HOW SERIES PROBLEMS INTEGRATING GEOMETRIC AND ARITHMETIC SCHEMES INFLUENCED PROSPECTIVE SECONDARY TEACHERS PEDAGOGICAL UNDERSTANDING

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In an undergraduate level mathematical problem-solving course, we conducted an experiment with a different methodology in the teaching of mathematical series problems to twenty-eight prospective secondary mathematics teachers. We supplemented the typical series instruction from an arithmetic focus to what we call a geo-arithmetic focus, one that focuses both on visual and analytic skills. What resulted were some inspiring revelations among these future high school teachers. We present the culminating geo-arithmetic series task, describe our interpretative methodology, and report the cases of three case-study students who reported, as a result of these tasks, initial cognitive dissonance, rich discussions in their learning groups, and ramifications for changes in their future teaching practices.

MOTIVATION

Mathematics students in sixth-century B.C. Greece concentrated on four very separate areas of mathematics (called *mathemata*): arithmetica (arithmetic), harmonia (music), geometria (geometry), and astrologia (astronomy). “This fourfold division of knowledge became known in the Middle Ages as the ‘quadrivium’” (Burton, 1997, p. 88). To these early Greeks, arithmetic and geometry were as separate as music and astronomy. Mathematicians soon realized that arithmetic and geometry are not separate, and that some intriguing mathematics lies at their intersection. This report attempts to explore the beauty and richness of viewing one problem from a geo-arithmetic perspective.

Studies (e.g., Vinner, 1989) have consistently shown that students' mathematics understanding is typically analytic and not visual. Two possible reasons for this are when the analytic mode, instead of the graphic mode, is pervasively used in instruction, or when students or teachers hold the belief that mathematics is the skillful manipulation of symbols and numbers. It is clear from the literature (e.g., Lesh, Post, & Behr, 1987; Janvier, 1987; NCTM, 2000) that having multiple ways – for example, visual and analytic – to represent mathematical concepts is beneficial.

Our argument is not that one student’s representational scheme is superior to another, only that students often construct vastly different personal and idiosyncratic representations that lead to different understandings of a concept. Because student-generated representations provide useful windows into students’ thinking, it is productive for teachers to value these personal representations. Moreover, there is a belief among mathematics educators (e.g., Janvier 1987; Lesh, Post, & Behr, 1987)

2005. In Chick, H. L. & Vincent, J. L. (Eds.). *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 2, pp. 89-96. Melbourne: PME.

that students benefit from being able to understand a variety of representations for mathematical concepts and to select and apply a representation that is suited to a particular mathematical task. The National Council of Teachers of Mathematics (NCTM) reinforces this belief: "Different representations support different ways of thinking about and manipulating mathematical objects. An object can be better understood when viewed through multiple lenses" (2000, p. 360).

Recently, Aspinwall and Shaw (2002) reported their work with two students with contrasting modes of mathematical thinking – Al, whose mode was primarily visual, and Betty, whose mode was almost entirely symbolic. Their assertion was that students often construct vastly different personal and idiosyncratic representations, which lead to different understandings of concepts. Given problems presented graphically, Betty generally found it nearly impossible to think about the problem in graphical terms; thus, she translated from the graphic representations to symbolic representations, or equations, in order to make sense of the problems. Once she completed analytic operations on the symbols, she translated the problem back to the graphic representations required for the tasks. Al, however, operated directly on the graphic representations without having first to translate to symbolic representations. Betty and Al showcased two very different ways of solving problems, but the study suggested that if students could move freely between the visual (geometria) and the symbolic (arithmetica), their mathematical understanding would be much richer and their problem-solving abilities more robust.

Krutetskii (1976) distinguished among three main types of mathematical processing by individuals: analytic, geometric, and harmonic. A student who has predominance toward the analytic relies strongly on verbal-logical processing and relies little on visual-pictorial processing. Conversely, a student who has predominance toward the geometric relies strongly on visual-pictorial processing predominating over above-average verbal-logical processing. A student who has predominance toward the harmonic relies equally on verbal-logical and visual-pictorial processes. Several aspects of Krutetskii's position are of relevance in our interpretation of the ways that our students, comprising both analytic and geometric, processed mathematical series problems demonstrated geometrically. The use of Krutetskii's categories permitted us to explore their thinking in the context of their cognitive processing.

The National Council of Teachers of Mathematics (NCTM, 2000) states that problem solving with an array of creative problems is an essential component in students' construction of meaningful mathematical content. "In high school, students' repertoires of problem-solving strategies expand significantly because students are capable of employing more-complex methods and their abilities to reflect on their knowledge and act accordingly have grown" (p. 334). The following is one of those creative problems that we developed to generate students' interests and to engage them in discussing mathematical content as well as geo-arithmetic issues of learning and teaching.

MATHEMATICAL PROBLEM

The teacher stands at the front of the room with a bag and begins to remove four cubes, with side lengths from 1 cm to 4 cm. After ensuring all the students see the four cubes, the teacher returns the cubes to the bag, shakes the bag, then slowly withdraws from the bag … the four cubes? No, she withdraws not four cubes but one single square with side length 10 cm. The students were amazed by this extraordinary feat of conversion of 4 cubes into a square. (For them, it represented a conversion of three-dimensional cubes into a two-dimensional square.)

From an arithmetic perspective, this problem can be represented by the following equation, $1^3 + 2^3 + 3^3 + 4^3 = 10^2$. One student remarked that the conversion was true when using 1, 2, or 3 cubes as well. Another student asked, "Does placing consecutively larger cubes into the magic bag always produce a square with this intriguing property; that is, does this equality always hold: $1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2$?" A mathematical induction approach is sufficient to show that this relationship is true for any natural number, n. We leave these familiar induction steps for the reader.

From a geo-arithmetic perspective, we can look at this generalized problem in a richer way. First we consider the square, in Figure 1, with size $(1 + 2 + 3 + \cdots + n)$ x $(1 + 2 + 3 + \cdots + n)$. We divide this large square into smaller squares and rectangles, and calculate the areas of these squares and rectangles based on their dimensions – lengths and widths. But we will add the areas separately based on their placement in groups that we will designate as the Diagonal, Bricked, Vertical-Line, Dotted-Line, and Horizontal-Line regions. Finally, we will demonstrate that the sum of each of these regions is a cube so that the area of the square is the sum of the cubes.

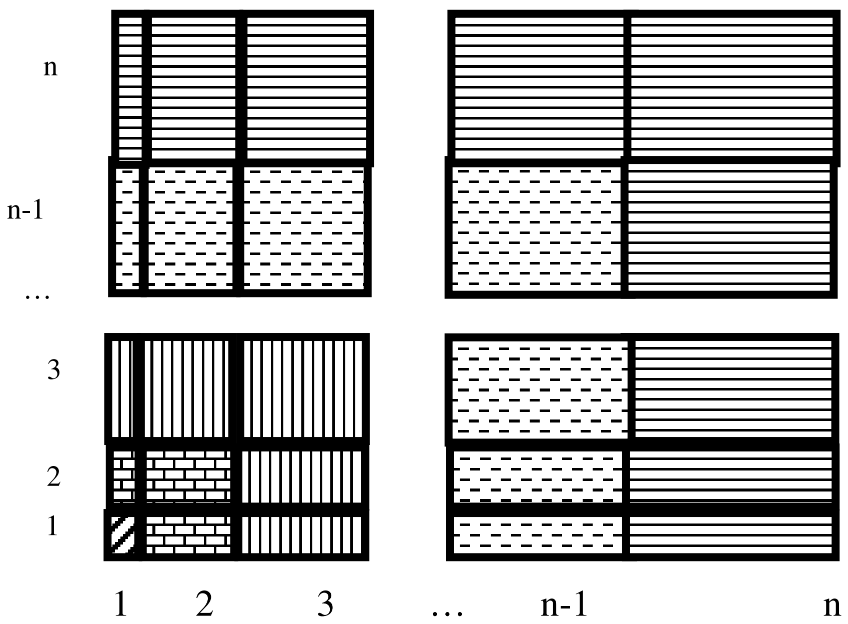

Figure 1: Generalized problem, regions of the square

Sum of the Diagonal Region

$$1 = 1^3$$

Sum of the Bricked Regions

$$1x2 + 2x2 + 2x1 = 2(1+2) + 2x1 = 2((2x3)/2) + 2x1 = 2(3+1) = 2x2^2 = 2^3$$

Sum of the Vertical-Line Regions

$$1x3 + 2x3 + 3x3 + 3x1 + 3x2 = 3(1+2+3) + 3(1+2) = 3[(3x4)/2] + 3[(2x3)/2] =$$
$$3[(3x4+2x3)/2] = 3x3(4+2)/2 = 3^2x3 = 3^3$$

Sum of the Dotted-Line Regions

$$1(n-1) + 2(n-1) + 3(n-1) +\ldots+ (n-1)(n-1) + 1(n-1) + 2(n-1) + 3(n-1) +\ldots+ (n-2)(n-1) =$$
$$(n-1)(1+2+3+\ldots+(n-1)) + (n-1)(1+2+3+\ldots+(n-2)) =$$
$$[(n-1)(n-1)n]/2 + [(n-1)(n-2)(n-1)]/2 = [(n-1)^2n]/2 + [(n-1)^2(n-2)]/2 =$$
$$[(n-1)^2(n+n-2)]/2 = [(n-1)^2(2n-2)]/2 = [(n-1)^22(n-1)]/2 = (n-1)^3$$

Sum of the Horizontal-Line Regions

$$1n + 2n + 3n +\ldots+ n(n-1) + nn + 1n + 2n + 3n +\ldots+ n(n-1) =$$
$$n(1+2+3+\ldots+n) + n(1+2+3+\ldots+n-1) = n[(n(n+1)/2] + n[(n-1)(n)/2] =$$
$$n^2(n+1)/2 + n^2(n-1)/2 = n^2[(n+1)+(n-1)]/2 = n^2(2n)/2 = n^3$$

Now, we have as the sum of the areas of the subdivided square:

+	Sum of the Diagonal Region:	1^3
+	Sum of the Bricked Regions:	2^3
+	Sum for the Vertical-Line Regions:	$3^3 + \ldots$
+	Sum for the Dotted-Line Regions:	$(n-1)^3$
+	Sum for the Horizontal-Line Regions:	n^3
=	Area of the square:	$(1+2+3+\ldots+n)^2 = 1^3 + 2^3 + 3^3 + \cdots + n^3$

A series of other geo-arithmetic problems, similar to this one, was presented to the students over a period of 6 weeks, culminating with the problem above. During the entire semester, students were negotiating these ideas within the small groups of the class, and although many students had valuable insights, we report the thinking of three students as they seemed representative of the students as a whole.

METHODOLOGY

Twenty-eight students (pre-service high school mathematics teachers) from one senior-level mathematical problem solving class participated in the study. Analyzing their responses to Presmeg's (1986) theoretical framework, we determined that some students were non-visual and that others tended to process information visually. Of the three students we chose for interviews, one was visual (Emily) and two non-visual (Ryan and Sara). Students in the class responded to written and oral tasks and questions, and the case studies consisted of students' responses to questions about the classroom activities. In general, the aims of our study were to arrive at a comprehensive understanding of the role of students' personal and idiosyncratic representations in their learning and to develop general theoretical statements about their learning processes.

We explored students' thinking on tasks designed to probe their different ways of understanding and representing series problems. Using multiple sources of qualitative data (e.g., audiotapes of interviews with students, transcripts of those tapes, researchers' fieldnotes, worksheets of case study students, and two researchers' journals), case study analyses were undertaken to identify patterns and changes in students' understanding. In particular, we report how their work on these series problems presented geo-arithmetically influenced the ways they thought about teaching. Analyses of taped sessions included coding of transcripts. We triangulated the data to identify common and distinct strands.

STUDENTS' EXPLORATIONS

As we began investigating these students' geo-arithmetic concepts, assertions in three domains arose from the data: Cognitive Perturbation, Learning Group Dynamics, and Pedagogical Implication. We discuss each of these below with data that support each assertion.

Cognitive Perturbation

Perturbation, although often characterized as negative, is an essential cognitive component of change; to learn and grow, teachers must face cognitive dissonance (Shaw & Jakubowski, 1991). Such dissonance may cause frustration, but can also lead to reflection. We found this task caused students a great deal of reflection as the task was geo-arithmetic and students tended to have a preference toward either the geometric (visual) or the arithmetic (analytic). Thus, non-visual students experienced cognitive dissonance thinking about the visual components, and, similarly, visual students thinking about the analytic (arithmetic) part of the problem saw this as a perturbation.

Ryan, the non-visual thinker above, was initially frustrated by our asking him to solve the series problems geometrically; he said he had always thought "in equations." Ryan said that being confronted with problems presented visually had altered the way he thought about mathematics and his future role as a teacher. But Emily stated that she was

> extremely visual. I have to see things done out; I am sometimes not confident in my mathematical abilities, my algebra skills. I know what I am doing but I am afraid [of mistakes]. If I can do it visually, I know I am on the right track.

She claims that she has a good "3-D mind" and that her "last resort is to write an equation out." She confessed that she looks at problems in creative ways and ways that are "out of the norm." She asserts it "is easier for me to conceptualize it that way." Though Emily was comforted by the blended visual/analytic problem, just because it was partly visual, she found herself mentally challenged as she tied together the visual and analytic aspects of the problem. She said, "I was struggling with the problem algebraically, I did not feel confident in myself."

Ryan said his first approach was to try to write an equation; but Emily's approach was much different. When we asked Emily whether she thought these series problems were algebraic or geometric in nature, she said, "It was a blend for me. You needed to know the algebra behind it, but you had to have that geometry, spatial sense, in order to see the problem." When we asked her how she thought about the problem presented above, she responded, "With the series problems, I had to picture a physical cube, with them lined up next to each other, and figure it out from there."

Sara reported that she found that the inductive proof to be easy, but had "a hard time visualizing it." She said she would "never have thought about the geometric aspect of it." She also stated that it "was confusing to me, and I would still solve them algebraically and then convert it." Recollecting the problem later, after we had given the students cubes for modeling the problems, she said,

> "Once we had the manipulatives, … I can remember working with the actual blocked cubes, colored blocked to build the cubes and then see how they unfolded to make the square. And when I actually had hands-on something to work with, it was a little easier for me to see it, because I wasn't having to depend on my spatial sense."

Here she notes that having physical manipulatives was an aid to her understanding as she had difficulties with mentally picturing the problem. Though the manipulatives were beneficial to her, she still relied on the analytic as her absolute,

> And I still think even though the visual representations were effective, they're not a proof to me. I would still have to do it algebraically for it to verifiably be true in every case.

Learning Group Dynamics

During group activity, Ryan reported he was able to see how some students process information geometrically as he worked through the problems. What was striking was that as a result of the group activities, he felt he would be a better teacher in relating to visual and non-visual learners. "They taught me how to think about a problem so that if you are trying to reach someone who does not think just in numbers, well, you can help the student to see the problem visually."

Sara was also influenced by working within her groups. She said,

> It showed me that there are more visual aspects to math than I ever would have thought…. In the past I tended to rely on algebraic methods to solve problems and now I might be more willing to look at it visually and to think about whether or not my answer makes sense geometrically and visually.

Sara contrasted the way her group partner worked the problem, "She was very visual and I was very non-visual, but together we somehow always seemed to find a solution… we could always find some way to make sense for both of us." She valued working with someone who was a visual thinker,

> I think that if I didn't have someone like that to work with, who looked at it completely different, I would've kept trying the same things over and over and over again, and never have found a solution.

Pedagogical Implication

Our students reported that the activities had altered the way they thought about their future careers in teaching high school. Ryan's experience with the geo-arithmetic problem "opened my eyes to a new way of seeing things that I had never been exposed to before. I consider myself to be not just a better problem solver, but a better teacher seeing how other students are going to see things." Furthermore, he explained,

> Before, I was only thinking of the equations, and I thought everyone else was too. My idea was that everyone was going to learn by my [symbolic] teaching. I wasn't open to visual teaching. Now I'm thinking differently, out of my comfort zone.

Emily reflected on her future teaching practice, "Before these problems, I would have had to just go by the book, teach by breaking the equations down into smaller parts algebraically." As a result of doing these geo-arithmetic problems, she asserted,

> I want to try to incorporate this (visual aspects) into my teaching, into as many lessons as possible. Because I now know I am that kind of thinker (visual), I know there are others like me. Based on this I want to try to accommodate all the different kinds of thinking. I will have to teach it purely algebraically for those who don't think visually. I want to try to incorporate as much visual as I can, and that will help the algebra (analytic) people to see it differently too. Maybe I can create a future engineer. And the people who are visual need to know the numbers, how the equations work and not have to see it visually.

Emily clearly saw a need to provide a balanced approach in teaching students both the analytic and the visual components of problems. Sara stated that,

> In teaching, definitely, I think that I would use more visual aspects, because at least for me as a student it was easier to see why things made sense, because you could visually look at it and tell, as opposed to algebraic methods where you had to think about it and see if it reasoned out.

Since Sara states that she is non-visual, we asked Sara specifically, "What are the ramifications for the non-visual students if she presented something visually?" Sara responded,

> I think you would have to show it the algebraic way, the inductive way, the proof way, and then show it visually to kind of illustrate why it works. And I think that, at least for me, as a non-visual thinker, it still made sense for me to look at it visually.

CONCLUSIONS

We believe students develop mathematical power by learning to recognize an idea embedded in a variety of different representational systems and to translate the idea from one mode of representation to another. A positive result of multiple instructional representations of concepts is that students who are prospective teachers learn to construct and to present representational schemes with which they might not be comfortable.

The geo-arithmetic problems had positive implications for each student in class and in particular, the three students that have been mentioned in this paper. The problems, along with the group interactions caused students to reflect on how they think, whether it be predominantly visual or analytic. They were able to see from their colleagues that not everyone thinks they way they do. The pedagogical discussions were rich in that these prospective teachers began to describe how they might deal with various modes of students' representations in their own classes, especially students who may have a predominance that differs from theirs. The authors intend to continue to investigate how geo-arithmetic problems positively perturb prospective mathematics teachers in their own thinking about mathematics learning and what impact these problems may have on their pedagogical content knowledge.

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