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MATHEMATICAL ASPECTS IN CELESTIAL MECHANICS, THE LAGRANGE AND EULER PROBLEMS IN THE LOBACHEVSKY SPACE

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Abstract. The generalization of a test particle motion in a central field of the two immovable point-like centers to the case of a constant curvature space, in the space of Lobachevsky, is studied in the paper. The bifurcation set in the plane of integrals of motion was constructed and the classification of the domains of possible motion was carried out. The Lagrange's problem on the pseudosphere: a mass point motion under the action of attracting center field and the analogue of a constant homogeneous field in a constant curvature space, is studied as well.

1. Introduction

For the first time the problem on the motion of dynamical systems in a constant curvature spaces was formulated by N. I. Lobachevsky. He generalized the Newton law of attraction for the space of negative curvature. Later E. Schrödinger found the spectrum of hydrogen atom (the Kepler problem on a three-dimensional sphere $\mathbb{S}^3$). Trajectories and modification of the Kepler laws of the body motion in the field of Newton's potential were investigated by P. W. Higgs [1]. The generalization of the Kepler laws for the classical problem to the spaces $\mathbb{S}^3$ and $\mathbb{H}^3$ (here $\mathbb{H}^3$ is an upper sheet of hyperboloid embedded into the Minkowski space) is given in the work of N. A. Chernikov [2]. The classification of the motion for the plane case was carried out by C. L. Charlier [3]. However his analysis turned out to be incomplete and partially incorrect so it was corrected twice by H. J. Tallqvist [4]and T. K. Badalyan [5] In the paper of V. V. Kozlov and O. A. Harin [6] the full integrability of the generalized Euler problem of the motion of a particle in the field of two fixed centers was proved for a two-dimensional sphere. For a sphere S^2 the potential has the following form $V = -\gamma_1 \cot\theta_1 - \gamma_2 \cot\theta_2$, where θ_i is a length of the arc of the great circle, connecting the tentative particle and the gravitation centers, γ_i are constant.

In the present paper the integrability of the Euler and Lagrange problems in the space $\mathbb{H}^3$ is proved. The classification of the domains of possible motion on the pseudosphere, which allows one to answer the question about the trajectory equivalence of the Euler and Lagrange problems in the plane and curved spaces, is carried out. The influence of a curvature of space on the integrability of the problem on the motion of a mass point in a constant curvature spaces is investigated.

2. The Two-Center Problem on the Pseudosphere

Let us consider an upper sheet of hyperboloid H^3 embedded into the four-dimensional Minkowski space $\mathbb{M}^4$ described by the coordinates x^1, x^2, x^3, x^4. Its equation is

$$(x^1)^2 - (x^2)^2 - (x^3)^2 - (x^4)^2 = 1\,. \tag{2.1}$$

Let the two fixed attracting centers be placed at the points with coordinates $r_1 = (\beta, \alpha, 0, 0)$ and $r_2 = (\beta, -\alpha, 0, 0)$ with $\beta^2 - \alpha^2 = 1$ (this can be achieved by a motion in H^3). The potential energy of a mass point in the field of these centers is

$$U = -\gamma_1 \coth\theta_1 - \gamma_2 \coth\theta_2\,, \tag{2.2}$$

where θ_i is an angle between radius-vectors connecting the test particle and one of the gravitational center. Here

$$\cosh\theta_i = \langle x, r_i\rangle = x^1 x_i^1 - x^2 x_i^2 - x^3 x_i^3 - x^4 x_i^4$$

$\langle\cdot\,,\cdot\rangle$ is a scalar product in the Minkowski space.

Let us pass to the pseudospherical coordinates. The transformation formulae are as follows

$$\begin{aligned} x^1 &= \cosh\theta\,, \qquad & x^2 &= \sinh\theta\cos\varphi\,, \\ x^3 &= \sinh\theta\sin\varphi\cos\psi\,, \qquad & x^4 &= \sinh\theta\sin\varphi\sin\psi\,. \end{aligned} \tag{2.3}$$

The metric induced on the unit three-dimensional pseudosphere is

$$\mathrm{d}s^2 = \mathrm{d}\theta^2 + \sinh^2\theta\,\mathrm{d}\varphi^2 + \sinh^2\theta\sin^2\varphi\,\mathrm{d}\psi^2. \tag{2.4}$$

Relative to these coordinates, the Lagrangian L is given by

$$L = \frac{1}{2}(\dot{\theta}^2 + \sinh^2\theta(\dot{\varphi}^2 + \sin^2\varphi\dot{\psi}^2) - U\,. \tag{2.5}$$

Theorem. *A mass point moves in the two-center problem in the space* $\mathbb{H}^3$ *in the same way as in the two-dimensional system (on the unit two-dimensional pseudosphere* $H^2 = y^2 - x^2 - z^2 = 1$*) with energy*

$$E = \frac{1}{2}(\dot{y}^2 - \dot{x}^2 - \dot{z}^2) + U_{\text{eff}}\,,$$

where the effective potential energy is defined by the formula

$$U_{\text{eff}} = -\gamma_1 \coth\theta_1 - \gamma_2 \coth\theta_2 + \frac{\beta_\varphi^2}{2z^2}\,.$$

Proof: Let us introduce a new variables (x, y, z, φ) by formulas

$$x^1 = x, \quad x^2 = y\,, \qquad x^3 = z\cos\varphi\,, \qquad x^4 = z\sin\varphi\,. \tag{2.6}$$

Then we have the Lagrangian

$$L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2 + z^2\dot{\varphi}^2) + \gamma_1\cot\theta_1 + \gamma_2\cot\theta_2\,. \tag{2.7}$$

It is obvious that the coordinate φ is ignorable. Let us introduce the constant of the angular momentum integral

$$\frac{\partial L}{\partial\dot{\varphi}} = z^2\dot{\varphi} = \beta_\varphi$$

and write out Routh's function

$$R = \frac{1}{2}(\dot{y}^2 - \dot{x}^2 - \dot{z}^2) - U_{\text{eff}}$$

where

$$U_{\text{eff}} = U + \frac{\beta_\varphi^2}{2z^2}\,.$$

We obtained the full energy of the system in the form

$$E = \frac{1}{2}(\dot{y}^2 - \dot{x}^2 - \dot{z}^2) + U + \frac{\beta_\varphi^2}{2z^2}\,. \tag{2.8}$$

Thus, we reduced our problem to the problem of the motion of a mass point on the unit two-dimensional pseudosphere $H^2 = y^2 - x^2 - z^2 = 1$ in the field with reduced potential. The theorem is proved. □

Let us perform the change of coordinates allowing to separate the variables. In this problem it is natural to pass from Cartesian coordinates to the pseudospheroconical coordinates ξ and η. We define ξ and η as the roots of the equation

$$f(\lambda) = \frac{x^2}{\lambda - \alpha^2} - \frac{y^2}{\lambda^- \beta^2} + \frac{z^2}{\lambda} = \frac{-(\lambda - \xi^2)(\lambda + \eta^2)}{\lambda(\lambda - \alpha^2)(\lambda - \beta^2)} . \tag{2.9}$$

The coordinate surfaces are described by the equations

$$\frac{x^2}{\xi^2 - \alpha^2} - \frac{y^2}{\xi^2 - \beta^2} + \frac{z^2}{\xi^2} = 0; \qquad \frac{x^2}{\eta^2 + \alpha^2} - \frac{y^2}{\eta^2 + \beta^2} + \frac{z^2}{\eta^2} = 0; \\ y^2 - x^2 - z^2 = 1. \tag{2.10}$$

Therefore, the coordinate lines are the lines of intersection of the pseudosphere and confocal cones. The formulae of coordinate transformation from Cartesian coordinates to the coordinates ξ, η can be found from (2.9)

$$\begin{aligned} x &= \operatorname{sign}(x) \frac{\sqrt{(\alpha^2 - \xi^2)(\alpha^2 + \eta^2)}}{\alpha} , \\ y &= \frac{\sqrt{(\beta^2 - \xi^2)(\beta^2 + \eta^2)}}{\beta} , \\ z &= \frac{\xi\eta}{\alpha\beta} . \end{aligned} \tag{2.11}$$

In this case:

$$-\alpha < \xi < \alpha , \qquad 0 < \eta < \infty . \tag{2.12}$$

Coordinates ξ and η are orthogonal in meaning of the metric (2.4).

The Hamiltonian function relative to the new coordinates is given by

$$H = \frac{1}{2} \left[\frac{(\alpha^2 - \xi^2)(\beta^2 - \xi^2)}{\xi^2 + \eta^2} p_\xi^2 + \frac{(\alpha^2 + \eta^2)(\beta^2 + \eta^2)}{\xi^2 + \eta^2} p_\eta^2 \right] + U_{\text{eff}} , \tag{2.13}$$

where

$$\begin{aligned} U_{\text{eff}} = & - \frac{\operatorname{sign}(x)(\gamma_1 - \gamma_2)\sqrt{(\alpha^2 - \xi^2)(\beta^2 - \xi^2)}}{\xi^2 + \eta^2} \\ & - \frac{(\gamma_1 + \gamma_2)\sqrt{(\alpha^2 + \eta^2)(\beta^2 + \eta^2)}}{\xi^2 + \eta^2} + \frac{\beta_\varphi^2 \alpha^2 \beta^2 (\xi^{-2} + \eta^{-2})}{2(\xi^2 + \eta^2)} . \end{aligned} \tag{2.14}$$

The sign "+" corresponds to the range $x > 0$ and the sign "−" refers to the range $x < 0$. This ambiguity is connected with different way of taking the root when we express x from the equation (2.9).

In order to find the integrals of motion let us define the Liouville system.

Definition. *A dynamical system is called a Liouville system if there exist coordinates q_i in which the Hamiltonian is written in the form*

$$H = T + U,$$

where the kinetic and potential energy have the form

$$T = \frac{1}{2} C \sum_{j=1}^{n} \frac{\dot{q}_j^2}{a_j} = \frac{1}{2C} \sum_{j=1}^{n} a_j p_j^2 \,, \qquad U = \frac{1}{C} \sum_{j=1}^{n} u_j \,, \qquad C = \sum_{j=1}^{n} c_j \,,$$

where the functions a_j, c_j, u_j depend on variables q_j. For these systems

$$\dot{q}_j = \frac{a_j}{C} p_j \,.$$

Such systems can be easily integrated. It is easy to check that the functions

$$I_j = \frac{1}{2} a_j p_j^2 + u_j - H c_j, \quad j = 1, 2, \ldots, n$$

are integrals of motion. These integrals are called Liouville integrals. It should be noted that only $n - 1$ integrals are independent since

$$\sum_{j=1}^{n} I_j = 0.$$

Accounting the Hamiltonian H we have n integrals of motion. It is obvious that all these integrals are in involution

$$\{I_j, I_k\} = 0,$$

and, therefore, the system is completely integrable in Hamilton's sense.

Our system is obviously a Liouville system, and we can write out the integrals of motion.

Thus, the full integrability of the problem on the motion of a mass point in the field of two centers in the space $\mathbb{H}^3$ is proved. It is obvious that the curvature of hyperboloid does not influence the integrability of the considered system.

3. The Qualitative Analysis of a Mass Point Motion in the Two-Center Problem on the Pseudosphere

Let us carry out the qualitative investigation of a material point motion on the Lobachevsky's plane in the field of two immovable centers. The qualitative investigation of the system is of great significance, because expressions of precise solutions are rather complicated and so do not give any visual understanding of how a test particle moves.

Let us consider a two-dimensional upper sheet of hyperboloid included into the space of Minkowski $\mathbb{M}^3$. We will study the trajectory equivalence of the two-center problem on the plane and pseudosphere.

In order to reduce the problem to the quadratures, let us use the following integrals of the motion: the energy integral

$$h = T + V \tag{3.1}$$

and the two Liouville's integrals

$$\begin{aligned} I_1 &= \frac{1}{2}\,\frac{\dot{\xi}^2(\xi^2+\eta^2)^2}{(\alpha^2-\xi^2)(\beta^2-\eta^2)} + \operatorname{sign}(x)\sqrt{(\alpha^2-\xi^2)(\beta^2-\xi^2)}(\gamma_1-\gamma_2) - h\xi^2 \\ I_2 &= \frac{1}{2}\,\frac{\dot{\eta}^2(\xi^2+\eta^2)^2}{(\alpha^2+\eta^2)(\beta^2+\eta^2)} + \sqrt{(\alpha^2+\eta^2)(\beta^2+\eta^2)}(\gamma_1+\gamma_2) - h\eta^2 \end{aligned} \tag{3.2}$$

(the sign"+" coresponds the range $x > 0$, the sign "$-$" refers to the range $x < 0$). As $I_1 + I_2 = 0$, let us denote $I_1 = l = -I_2$.

By introducing instead of t the new independent variable using the substitution $\mathrm{d}t = \dfrac{\xi^2+\eta^2}{\sqrt{2(\gamma_1+\gamma_2)}}\,\mathrm{d}\tau$ and making the change $h \to \dfrac{h}{\gamma_1+\gamma_2}$, $l \to \dfrac{l}{\gamma_1+\gamma_2}$.

Let us reduce the motion equation to the form

$$\frac{\mathrm{d}\xi}{\mathrm{d}\tau} = \sqrt{R_\pm(\xi)}\,; \qquad \frac{\mathrm{d}\eta}{\mathrm{d}\tau} = \sqrt{S(\eta)}\,, \tag{3.3}$$

where $R_\pm(\xi)$ and $S(\eta)$ are the irrational functions

$$\begin{aligned} R_\pm(\xi) &= (\alpha^2-\xi^2)(\beta^2-\xi^2)R^*_\pm(\xi)\,, \\ S(\eta) &= (\alpha^2+\eta^2)(\beta^2+\eta^2)S^*(\eta)\,, \end{aligned} \tag{3.4}$$

and define

$$\begin{aligned} R^*_\pm(\xi) &= l + h\xi^2 + \operatorname{sign}(x)K\sqrt{(\alpha^2-\xi^2)(\beta^2-\xi^2)}\,, \\ S^*(\eta) &= \left(-l + h\eta^2 + \sqrt{(\alpha^2+\eta^2)(\beta^2+\eta^2)}\right)\,. \end{aligned}$$

Here $K = \dfrac{\gamma_1 - \gamma_2}{\gamma_1 + \gamma_2}$ is a parameter, h and l are the integrals depending on the initial conditions. The functions R_+ and R_- describe the motion in the ranges $x > 0$ and $x < 0$ respectively: on reaching of the value $\pm\alpha$ by coordinate ξ a particle passes from one range to another (see Eq. (2.11)). At these values the function R describing the variation of ξ have to be changed. It is clear that the function S is the same in the both ranges.

Analysis of the functions satisfying the equations (3.3) and, therefore investigation of the properties of motion of a mass point is based on the obvious states. In any real motion of a mass point the pseudospherical coordinates and their derivatives being real in the initial moment of time remain real for any values of t (or τ). Therefore, in the domains where the motion is possible the conditions $R(\xi) \geq 0$, $S(\eta) \geq 0$ must be fulfilled. Besides, the conditions (2.12) must be fulfilled as well. The signs of the irrational functions depend on mutual positions of the initial values ξ_0 and η_0 and the corresponding roots of the functions.

Let us introduce some definitions.

Let $v = \operatorname{sgrad} h$ be a Hamiltonian system, integrable in the Liouville sense, on a symplectic manifold M^{2n}. Let the system v has a complete set of commuting integrals $f_1 = H, f_2, \ldots, f_n$ such that $\{f_i, f_j\} = 0$ and all functions are independent on M^{2n}.

By

$$F : M^{2n} \to \mathbb{R}^n$$

we denote the smooth mapping defined by

$$F(x) = (f_1(x), \ldots, f_n(x)) .$$

Definition. *The mapping F is called the momentum mapping of the system.*

Definition. *The point $x \in M^{2n}$ is called a critical point for the mapping F if rank $\mathrm{d}F(x) < n$. The image of the point $F(x)$ in $\mathbb{R}^n$ is called a critical value.*

Let K be the set of all critical points for the mapping F.

Definition. *The image of F, i. e., the set*

$$\Sigma = F(K) \subset \mathbb{R}^n$$

is called the bifurcation diagram of the momentum mapping.

Thus, the bifurcation diagram is the set of all critical values of the momentum mapping F.

Let us construct the bifurcation set in the plane of the integrals h and l, i. e. the set h and l, at which the form of the functions R and S changes. They specify the bifurcations.

$$\gamma_1\colon 4h^2\alpha^2\beta^2 + 4lh(\alpha^2+\beta^2) + K^2 + 4l^2 = 0\,,$$
$$\gamma_2\colon 4h^2\alpha^2\beta^2 + 4lh(\alpha^2+\beta^2) + 1 + 4l^2 = 0\,,$$
$$\gamma_3\colon l = -K\alpha\beta, \quad \gamma_4\colon h=-1, \quad \gamma_5\colon l=\alpha\beta\,, \quad \gamma_6\colon l = K\alpha\beta\,.$$

The bifurcation diagram is presented in Fig. 1. The domains of possible motion are denoted by the numbers, the domains of prohibited motion are shaded by vertical lines.

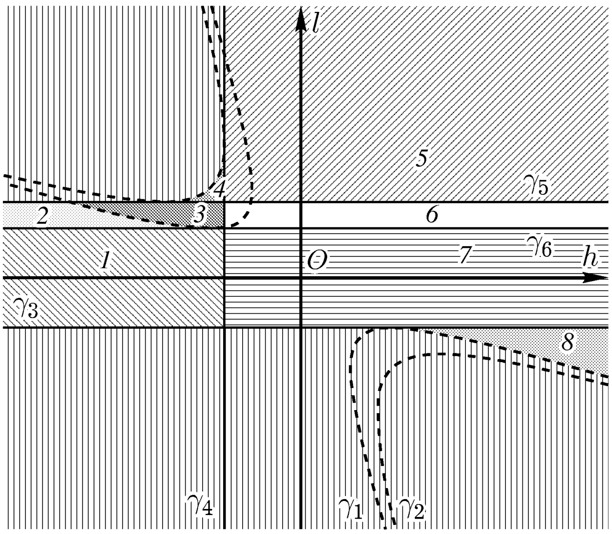
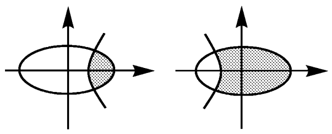
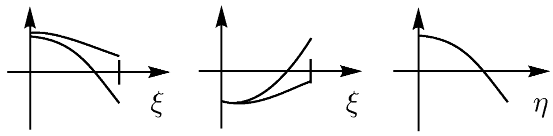
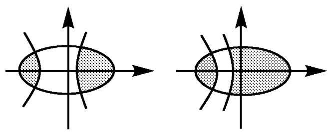
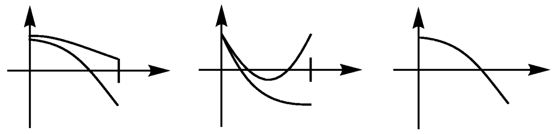
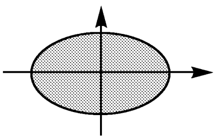
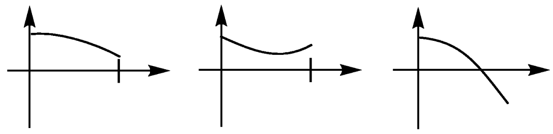
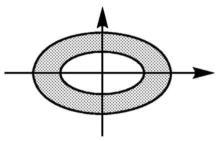
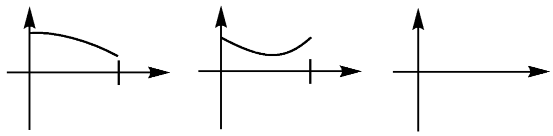
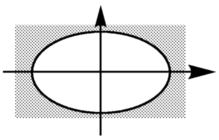
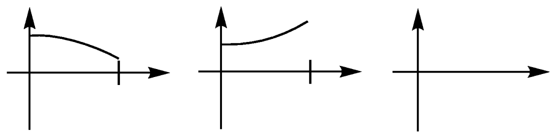
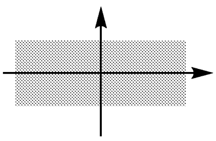
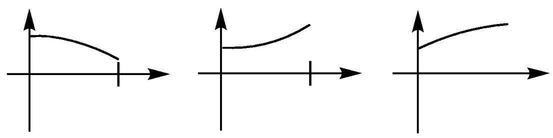
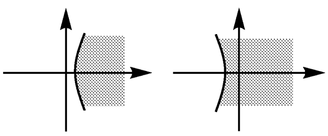
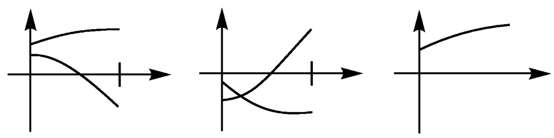
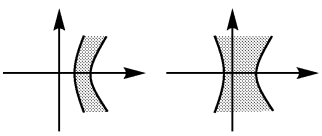
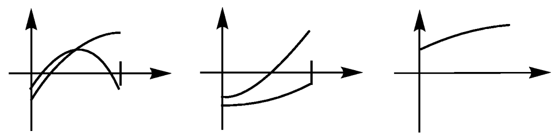


Figure 1. The bifurcation diagram in the two-center problem. The domains of possible motion are denoted by the numbers, the domains of prohibited motion are shaded by vertical lines

The classification of the domains of possible motion and the form of the functions R and S are presented in Table 1. In the domains *1*, *2*, *7*, *8* the function $R^*_\pm$ has two forms that corresponds to points above and below of the line $l=-h\alpha^2$. Just as in the plane case (Fig. 2) we have 8 different domains. However it is seen from Figs 1 and 2 that they are placed unlikely in the curved space: in the curved space the domain *5* bounds with the domain of prohibited motion and in the plane case the one does not have a common points. Therefore, these two problems are trajectory nonequivalent. In the case of repulsing centers we shall obtain the analogous results under reflection with respect to the axis OZ. If the centers are identical, i. e. $K=0$, then the hyperbola γ_1 disappears and the straights γ_4 and γ_7 are joined into the straight $l=0$.

In conclusion of this section let us present the most interesting types of the motions, corresponding to the values (h,l) from the bifurcation set — the curves γ_i. (For brevity the line of intersection of a hyperboloid (a pseudosphere) with an elliptic cone and with a hyperbolic cone we shall call the “ellipse” and the “hyperbola”).

Table 1

No	Domains of possible motion	R^*, S^*
1		
2		
3		
4		
5		
6		
7		
8		

1. The multiple roots $(\eta_1 = \eta_2)$ of the function S correspond to points on the curve γ_2 (Fig. 1), at this the "elliptic" ring (see Table 1, No. 4) degenerates into the "ellipse" and a particle accomplishes the periodic motion along its. If the roots equal zero $(\eta_1 = \eta_2 = 0$, the point of contact of the curve γ_2 and $\gamma_6)$, then a particle moves along the line connecting the centers until falls into one of its.
2. If one root of the function S equals zero (the line segment separating the domains 4 and 3), then a mass point moves along a spiral, which is bounded above by "ellipse" and is winded around the line, connecting the centers.
3. The multiple roots of the function R $(\xi_1 = \xi_2)$ and the unique root of the function S on the interval $(0, \infty)$ correspond to points on the part of the curve γ_1, separating the domains *2* and *3*, from the point of intersection of the curve γ_1 and the straight γ_6 up to the point of contact with the straight γ_7. Here we observe the oscillating motions inside the "ellipse", at this a mass point asymptotically tends to the "hyperbola". (see Table 1, No. 2 — in this case "hyperbolas" are joined.)
4. If the root of the function R equals zero (the part of the straight γ_4 from the point of contact with the curve γ_1), then one of "hyperbolas" (Table 1, No. 8) degenerates into the positive part of the axis OX with the beginning in the corresponding center (the projection of the curve $x^2 - y^2 = -1$ on the plane ZOX). A mass point crosses the line connecting the centers $(0 < \eta < \infty)$ and asymptotically approaches to the axis OX (the motion is bounded by second "hyperbola"), touching the bounding "hyperbola".

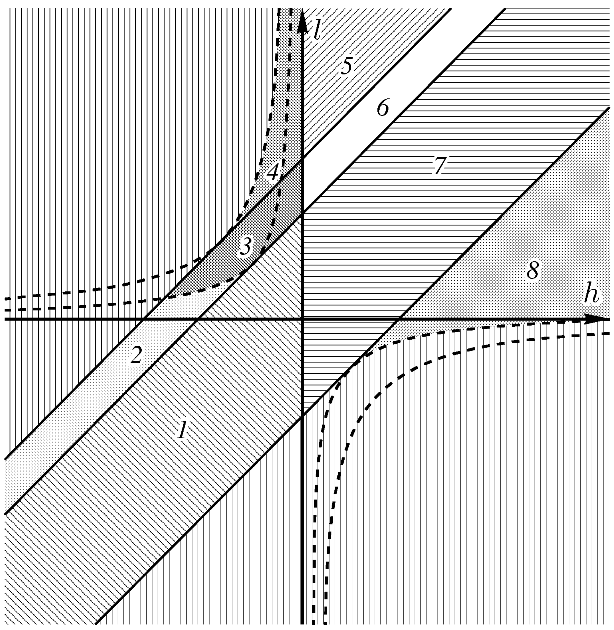


Figure 2. The bifurcation diagram in the two-center problem in the plane case

4. The Lagrange's Problem on the Lobachevsky's Plane

For the plane $\mathbb{R}^2$ there is the limiting case of the two-center problem, when one of the center is removed at infinity and its intensity tends to infinity also (so that the relation of intensity to a distance up to the origin of coordinates remains finite). The problem on a material point motion in the field of Newton's center and in the homogeneous field arises.

This problem was considered for the first time by Lagrange, the classification of the possible motion domains is given in the book by V. V. Beletsky [7]. There is an analogue of the Lagrange's problem, i. e. the problem of the splitting of the energy levels of hydrogen atom in the homogeneous electrical field, known in quantum mechanics as the Stark effect [8].

Theorem. *A mass point moves in the Lagrange problem in the space $\mathbb{H}^3$ in the same way as in the two-dimensional system (on the unit two-dimensional pseudosphere $H^2 = y^2 - x^2 - z^2 = 1$) with energy*

$$E = \frac{1}{2}(\dot{y}^2 - \dot{x}^2 - \dot{z}^2) + U_{\text{eff}}\,,$$

where the effective potential energy is defined by formula

$$V_{\text{eff}} = -\gamma_1 \frac{y}{\sqrt{y^2-1}} - \gamma_2 \frac{1}{(y-x)^2} + \frac{\beta_\varphi^2}{2z^2}\,.$$

Proof: Let us consider the analogous passage to the limit in the two-center problem in the space $\mathbb{H}^3$, removing one center at infinity. Let us place the attracting centers in the points with coordinates $r_1 = (1,0,0,0)$ and $r_2 = (\cosh\varphi, \sinh\varphi, 0, 0)$.

The potential energy of any point can be written as a sum of two values: the potential energy defined by the center placed in the pole of hyperboloid and the potential energy of the center removed at infinity

$$\begin{aligned} V_1 &= -\gamma_1 \coth\theta_1 = -\gamma_1 \frac{x^1}{\sqrt{(x^1)^2-1}}\,, \\ V_2 &= -\gamma_2 \coth\theta_2 = -\gamma_2 \frac{x^1\cosh\xi - x^2\sinh\xi}{\sqrt{(x^1\cosh\xi - x^2\sinh\xi)^2-1}}\,, \\ V &= V_1 + V_2\,. \end{aligned} \tag{4.1}$$

By making the change $2\exp(-2\varphi)\gamma_2 \to \gamma_2$, rejecting the constant term and passing to the limit $\varphi \to \infty$, we will obtain the expression for the potential of the infinitely removed center, the analogue of the homogeneous field in the curved space

$$V_2 = -\gamma_2 \frac{1}{(x^1 - x^2)^2} \,. \tag{4.2}$$

Let us introduce a new coordinates by formulas (2.6). It is clear that φ is a cyclic coordinate, so we can exclude it by the Rouhth's method. Thus, we reduce our problem to the problem on the motion of a mass point on a two-dimensional pseudosphere $y^2 - x^2 - z^2 = 1$ in the reduced field

$$V_{\text{eff}} = -\gamma_1 \frac{y}{\sqrt{y^2-1}} - \gamma_2 \frac{1}{(y-x)^2} + \frac{\beta_\varphi^2}{2z^2} \,. \tag{4.3}$$

The theorem is proved.

Let us perform the change of coordinates allowing to separate the variables. In this problem it is natural to pass from Cartesian coordinates to pseudoparabolic coordinates μ and ν. The coordinate surfaces have the form

$$\begin{aligned} 2x(y-x) &= \mu^2(y-x)^2 - \frac{z^2}{\mu^2} \,, \\ 2x(y-x) &= -\nu^2(y-x)^2 + \frac{z^2}{\nu^2} \,, \\ x^2 - y^2 + z^2 &= -1 \,. \end{aligned} \tag{4.4}$$

The new coordinates are defined so that the following conditions are realized

$$0 < \nu < 1, \qquad 0 < \mu < \infty \,. \tag{4.5}$$

The relations between these coordinates are

$$\begin{aligned} x &= \frac{1}{2} \frac{\mu^2 - \nu^2}{\sqrt{(1-\nu^2)(1+\mu^2)}} \,, & y &= \frac{2 + \mu^2 - \nu^2}{2\sqrt{(1-\nu^2)(1+\mu^2)}} \,, \\ z &= \frac{1}{2} \frac{\mu\nu}{\sqrt{(1-\nu^2)(1+\mu^2)}} \,, & y - x &= \frac{1}{\sqrt{(1-\nu^2)(1+\mu^2)}} \,. \end{aligned} \tag{4.6}$$

Let us write out our Hamiltonian in the variables μ, ν

$$\begin{aligned} H = \frac{(1-\nu^2)(1+\mu^2)}{\mu^2+\nu^2} \Bigg[& \frac{1}{2} \left((1+\mu^2) p_\mu^2 + (1-\nu^2) p_\nu^2 \right) \\ & + \gamma_1 \left(\frac{1}{1+\mu^2} + \frac{1}{1-\nu^2} \right) + \gamma_2(\mu^2+\nu^2) + \frac{\beta_\varphi^2}{2} \left(\mu^{-2} + \nu^{-2} \right) \Bigg] \,. \end{aligned} \tag{4.7}$$

Now our purpose is to construct the bifurcation set, to carry out the classification of motion of a mass point on a pseudosphere and to compare it with the plane case. This allows to answer the question of how the curvature of the Lobachevsky's space influences the trajectory equivalence.

Let us consider the Lobachevsky's plane (the two-dimensional upper sheet of the hyperboloid). It is clear that the variables in this problem are separated and it is possible to write the energy integral and the two additional Liouville's integrals (see the definition of a Liouville system) as:

$$
\begin{aligned}
h &= T + V\,,\\
I_1 &= \frac{1}{2}\frac{\dot\mu^2\left(\frac{1}{1-\nu^2}-\frac{1}{1+\mu^2}\right)^2}{1+\mu^2} - \gamma_1\frac{1}{1+\mu^2} - \gamma_2\mu^2 + h\frac{1}{1+\mu^2}\,,\\
I_2 &= \frac{1}{2}\frac{\dot\nu^2\left(\frac{1}{1-\nu^2}-\frac{1}{1+\mu^2}\right)^2}{1-\nu^2} - \gamma_1\frac{1}{1-\nu^2} - \gamma_2\nu^2 - h\frac{1}{1-\nu^2}\,.
\end{aligned}
\tag{4.8}
$$

Since $I_1 + I_2 = 0$, let us denote $I_1 = l = -I_2$. Introducing the new variable τ

$$
\mathrm{d}t = \frac{1}{\sqrt{2\gamma_2}}\left(\frac{1}{1-\nu^2}-\frac{1}{1+\mu^2}\right)\mathrm{d}\tau
$$

and making the changes $l \to \dfrac{l}{\gamma_2}$, $h \to \dfrac{h}{\gamma_2}$, $K = \dfrac{\gamma_1}{\gamma_2}$, we'll reduce the problem to the quadratures

$$
\frac{\mathrm{d}\nu}{\mathrm{d}\tau} = \sqrt{R(\nu)}\,, \qquad \frac{\mathrm{d}\mu}{\mathrm{d}\tau} = \sqrt{S(\mu)} \tag{4.9}
$$

where

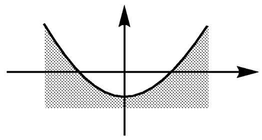
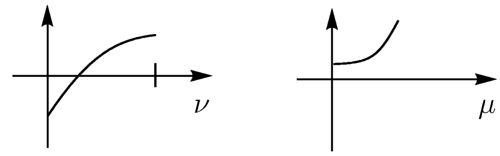
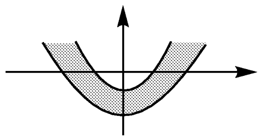
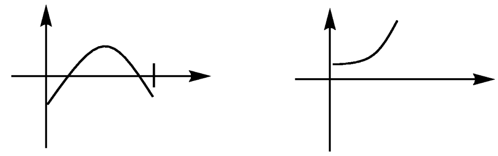
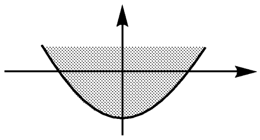
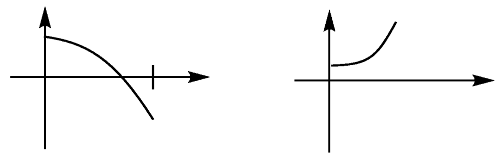
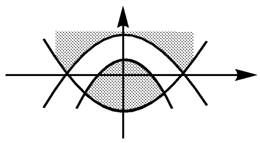
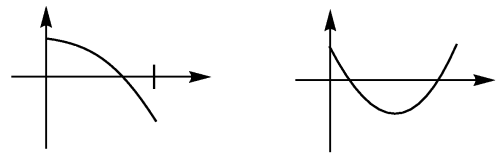
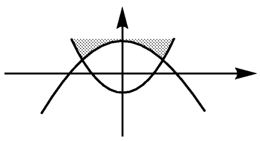
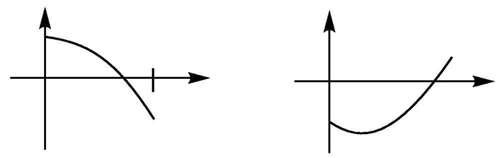
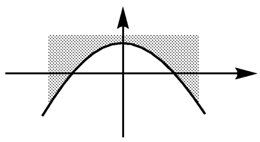
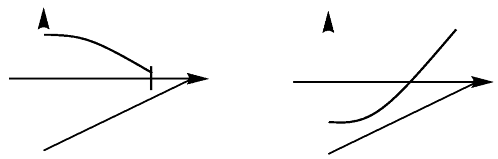
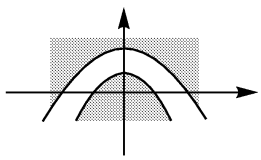
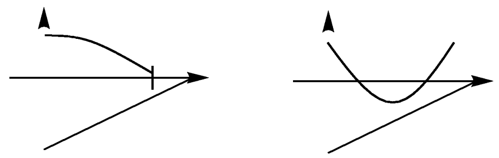
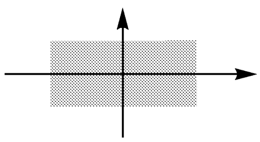
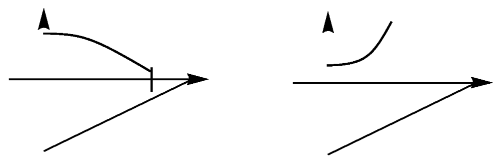
$$
\begin{aligned}
R(\nu) &= -l(1-\nu^2) + h + K + \nu^2(1-\nu^2)\,,\\
S(\mu) &= l(1+\mu^2) - h + K + \mu^2(1+\mu^2)\,.
\end{aligned}
$$

For the qualitative analysis of a material point motion in the present problem let us construct the bifurcation set in the plane of the integrals h and l in the same way as in the case of two centers (see Fig. 3).

$$
\begin{gathered}
\gamma_1\colon (l-1)^2 + 4h - 4K = 0\,, \qquad \gamma_2\colon (l-1)^2 + 4h + 4K = 0\,,\\
\gamma_3\colon l = h + K\,, \qquad \gamma_4\colon l = h - K\,, \qquad \gamma_5\colon h = -K\,.
\end{gathered}
\tag{4.10}
$$

The classification of the possible motion domains and the form of polynomials R and S are presented in Table 2. At the presence of only one field (in expression (14) $\gamma_1 = 0$) the straight lines γ_3 and γ_4, and the parabolas γ_1 and

Table 2

No	Domains of possible motion	R, S
1		
2		
3		
4		
5		
6		
7		
8		

γ_2 are joined. Respectively the domains placed between these curves are closed up (*3*, *4*, *7*, *8*). The domain, where the motion has not limit, is presented by the straight lines $l = h$. Thus, even at the negligible variation of the integrals h and l a material point "falls down" either in the domain *1* or in the domain *4*, in which the field of infinitely removed center curves the trajectory of a mass point, being moved, for example, initially along the axes X (see Fig. 3).

Comparing the bifurcation diagram of the Langrage's problem for the plane case [7] and its generalization to the constant curvature space (Fig. 3), it can be concluded that these problems are trajectory nonequivalent.

Thus, the interesting situation arises: at the curving of a space in the considered problems the integrability is preserved although even at the small curvature the trajectory equivalence respect to the plane case disappears.

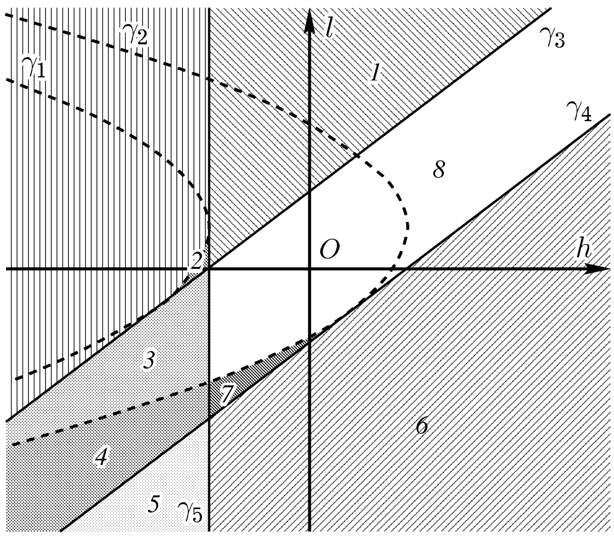


Figure 3. The bifurcation diagram in the Lagrange's problem on the pseudosphere. The domain of prohibited motion is shaded by the vertical lines while the domains of possible motion are denoted by the numbers

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