, Christopher was encouraged to work with the array and articulate his ideas in ways which seemed to make the underlying structure more visible, so that he could describe the numbers 'travelling' through the array.

While some pupils were able to appreciate and articulate the utility of standard notation for clarifying the way in which the total number was calculated, others who were focussing on justifying their strategy for finding the highest total were happy to use generic numerical examples, or generalised descriptions to do this. We conclude that perceptions of the purpose of a task affect the ways in which tools are used within it, and thus the extent to which these tools become transparent for the users.

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