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EXAMPLES OF HYPER-KÄHLER CONNECTIONS WITH TORSION

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1. It has been known for a fairy long time that when the Wess-Zumino term is present in the $N = 4$ supersymmetric one-dimensional sigma model, the internal space has a torsionful linear connection with holonomy in $\mathrm{Sp}(n)$ [3]. Such geometry also arises when one considers T-duality of toric hyper-Kähler manifolds [5]. In this conference, G. Papadopoulos explains the role of HKT-geometry in M-theory, or more specifically in IIA Superstring theory [10].

In this lecture, we discuss HKT-geometry entirely from a mathematical point of view, and present several methods to produce series of examples that may interest mathematicians.

2. The background object of HKT-geometry is a hyper-Hermitian manifold. Three complex structures I_1, I_2 and I_3 on a smooth manifold M form a hypercomplex structure if

$$I_1^2 = I_2^2 = I_3^2 = -1, \quad \text{and} \quad I_1 I_2 = I_3 = -I_2 I_1. \tag{0.1}$$

A triple of such complex structures is equivalent to the existence of a 2-sphere worth of integrable complex structures: $\mathcal{I} = \{a_1 I_1 + a_2 I_2 + a_3 I_3 : a_1^2 + a_2^2 + a_3^2 = 1\}$. When g is a Riemannian metric on the manifold M such that it is Hermitian with respect to every complex structure in the hypercomplex structure, $(M, \mathcal{I}, g)$ is called a hyper-Hermitian manifold.

On a hyper-Hermitian manifold, there are two natural torsion-free connections, namely the Levi-Civita connection and the Obata connection. However, in general the Levi-Civita connection does not preserve the hypercomplex structure and the

Obata connection does not preserve the metric. We are interested in the following type of connections.

Definition 1. A linear connection ∇ on a hyper-Hermitian manifold $(M, \mathcal{I}, g)$ is hyper-Hermitian if $\nabla g = 0$, and $\nabla I_1 = \nabla I_2 = \nabla I_3 = 0$.

Although a general hyper-Hermitian connection has torsion, physical requirement limits our discussion to a special type of hyper-Hermitian connections. We follow physicists' conventions of definitions. Recall that when T^∇ is the torsion tensor for a connection ∇, we can construct the $(3,0)$-tensor $c(X, Y, Z) = g(X, T^\nabla(Y, Z))$.

Definition 2. A linear connection ∇ on a hyper-Hermitian manifold $(M, \mathcal{I}, g)$ is hyper-Kähler with torsion (HKT) if it is hyper-Hermitian and its torsion $(3,0)$-tensor is totally skew-symmetric. It is a strong HKT-connection if its torsion 3-form is closed.

Example 1. Let q be the quaternion coordinate for the one-dimensional quaternion module $\mathbf{H}$. Through left multiplications by the unit quaterions i, j and k, one obtains a hypercomplex structure $\mathcal{I}$ on $\mathbf{H}$. The Euclidean metric $g = dqd\overline{q}$ is hyper-Kähler. The Levi-Civita connection is a HKT-connection.

A less obvious and more relevant example for us is to consider the following metric on $\mathbf{H}\backslash\{0\}$.

$$\hat{g} = \frac{dqd\overline{q}}{|q\overline{q}|}. \tag{0.2}$$

Considering the diffeomorphis $\mathbf{H}\backslash\{0\} = \mathbf{R}^+ \times S^3$, we choose a spherical coordinate (r, θ, ϕ, ψ). Let g_S be the metric on the round unit-sphere. Then

$$\hat{g} = \frac{dr^2}{r^2} + g_S. \tag{0.3}$$

Now, $(\mathbf{H}\backslash\{0\}, \mathcal{I}, \hat{g})$ is a HKT-structure. The torsion form c is the volume form of the sphere S^3. It is also a closed 3-form.

If one chooses to study the Hermitian geometry for one of the complex structures J in the hypercomplex structure, one should note that Gauduchon found a collection of canonical Hermitian connections on any Hermitian manifold. The collection forms an affine subspace of the space of linear connections [4]. This collection of Hermitian connections include Chern connection and Lichnerowicz's first canonical connection. Within this family, there exists exactly one connection whose torsion $(3,0)$-tensor is a 3-form. To describe it, we recall the following definitions and convention. For any n-form ω, $d^c\omega := (-1)^n JdJ\omega$ where $(J\omega)(X_1, \ldots, X_n) := (-1)^n\omega(JX_1, \ldots, JX_n)$. Then $\partial = \frac{1}{2}(d + id^c)$ and $\overline{\partial} = \frac{1}{2}(d - id^c)$. By [4], the Hermitian connection with totally skew-symmetric torsion $(3,0)$-tensor c is uniquely determined by the following identity.

$$c(X, Y, Z) = -\frac{1}{2}d^cF(X, Y, Z), \tag{0.4}$$

where $F(X,Y) = g(JX,Y)$ is the Kähler form for the complex structure J.

Now the HKT-connection serves as such a unique connection for each complex structure in the hypercomplex structure. Therefore, if we use F_a and d_a to represent the Kähler form and complex exterior differential for the complex structure I_a, $a = 1,2,3$, we have the following observation.

Proposition 1. A hyper-Hermitian manifold $(M, \mathcal{I}, g)$ admits a HKT-connection if and only if $d_1 F_1 = d_2 F_2 = d_3 F_3$. If it exists, it is unique.

In view of the uniqueness, we say that $(M, \mathcal{I}, g)$ is a HKT-structure if it admits a HKT-connection.

Example 2. A non-trivial class of HKT-structures can be found on semi-simple Lie groups and homogeneous spaces [13] [9]. For instance, the Killing-Cartan form $-B$ on the Lie group $\mathrm{SU}(2n+1)$ defines a bi-invariant metric $g = -B$. This group has a left-invariant hypercomplex structure $\mathcal{I}$ so that with the bi-invariant metric g, it forms a HKT-structure. The HKT-connection is the left-invariant connection defined by having all left-invariant vector fields to be parallel. The torsion of this connection is the Lie bracket, and the torsion tensor $c(X,Y,Z) = -B(X,[Y,Z])$ is totally skew-symmetric. Similar constructions can be applied to $\mathrm{U}(1) \times \mathrm{SU}(2n)$ and other homogeneous spaces.

3. To further our analysis of HKT-geometry, we note a holomorphic characterization of HKT-structures.

Proposition 2. Let $(M, \mathcal{I}, g)$ be a hyper-Hermitian manifold and F_a be the Kähler form for (I_a, g). Then $(M, \mathcal{I}, g)$ is a HKT-structure if and only if $\partial_1(F_2 + iF_3) = 0$; or equivalently $\overline{\partial}_1(F_2 - iF_3) = 0$.

Applying this proposition to any complex structure in the given hypercomplex structure, one obtains a section of twisted 2-form on the twistor space of the hypercomplex structures. However, this 2-form is only J_2-holomorphic in the sense of Eells-Salamon [2]. Since the almost complex structure J_2 is never integrable [2], we shall concentrate on the holomorphic characterization given above and ignore the twistor characterization.

Due to the absence of type $(3,0)$-form with respect to any complex structure on any real four-dimensional manifold, it is now apparent that any four-dimensional hyper-Hermitian manifold is a HKT-structure.

The holomorphic characterization also yields new examples of HKT-structures.

Example 3. Let $\{X_1, ..., X_{2n}, Y_1, ..., Y_{2n}, Z\}$ be a basis for $\mathbf{R}^{4n+1}$. Define commutators by $[X_i, Y_i] = 4Z$, and all others are zero. These commutators define on $\mathbf{R}^{4n+1}$ the structure of the Heisenberg Lie algebra $\mathfrak{h}$. Let $\mathbf{R}^3$ be the 3-dimensional Abelian algebra. The direct sum $\mathfrak{n} = \mathfrak{h} \oplus \mathbf{R}^3$ is a 2-step nilpotent algebra whose center is four-dimensional. Fix a basis $\{E_1, E_2, E_3\}$ for $\mathbf{R}^3$. Consider the endomorphisms I_1,

I_2 and I_3 of $\mathfrak{n}$ defined by left multiplications of the quaternions i, j and k on the module of quaternions $\mathbf{H}$, and the identifications

$$(0.5)\qquad \begin{aligned} x_0X_{2a-1}+x_1X_{2a-1}+x_2Y_{2a-1}+x_3Y_{2a} &\to x_0+x_1i+x_2j+x_3k;\\ x_0Z+x_1E_1+x_2E_2+x_3E_3 &\to x_0+x_1i+x_2j+x_3k. \end{aligned}$$

Through left translations, these endomorphisms define almost complex structures on the product of the Heisenberg group and the Abelian group $N=H\times\mathbf{R}^3$. It is clear from the definition that these almost complex structures satisfy the algebra (0.1). Moreover, for $a=1,2,3$ and $X,Y\in\mathfrak{n}$,

$$(0.6)\qquad [I_aX, I_aY]=[X,Y]$$

so I_a are Abelian complex structures on $\mathfrak{n}$ in the sense of [1]. In particular, they are integrable. It implies that $\{I_a : a=1,2,3\}$ is a left-invariant hypercomplex structure on the Lie group N. It is known [12] that the complex structures I_a on $\mathfrak{n}$ satisfy $d(\Lambda^{1,0}_{I_a}\mathfrak{n}^*)\subset\Lambda^{1,1}_{I_a}\mathfrak{n}^*$ where $\mathfrak{n}^*$ is the space of left-invariant 1-forms on N and $\Lambda^{i,j}_{I_a}\mathfrak{n}^*$ is the (i,j)-component of $\mathfrak{n}^*\otimes\mathbf{C}$ with respect to I_a. But then we have $d(\Lambda^{2,0}_{I_a}\mathfrak{n}^*)\subset\Lambda^{2,1}_{I_a}\mathfrak{n}^*$ and any left-invariant (2,0)-form is ∂_1-closed. Now consider the invariant metric on N for which the basis $\{X_i, Y_i, Z, E_a\}$ is orthonormal. Since it is compatible with the complex structures I_a, in view of the holomorphic characterization of HKT-geometry, we obtain a left-invariant HKT-structure on N.

Example 4. Based on the above computation, we could also see that there is a left-invariant HKT-structure on the product of the $4n+1$-dimensional Heisenberg group and the compact simple Lie group SU(2), an interesting mixture of the last example and Example 2.

Recall that the underlying manifold of the Heisenberg group H_{4n+1} is the manifold $\mathbf{R}^{4n+1}$. Consider it as the product space $\mathbf{R}^{2n}\times\mathbf{R}^{2n}\times\mathbf{R}$, the group law for the Heisenberg group is

$$(0.7)\qquad (\vec{x},\vec{y},z)*(\vec{x}',\vec{y}',z')=(\vec{x}+\vec{x}',\vec{y}+\vec{y}',z+z'+2\sum_{i,j=1}^{2n}(x_iy'_j-y_ix'_j)).$$

The 1-forms $\alpha_j=dx_j,\beta_j=dy_j,\gamma=dz+2\sum(y_jdx_i-x_jdy_i)$ are left-invariant. Let $\{X_j,Y_j,Z\}$ be the dual left-invariant vector fields.

On SU(2), choose left-invariant vector fields A_1, A_2 and A_3 such that $[A_1,A_2]=2A_3$, etc., then the dual left-invariant 1-forms σ_1, σ_2 and σ_3 satisfy the identities

$$(0.8)\qquad d\sigma_1=2\sigma_2\wedge\sigma_3, d\sigma_2=2\sigma_3\wedge\sigma_1, d\sigma_3=2\sigma_1\wedge\sigma_2.$$

Now, using $\{A_1,A_2,A_3\}$ instead of $\{E_1,E_2,E_3\}$, we define endomorphisms I_1, I_2 and I_3 on $\mathfrak{h}\oplus\mathfrak{su}(2)$ as in (0.5). Through left translation, we define three almost complex structures on the product group $H\times\mathrm{SU}(2)$ satisfying the identities (0.1). To prove that these almost complex structures are integrable, one first notes that when $\mathfrak{c}$ is the center of the Heisenberg algebra, then the vector space $\mathfrak{h}\oplus\mathfrak{su}(2)$

has a direct sum decomposition $\mathfrak{t}_{4n} \oplus \mathfrak{c} \oplus \mathfrak{su}(2)$ where $\mathfrak{t}_{4n}$ is the linear span of all the X_j and Y_j. On $\mathfrak{t}_{4n}$, the almost complex structures satisfy the identity (0.6). Therefore, the Nijenhuis tensor vanishes on $\mathfrak{t}_{4n}$. On $\mathfrak{c} \oplus \mathfrak{su}(2)$, the almost complex structures are the standard ones for $\mathbf{H}\backslash\{0\}$. Therefore, the Nijenhuis tensor vanishes on this summand. Since $\mathfrak{c} \oplus \mathfrak{su}(2)$ commutes with $\mathfrak{t}_{4n}$, and both $\mathfrak{t}_{4n}$ and $\mathfrak{c} \oplus \mathfrak{su}(2)$ are invariant of the endomorphisms I_1, I_2 and I_3, the Nijenhuis tensor vanishes completely. Therefore, the left-invariant almost complex structures I_1, I_2 and I_3 define a left-invariant hypercomplex structure on the product group $H \times \mathrm{SU}(2)$. However the hypercomplex structure is no longer Abelian.

We define a left-invariant metric g on the product group by requiring the left-invariant vector fields $\{X_1, ..., X_{2n}, Y_1, ..., Y_{2n}, Z, E_1, E_2, E_3\}$ to be an orthonormal frame. Equivalently,

$$g = \sum_{a=1}^{n} (\alpha_{2a-1}^2 + \alpha_{2a}^2 + \beta_{2a-1}^2 + \beta_{2a}^2) + \gamma^2 + \sigma_1^2 + \sigma_2^2 + \sigma_3^2.$$

This metric, along with the left-invariant hypercomplex structure, is a HKT-structure on the product group $H \times \mathrm{SU}(2)$. Indeed, when F_1, F_2 and F_3 are the three Kähler forms for the complex structures I_1, I_2 and I_3, $dF_a = 2i(d\gamma \wedge \sigma_a - \gamma \wedge \sigma_b \wedge \sigma_c)$, where (abc) is any even permutation of (123). Since $d\gamma = -4\sum_{a=1}^{n}(\alpha_{2a-1} \wedge \beta_{2a-1} + \alpha_{2a} \wedge \beta_{2a})$,

$$I_1 dF_1 = I_2 dF_2 = I_3 dF_3 = -2i(\gamma \wedge d\gamma + \sigma_1 \wedge \sigma_2 \wedge \sigma_3). \tag{0.9}$$

Therefore, we have a HKT-structure. Since the torsion 3-form $c = i(\gamma \wedge d\gamma + \sigma_1 \wedge \sigma_2 \wedge \sigma_3)$

$$dc = i d\gamma \wedge d\gamma. \tag{0.10}$$

This is not a closed 3-form, the corresponding HKT-structure is weak.

4. The holomorphic characterization shows that the form $F_2 + iF_3$ has a locally defined $(1,0)$-form β as its potential. Although the $(0,1)$-form $I_2\beta$ is not a priori ∂_1-closed, we consider the case when it is. From this observation, we extract the following definition.

Definition 3. Let $(M, \mathcal{I}, g)$ be a HKT-structure with Kähler forms F_1, F_2 and F_3. A possibly locally defined function μ is a potential function for the HKT-structure if

$$F_1 = \frac{1}{2}(dd_1 + d_2 d_3)\mu, \quad F_2 = \frac{1}{2}(dd_2 + d_3 d_1)\mu, \quad F_3 = \frac{1}{2}(dd_3 + d_1 d_2)\mu. \tag{0.11}$$

Referring to the holomorphic characterization of HKT-geometry, we reformulate the definition of HKT-potential in the following way.

Proposition 3. Let $(M, \mathcal{I}, g)$ be a HKT-structure with Kähler form F_1, F_2 and F_3. A possibly locally defined function μ is a potential function for the HKT-structure if

$$F_2 + iF_3 = 2\partial_1 I_2 \overline{\partial}_1 \mu. \tag{0.12}$$

Example 5. On the complex vector space $(\mathbf{C}^n \oplus \mathbf{C}^n)\backslash\{0\} \cong \mathbf{H}^n\backslash\{0\}$, let (z_α, w_α), $1 \leq \alpha \leq n$, be its coordinates. Define a hypercomplex structure $\mathcal{I}$ by right multiplication of the pure quaternions i, j and k. Let g be the flat metric. It is a hyper-Kähler metric with hyper-Kähler potential $\mu = \frac{1}{2}(|z|^2 + |w|^2)$. Consider a new metric

$$\hat{g} = \frac{1}{\mu}g - \frac{1}{4\mu^2}(d\mu \otimes d\mu + I_1 d\mu \otimes I_1 d\mu + I_2 d\mu \otimes I_2 d\mu + I_3 d\mu \otimes I_3 d\mu).$$

Then the hyper-Hermitian structure $(\mathcal{I}, \hat{g})$ is a HKT-structure. Moreover, the function $\ln(\mu)$ is its potential.

Now, for any real number r, with $0 < r < 1$, and $\theta_1, \dots, \theta_n$ modulo 2π, we consider the integer group $< \gamma >$ generated by the following action on $(\mathbf{C}^n \oplus \mathbf{C}^n)\backslash\{0\}$.

$$\gamma(z_\alpha, w_\alpha) = (re^{i\theta_\alpha} z_\alpha, re^{-i\theta_\alpha} w_\alpha). \tag{0.13}$$

Since γ is a hyper-holomorphic isometry, the HKT-structure on $(\mathbf{C}^n \oplus \mathbf{C}^n)\backslash\{0\}$ descends to a HKT-structure on the quotient space with respect to the group $< \gamma >$. As this quotient space is diffeomorphic to $S^1 \times S^{4n-1}$ [11], and the quotient hypercomplex structure is not homogeneous, we obtain a family of inhomogeneous HKT-structures on the manifold $S^1 \times S^{4n-1}$.

It should be noted that this method of generating HKT-geometry through a transformation from HKT-potentials to HKT-potentials can easily generate large classes of inhomogeneous HKT-structures on homogeneous manifolds especially when we start from well known hyper-Kähler metrics with hyper-Kähler potentials.

Remark To produce more examples, one may develop a reduction theory along the line of hyper-Kähler reduction [7]. One can also prove that Joyce's twist construction of hypercomplex manifolds [8] carries HKT-manifolds to HKT-manifolds. We do not present details of these theories here. Details of our work can be found in [6].

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