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THE GEOMETRY OF PARTIAL DIFFERENTIAL HAMILTONIAN SYSTEMS

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Abstract. Partial differential Hamiltonian systems have been recently introduced by the author in arXiv:0903.4528. They are field theoretic analogues of Hamiltonian systems on abstract symplectic manifolds. We will present here their main geometry, consisting of partial differential Hamilton equations, partial differential Noether theorem, partial differential Poisson bracket, etc..

1. Introduction

Lagrangian mechanics can be naturally generalized to first order Lagrangian field theory. Moreover, the latter can be presented in a very elegant and precise algebro-geometric fashion [12]. On the other hand, it seems to be a bit harder to understand what the most "reasonable, unambiguous, field theoretic generalization" of Hamiltonian mechanics on abstract symplectic manifolds is. Actually, there exists an universally accepted field theoretic version of Hamiltonian mechanics on the cotangent bundle T^*Q of a configuration manifold Q, i.e., multisymplectic field theory on the multimomentum bundle of a configuration bundle. However, T^*Q is just a very special example of (pre)symplectic manifold. Hamiltonian mechanics can (and should, in some cases [8]) be formulated on abstract (pre)symplectic manifolds. Similarly, it is natural to wonder if there exists the concept of abstract multi(pre)symplectic manifolds in such a way that Hamiltonian field theory could be reasonably formulated on them. In the literature there can be found some proposals of should be abstract multi(pre)symplectic manifolds (see, for instance, [1, 3]). Recently, the author presented his own proposal about what should be an abstract, first order, Hamiltonian field theory, and called it the theory of **partial**

differential (PD in the following) **Hamiltonian systems** [14]. Unlike most multisymplectic field theories, a PD Hamiltonian system encodes both the kinematics and the dynamics which appear as just different components of one single geometric object. Notice that this idea is already present in literature [10]. However, our formalism differs from the one in [10] in that it is adapted to the fibered structure of the "manifold of field variables". In this paper we present the main geometry underlying the theory of PD Hamiltonian systems. We will omit the proofs referring to [14] for them.

Let P be a smooth manifold, $\ldots, x^i, \ldots, y^a, \ldots$ coordinates on it, X a vector field and $F : P_1 \longrightarrow P$ a smooth map of manifolds. In the following we put $\ldots, \partial_i := \frac{\partial}{\partial x^i}, \ldots, \partial_a := \frac{\partial}{\partial y^a}, \ldots$. The graded algebra of differential forms on P is denoted by $\Lambda(P) = \bigoplus_k \Lambda^k(P)$, $\Lambda^k(P)$ being its k-th homogeneous component, i.e., the space of differential k-forms on P. We denote by d (respectively i_X, L_X) the exterior differential (respectively the insertion of X into, the Lie derivative along X of differential forms). Moreover, we denote by $F^* : \Lambda(P) \longrightarrow \Lambda(P_1)$ the pull-back. If V is a vector space, we denote by $\Lambda^\bullet V = \bigoplus_k \Lambda^k V$ its exterior algebra, $\Lambda^k V$ being the k-th exterior power of V. We put in square bracket skew-symmetrized indexes, for instance $A_{[a}B_{b]}$ means $\frac{1}{2}(A_a B_b - A_b B_a)$. Finally, we will always understand the sum over upper-lower pairs of repeated indexes.

2. Affine Forms on Fiber Bundles

In this sections we define what we call *affine forms* on fiber bundles. The introduction of affine forms can be motivated as follows. In Hamiltonian mechanics motions are curves on a manifold and velocities are tangent vectors. In their turn, tangent vectors can be inserted into differential forms and, in particular, a symplectic one, and the Hamilton equations can be written in terms of such an insertion. On the other hand, in field theory motions are sections of a fiber bundle $\alpha : P \longrightarrow M$ and velocities are points in the first jet space $J^1\alpha$ of α. In their turn, points of $J^1\alpha$ can be inserted into affine forms and, in particular, a PD Hamiltonian system, and PD Hamilton equations can be written in terms of such an insertion (see Section 3). Recall now that the natural projection $J^1\alpha \longrightarrow P$ is an affine bundle whose sections are naturally interpreted as (Ehresmann) connections in α. Thus, connections and affine geometry play a prominent role in the theory of PD Hamiltonian systems.

Let $\alpha : P \longrightarrow M$ be a fiber bundle, A the algebra of smooth, real valued functions on P, $x^1, \ldots, x^n$ coordinates on M, $\dim M = n$, and $y^1, \ldots, y^m$ fiber coordinates on P, $\dim P = n + m$. Denote by $\Lambda_1 = \bigoplus_k \Lambda_1^k$ the (graded) differential ideal in $\Lambda(P)$ made of differential forms vanishing when pulled-back to fibers of the projection α. For all k, Λ_1^k is made of differential k-forms ω on P that are locally of the form

$$\omega = \sum_{l\geq 0} \omega_{a_1\cdots a_{k-l-1}i_1\cdots i_{l+1}} \mathrm{d}y^{a_1} \wedge \cdots \wedge \mathrm{d}y^{a_{k-l-1}} \wedge \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_{l+1}}$$

with $\omega_{a_1\cdots a_{k-l-1}i_1\cdots i_{l+1}}$ being some local functions on P, $a_1, \ldots, a_{k-l-1}$ run over $1, \ldots, m$ and $i_1, \ldots, i_{l+1} = 1, \ldots, n$.

Denote by $V\Lambda = \bigoplus_k V\Lambda^k$ the quotient (graded) differential algebra $\Lambda(P)/\Lambda_1$, $\mathrm{d}^V : V\Lambda \longrightarrow V\Lambda$ its differential. An element ρ in $V\Lambda^k$ is locally of the form

$$\rho = \rho_{a_1\cdots a_k} \mathrm{d}^V y^{a_1} \wedge \cdots \wedge \mathrm{d}^V y^{a_k}$$

with $\rho_{a_1\cdots a_k}$ being local functions on P and $\mathrm{d}^V\rho$ is locally given by

$$\mathrm{d}^V \rho^V = \partial_{[a}\rho_{a_1\cdots a_k]} \mathrm{d}^V y^a \wedge \mathrm{d}^V y^{a_1} \wedge \cdots \wedge \mathrm{d}^V y^{a_k}.$$

Denote by $\overline{\Lambda} = \bigoplus_k \overline{\Lambda}^k := \bigoplus_k \Lambda^k_k$ the A-subalgebra of $\Lambda(P)$ generated by Λ^1_1. An element $\omega \in \overline{\Lambda}^k$ is locally of the form

$$\omega = \omega_{i_1\cdots i_k} \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}$$

with $\omega_{i_1\cdots i_k}$ being local functions on P. Now, let ∇ be a connection in α. It is well-known that ∇ determines a splitting $\Sigma_\nabla : V\Lambda^1 \longrightarrow \Lambda^1(P)$ of the short exact sequence $0 \longrightarrow \overline{\Lambda}^1 \longrightarrow \Lambda^1(P) \longrightarrow V\Lambda^1 \longrightarrow 0$, locally given by $\Sigma_\nabla(\mathrm{d}^V y^a) = \mathrm{d}^V_\nabla y^a := \mathrm{d}y^a - \nabla^a_i \mathrm{d}x^i$, ∇^a_i being the symbols of ∇. Accordingly, for any p, there is an obvious projection $\mathfrak{p}^p_\nabla : \Lambda(P) \longrightarrow V\Lambda \otimes_A \overline{\Lambda}^p$, locally given by

$$\mathfrak{p}^p_\nabla(\omega) = \sum_q \omega_{a_1,\ldots,a_q,i_1,\ldots,i_p} \mathrm{d}^V y^{a_1} \wedge \cdots \wedge \mathrm{d}^V y^{a_q} \otimes \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_p}$$

if

$$\omega = \sum_{q,p'} \omega_{a_1,\ldots,a_q,i_1,\ldots,i_{p'}} \mathrm{d}^V_\nabla y^{a_1} \wedge \cdots \wedge \mathrm{d}^V_\nabla y^{a_q} \wedge \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_{p'}} \in \Lambda(P).$$

Now, let $\Omega(P, \alpha)$ (or, simply, Ω, if this does not lead to confusion) be the $(n-1)$th exterior power of Λ_1 in $\Lambda(P)$, i.e.,

$$\Omega := \{\textstyle\sum \omega_1 \wedge \cdots \wedge \omega_{n-1};\, \omega_1, \ldots, \omega_{n-1} \in \Lambda_1\} \subset \Lambda.$$

Denote by $\Omega^q(P, \alpha)$ (or, simply, Ω^q) the $(q+n-1)$th homogeneous component of Ω. Then, $\Omega = \bigoplus_q \Omega^q$ and any element $\omega \in \Omega^q$ is locally of the form

$$\omega = \omega^i_{a_1\ldots a_q} \mathrm{d}y^{a_1} \wedge \cdots \wedge \mathrm{d}y^{a_q} \wedge \mathrm{d}^{n-1}x_i - \omega_{a_1\ldots a_{q-1}} \mathrm{d}y^{a_1} \wedge \cdots \mathrm{d}y^{a_{q-1}} \wedge \mathrm{d}^n x$$

where $\mathrm{d}^n x := \mathrm{d}x^1 \cdots \mathrm{d}x^n$, $\mathrm{d}^{n-1}x_i := i_{\partial_i}\mathrm{d}^n x$, $i = 1, \ldots, n$, and $\omega^i_{a_1\ldots a_q}$, $\omega_{a_1\ldots a_{q-1}}$ are local functions on P. Elements in Ω are conventionally called by the author

affine forms. Such convention can be motivated as follows. Let $\omega \in \Omega$ and ∇ be a connection in α. We define the *insertion* $i_\nabla\omega$ *of* ∇ *into* ω as

$$i_\nabla\omega := \mathfrak{p}^n_\nabla(\omega) \in V\Lambda \otimes \overline{\Lambda}^n. \tag{1}$$

Now, recall that the set of all connections in α is an affine space. It can be proved that, for any $\omega \in \Omega$, $i_\nabla\omega$ is affine in the argument ∇. For instance, if $\omega \in \Omega^2$ is locally given by

$$\omega = \omega^i_{ab}\mathrm{d}y^a \wedge \mathrm{d}y^b \wedge \mathrm{d}^{n-1}x_i - \omega_a \mathrm{d}y^a \wedge \mathrm{d}^n x \tag{2}$$

then $i_\nabla\omega$ is locally given by the formula

$$i_\nabla\omega = (2\omega^i_{ab}\nabla^b_i - \omega_a)\mathrm{d}^V y^a \otimes \mathrm{d}^n x$$

which is manifestly affine in the ∇^b_i's. As a consequence, the operation defined in (1) is actually point-wise. Namely, let $y \in P$, $x = \alpha(y) \in M$ and $\Pi \subset T_yP$ be an n-dimensional subspace transversal to the fiber $F_x := \alpha^{-1}(x)$ of α through x. Then, it can be defined in an analogous way as in (1) the insertion $i_\Pi\omega_y$ of Π into the form ω at the point y, and $i_\Pi\omega_y \in \Lambda^\bullet T^*_yF_x \otimes_\mathbb{R} \Lambda^n T^*_x M$. Finally, let $\sigma : M \longrightarrow P$ be a (local) section of α. For any $x \in M$, $\dot\sigma(x) := T_{\sigma(x)}\,\mathrm{im}\,\sigma \subset T_{\sigma(x)}P$ is precisely an n-dimensional subspace transversal to F_x. Denote by $i_{\dot\sigma}\omega|_\sigma$ the map $M \in x \longmapsto i_{\dot\sigma(x)}\omega_{\sigma(x)} \in \Lambda^\bullet T^*_{\sigma(x)}F_x \otimes_\mathbb{R} \Lambda^n T^*_x M$.

We conclude this section remarking that if $\omega \in \Omega$ then 1) $\mathrm{d}\omega \in \Omega$, 2) $i_Y\omega \in \Omega$ for any vertical vector field on P, 3) $L_X\omega \in \Omega$ for any vector field X on P that can be projected to M via α, 4) $F^*(\omega) \in \Omega(P_1, \alpha_1)$ for any bundle $\alpha_1 : P_1 \longrightarrow M$ and any bundle morphism $F : P_1 \longrightarrow P$. In particular, the exterior differential can be restricted to affine forms. We denote by δ the restricted differential. Thus

$$0 \longrightarrow \Omega^0 \xrightarrow{\delta} \Omega^1 \xrightarrow{\delta} \cdots \longrightarrow \Omega^q \xrightarrow{\delta} \Omega^{q+1} \xrightarrow{\delta} \cdots$$

is a subcomplex of the de Rham complex of P.

Theorem 1 (Affine Poincaré Lemma)**.** *Let* $q \geq 0$ *and* $\omega \in \Omega^q$ *be* δ*-closed, i.e.,* $\delta\omega = 0$*. Then,* 1) *if* $q = 0$*,* ω *is locally of the form* $\alpha^*(\eta)$ *for some* $\eta \in \Lambda^{n-1}(M)$*,* 2) *if* $q > 0$*, then* ω *is locally* δ*-exact, i.e.,* ω *is locally of the form* $\delta\theta$*,* θ *being a local element in* Ω^{q-1}*.*

3. PD Hamiltonian Systems

In this section we introduce what we think should be understood as the partial differential, i.e., field theoretic, analogue of a Hamiltonian system on an abstract (pre)symplectic manifold.

Let $\alpha : P \longrightarrow M$ be as in the previous section.

Definition 1. *A* PD Hamiltonian system *on the fiber bundle* $\alpha : P \longrightarrow M$ *is a* δ*-closed element* $\omega \in \Omega^2(P, \alpha)$*.*

Let $\theta \in \Omega^1$ be locally given by $\theta = \theta^i_a \mathrm{d}y^a \wedge \mathrm{d}^{n-1}x_i - H\mathrm{d}^n x$, θ^i_a, H being local functions on P. Then $\delta\theta$ is locally given by

$$\delta\theta = \partial_{[a}\theta^i_{b]}\mathrm{d}y^a \wedge \mathrm{d}y^b \wedge \mathrm{d}^{n-1}x_i - (\partial_a H + \partial_i \theta^i_a)\mathrm{d}y^a \wedge \mathrm{d}^n x.$$

Similarly, let $\omega \in \Omega^2$ be locally given by (2), then $\delta\omega$ is locally given by

$$\delta\omega = \partial_{[a}\omega^i_{bc]}\mathrm{d}y^a \wedge \mathrm{d}y^b \wedge \mathrm{d}y^c \wedge \mathrm{d}^{n-1}x_i + (\partial_i \omega^i_{ab} - \partial_{[a}\omega_{b]})\mathrm{d}y^a \wedge \mathrm{d}y^b \wedge \mathrm{d}^n x$$

so that ω is a PD Hamiltonian system iff

$$\partial_{[a}\omega^i_{bc]} = 0, \qquad \partial_i \omega^i_{ab} = \partial_{[a}\omega_{b]} \tag{3}$$

or, which is the same (see Theorem 1)

$$\omega^i_{ab} = \partial_{[a}\theta^i_{b]}, \qquad \omega_a = \partial_a H + \partial_i \theta^i_a \tag{4}$$

for some θ^i_a, H local functions on P.

Let ω be a PD Hamiltonian system on α. We can search for some (local) sections $\sigma : M \longrightarrow P$ of α such that

$$i_{\dot\sigma}\omega|_\sigma = 0. \tag{5}$$

Definition 2. *Equations* (5) *are called the* ***PD Hamilton equations*** (*determined by the* PD *Hamiltonian system* ω).

If ω is locally given by (2), then it determines PD Hamilton equations which are locally given by

$$2\omega^i_{ab}\partial_i y^b = \omega_a. \tag{6}$$

Conversely, a system of PDEs in the form (6) is a system of PD Hamilton equations for some PD Hamiltonian system iff the coefficients ω^i_{ab}, ω_a satisfy (3) (or, which is the same, (4)). Notice that, in view of (6), a general PD Hamiltonian system ω encodes both "kinematical information", which can be identified with the coefficients ω^i_{ab} and "dynamical information", which can be identified with the coefficients ω_a. However, the two cannot be disentangled since "they mix under a general change of coordinates". Notice also that, when $n = 1$ and the θ^i_a's are zero, then Equations (6) reduce to Hamilton equations of a Hamiltonian system with Hamiltonian H on a suitable (pre)symplectic manifold.

Searching for solutions of PD Hamilton equations of a PD Hamiltonian system ω on α, we could proceed in two steps

1. search for a connection ∇ in α such that $i_\nabla \omega = 0$
2. search for flat sections with respect to ∇.

However, the first problem needs no to possess solutions. Therefore, in general, we are led to weaken it and search for connections ∇_1 in a subbundle $P_1 \subset P$ such that $i_{\nabla_1}\omega|_{P_1} = 0$ (here $\omega|_{P_1}$ denotes the restriction of ω to P_1). As shown in the next theorem, there is always an "algorithmic" way to find a maximal subbundle α' :

$P' \longrightarrow M$ of α such that the equation $i_{\nabla'}\omega|_{P'} = 0$ admits at least one solution. We refer to the above mentioned "algorithm" as the PD **constraint algorithm** (see also [4, 8]), since it is the PD version of the standard constraint algorithm in Hamiltonian mechanics.

Theorem 2 (Existence of a PD Constraint Algorithm). *Let ω be a PD Hamiltonian system on α. For $s \geq 0$ define recursively (if possible)*

$$\begin{aligned} P_{(s+1)} &:= \{y \in P_{(s)}\,;\, i_\Pi \omega_y = 0 \textit{ for some } n\textit{-dimensional subspace } \Pi \subset T_y P_s \\ &\qquad \textit{transversal to } F_{\alpha(y)}\} \subset P \\ \alpha_{(s+1)} &:= \alpha|_{P_{(s+1)}} : P_{s+1} \longrightarrow M \end{aligned}$$

where $P_{(0)} := P$, $\alpha_{(0)} := \alpha$. Then there exist a subbundle $\alpha' : P' \longrightarrow M$ of α and an s_0 such that $P_{(s)} = P'$ for all $s \geq s_0$. Moreover, α' is a maximal subbundle such that $i_{\nabla'}\omega|_{P'} = 0$ for some connection ∇' in α'.

Notice that P' as in the above theorem can be empty and, in this case, PD Hamilton equations do not possess solutions. Moreover, the image of any solution of the PD Hamilton equations is contained into P'. The converse is, a priori, only true for $n = 1$. Namely, we may wonder if for any $y \in P'$ there is a solution σ of the PD Hamilton equations such that $y \in \operatorname{im}\sigma$. We know that there is a connection ∇' in P' which is "a solution of PD Hamilton equations up to first order", i.e., $i_{\nabla'}\omega|_{P'} = 0$. Flat sections with respect to ∇' are clearly images of solutions of the PD Hamilton equations. If $n = 1$, ∇' is trivially flat and Frobenius theorem guarantees that for any $y \in P'$ there is some flat section (with respect to ∇') "through y". The same is a priori untrue for $n \geq 2$.

4. PD Noether Symmetries and Currents

The multisymplectic analogues of Hamiltonian vector fields and Poisson bracket in symplectic geometry have been longly investigated (see, for instance, [5, 9, 11]). We here present the natural definitions for general PD Hamiltonian systems. Notice that our definitions have actually got a dynamical content, not only a kinematical one, so that, for instance, we can prove a PD version of (Hamiltonian) Noether theorem.

Let ω be a PD Hamiltonian system on the bundle $\alpha : P \longrightarrow M$. In the following we assume α to have connected fibers.

Definition 3. *Let Y be a vertical vector field on P and $f \in \Omega^0$. If $i_Y\omega = \delta f$, then Y and f are said to be a* ***PD Noether symmetry*** *and a* ***PD Noether current*** *of ω (relative to each other), respectively.*

Denote by $\mathcal{S}(\omega)$ and $\mathcal{C}(\omega)$ the sets of PD Noether symmetries and PD Noether currents of ω, respectively. A PD Noether symmetry Y (relative to a PD Noether current f) is a symmetry of ω in the sense that

$$L_Y\omega = i_Y\delta\omega + \delta i_Y\omega = \delta\delta f = 0.$$

The next theorem clarifies in which sense a PD Noether current is a conserved current for ω.

Theorem 3 (PD Noether). *Let $f \in \mathcal{C}(\omega)$, σ be a solution of the* PD *Hamilton equations, and $\Sigma \subset M$ an hypersurface. Then $\int_\Sigma \sigma^*(f)$ is a* conserved charge, *i.e., it is independent of the choice of Σ in a homology class.*

We are now in the position to introduce a Lie bracket among PD Noether currents.

Theorem 4. *Let $Y_1, Y_2 \in \mathcal{S}(\omega)$ be* PD *Noether symmetries relative to the* PD *Noether currents $f_1, f_2 \in \mathcal{C}(\omega)$, respectively. Then we have $[Y_1, Y_2] \in \mathcal{S}(\omega)$, $f := L_{Y_1} f_2 \in \mathcal{C}(\omega)$, and they are relative to each other. Moreover, f is independent of the choice of Y_1 among the* PD *Noether symmetries relative to the* PD *Noether current f_1. Finally, the $\mathbb{R}$-bilinear map*

$$\mathcal{C}(\omega) \times \mathcal{C}(\omega) \ni (f_1, f_2) \longmapsto \{f_1, f_2\} := L_{Y_1} f_2 \in H(\omega)$$

is a well-defined Lie bracket.

We call **PD Poisson bracket** the Lie bracket among PD Noether currents.

PD Noether symmetries and PD Noether currents of a PD Hamiltonian system constitute, in general, very small Lie subalgebras of the Lie algebras of higher symmetries and conservation laws of PD Hamilton equations, for which there have been given fully satisfactory definitions and have been developed many infinite jet based computational techniques [2]. Nevertheless, it is worthy to give Definition 3 and to carefully analyse it, independently on infinite jets, in view of the possibility of developing a "(multi)symplectic theory" of higher symmetries and conservation laws (see, for instance, [13]).

Let us now focus on PD Noether symmetries relative to the trivial PD Noether current $0 \in \mathcal{C}(\omega)$. We call such symmetries *trivial*. Indeed, from a physical point of view, they are infinitesimal gauge transformations and therefore should be quotiented out via a reduction of the system. Now on we assume, for the sake of simplicity, that ω is an unconstrained PD Hamiltonian system, i.e., there exists at least one connection ∇ in α such that $i_\nabla\omega = 0$. As a further regularity condition, we assume that trivial PD Noether symmetries span a smooth (vertical) distribution G on P, whose leaves form a fiber bundle $\widetilde{P}$ over M, with projection $\widetilde{\alpha} : \widetilde{P} \longrightarrow M$, in such a way that the canonical projection $\mathfrak{p} : P \longrightarrow \widetilde{P}$ is a smooth bundle itself. The last condition is always fulfilled at least locally.

Theorem 5 (Gauge Reduction). *There exists a unique* PD *Hamiltonian system* $\widetilde{\omega}$ *in* $\widetilde{\alpha}$ *such that* 1) $\widetilde{\omega}$ *doesn't possess trivial* PD *Noether symmetries,* 2) $\omega = \mathfrak{p}^*(\widetilde{\omega})$, 3) *a* (*local*) *section* σ *of* α *is a solution of the* PD *Hamilton equations determined by* ω *iff* $\mathfrak{p} \circ \sigma$ *is a solution of the* PD *Hamilton equations determined by* $\widetilde{\omega}$.

5. Two Examples

Standard examples of PD Hamiltonian systems come from Lagrangian field theory. As a first instance, consider the PD Hamiltonian system

$$\omega := T^{-1}(\delta^{ij} - T^{-2}u^i u^j)\mathrm{d}u_i \wedge (\mathrm{d}u \wedge \mathrm{d}^{n-1}x_j - u_j \mathrm{d}^n x)$$

where $T := \sqrt{1 + \delta^{ij}u_i u_j}$, d δ^{ij} is the Kronecker symbol, $i, j = 1, 2$, and $u^i = \delta^{ij}u_j$, on the bundle

$$(x^1, x^2, u, u_1, u_2) \longmapsto (x^1, x^2). \tag{7}$$

The PD Hamilton equations read

$$(\delta^{ij} - T^{-2}u^i u^j)\partial_j u_i = 0, \qquad \partial_i u = u_i$$

which are equivalent to the minimal surface equation.

Theorem 6. *A vertical vector field* Y *with respect to projection (7) and* $f \in \Omega^0$ *are a* PD *Noether symmetry and a* PD *Noether current for* ω*, relative to each other, respectively, iff*

$$Y = U\frac{\partial}{\partial u}, \qquad f = U T^{-1}(u_2 \mathrm{d}x^1 - u_1 \mathrm{d}x^2) + \mathrm{d}B$$

where $U = \mathrm{const}$ *and* $B = B(x^1, x^2)$. *Moreover, the PD Poisson bracket determined by* ω *is trivial.*

Finally, we provide an example of gauge reduction of a PD Hamiltonian system. Let $\mathbb{M}$ be the Minkowski space. In particular, $\mathbb{M}$ is endowed with the metric $g := g_{ij}\mathrm{d}x^i\mathrm{d}x^j$, $i, j = 1, \ldots, 4$, where

$$(g_{ij}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

In the following we will raise and lower indexes using g. Consider the cotangent bundle $\pi : T^*\mathbb{M} \ni A_i \mathrm{d}x^i|_{(x^1,\ldots,x^4)} \longmapsto (x^1, \ldots, x^4) \in \mathbb{M}$ and let

$$\alpha := \pi_1 : (x^1, \ldots, x^4, \ldots, A_i, \ldots, A_{i,j}, \ldots) \ni J^1\pi \longmapsto (x^1, \ldots, x^n) \in \mathbb{M}$$

be its first jet bundle. The differential form on $J^1\pi$

$$\omega := \mathrm{d}A^{[j,i]} \wedge (A_{i,j}\mathrm{d}^n x - 2\mathrm{d}A_i \wedge \mathrm{d}^{n-1}x_j)$$

is an unconstrained PD Hamiltonian system on α whose PD Hamilton equations read

$$\partial_k A^{[i,k]} = 0, \qquad \partial_{[j} A_{i]} = A_{[i,j]}, \qquad i,j = 1,\ldots,4$$

which are equivalent to Maxwell equations for the vector potential. Notice that

$$G = \left\langle \ldots, \frac{\partial}{\partial A_{i,j}} + \frac{\partial}{\partial A_{j,i}}, \ldots \right\rangle .$$

Moreover, leaves of G are given by $A_{[i,j]} = \mathrm{const}$. We conclude that the bundle $J^1\pi$ "reduces" via

$$\mathfrak{p} : J^1\pi \longrightarrow T^*\mathbb{M} \times_M \Lambda^2 T^*\mathbb{M} \simeq \mathbb{R}^{14}$$

$$(x^1, \ldots, x^4, \ldots, A_i, \ldots, A_{i,j}, \ldots) \longmapsto (x^1, \ldots, x^4, \ldots, A_i, \ldots, F_{ij}, \ldots)$$

where $F_{ij} = F_{[ij]}$, $\mathfrak{p}^*(F_{ij}) := 2A_{[j,i]}$ and $\omega = \mathfrak{p}^*(\widetilde{\omega})$, with

$$\widetilde{\omega} := \tfrac{1}{4} \mathrm{d}F^{ij} (F_{ji} \mathrm{d}^n x - 4\mathrm{d}A_i \wedge \mathrm{d}^{n-1} x_j)$$

which is a PD Hamiltonian system without trivial PD Noether symmetries on the bundle

$$\widetilde{\alpha} : \mathbb{R}^{14} \ni (x^1, \ldots, x^4, \ldots, A_i, \ldots, F_{ij}, \ldots) \longmapsto (x^1, \ldots, x^4) \in \mathbb{M}$$

whose PD Hamilton equations read

$$\partial_k F^{ik} = 0, \qquad \partial_{[j} A_{i]} = 2F_{ji}$$

which are just Maxwell equations for the field strength.

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