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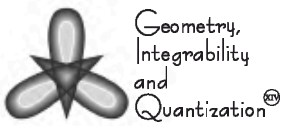

SOME CONSTRAINTS AND SYMMETRIES IN DYNAMICS OF HOMOGENEOUSLY DEFORMABLE ELASTIC BODIES

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Abstract. Our work has been inspired among others by the work of Arnold, Kozlov and Neihstadt. Our goal is to carry out a thorough analysis of the geometric problems we are faced with in the dynamics of affinely rigid bodies. We examine two models: classical dynamics description by d'Alembert and vakonomic one. We conclude that their results are quite different. It is not yet clear which model is practically better.

1. Introduction

One of the examples, which was very interesting for us, was an affinely rigid body, i.e., a body rigid in the sense of affine geometry, in other words, homogeneously deformable body. So, the subject of our interest is the case of uniformly deformable objects. There are usually some groups responsible for the geometry of the physical space or space-time. Mostly it is such groups like the isometry group, affine group, conformal group, Poincare group, Galilei group, etc. Configuration spaces of various constrained continua very often happen to be homogeneous spaces of those groups. One of the examples, which was very interesting for us, was a body rigid in the sense of affine geometry. Such a body we call affinely rigid body. It can be for instance the model of internal degrees of freedom in Eringen's micromorphic continuum. There are also other interesting examples like, e.g., molecular vibrations. Let us notice there is plenty of misunderstandings here. Often one does not distinguish between two procedures: the first one of finding special solutions of continua in terms of affine motion and the second one of the dynamically restricted problem of affine motion. We are looking for the special solutions of unconstrained problems, rather then of the constrained dynamics with its characteristic reaction

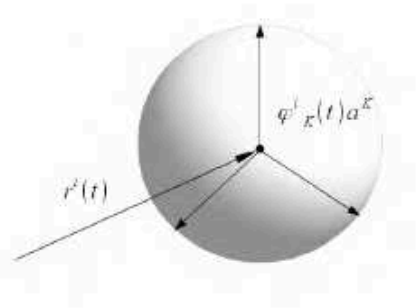

Figure 1. Degrees of freedom of an affine body.

forces. An interesting argument is that both the deformation and stress tensors are constant within the homogeneously deformable body.

Obviously, finite bodies with boundary cannot have a constant deformation tensor, except theirs interior. In virtue of the d'Alembert principle, reactions responsible for the affine rigidity do not vanish. However, their monopole and dipole distributions do vanish. It means that the total reaction force and the dipole distribution of reactions do vanish. Because of this, if we describe the configuration of affine body by

$$x^i(r,\varphi;t) = r^i(t) + \varphi^i{}_K(t)a^K$$

then r^i are coordinates of the centre of mass, $\varphi^i{}_K$ are internal (relative) parameters, and a^K are material variables, as it is shown on the picture below.

To describe equations of motion we use the following symbols

- M is the **total mass** of the body

$$M = \int \mathrm{d}\mu$$

- J^{KL} is the co-moving **tensor of inertia** in the material space, thus, constant

$$J^{KL} = \int a^K a^L \mathrm{d}\mu(a)$$

- the **centre of mass** is placed at $a^K = 0$, hence

$$J^K = \int a^K \mathrm{d}\mu(a) = 0$$

- F^i is the **total force**

$$F^i = \int \mathcal{F}^i(a)\mathrm{d}\mu(a)$$

- N^{KL} is the co-moving dipole of forces distribution, therefore, its spatial (Eulerian) components are given by

$$N^{ij} = \int \varphi^i{}_K \varphi^j{}_L a^K a^L \mathrm{d}\mu(a) = \varphi^i{}_K \varphi^j{}_L \int a^K a^L \mathrm{d}\mu(a).$$

Then using the above symbols we obtain that the equations of motion have the form

$$M\frac{\mathrm{d}^2 r^i}{\mathrm{d}t^2} = F^i, \qquad \varphi^i{}_K \frac{\mathrm{d}^2 \varphi^j{}_L}{\mathrm{d}t^2} J^{KL} = N^{ij}.$$

Let us quote some alternative balance forms of the above equations of motion

$$\frac{\mathrm{d}p^i}{\mathrm{d}t} = F^i, \qquad \frac{\mathrm{d}K^{ij}}{\mathrm{d}t} = \frac{\mathrm{d}\varphi^i{}_K}{\mathrm{d}t}\frac{\mathrm{d}\varphi^j{}_L}{\mathrm{d}t} J^{KL} + N^{ij}$$

where p^i is a **linear momentum** and K is an **affine spin**. Respectively they are given in the form

$$p^i = M\frac{\mathrm{d}r^i}{\mathrm{d}t}, \qquad K^{ij} = \varphi^i{}_K \frac{\mathrm{d}\varphi^j{}_L}{\mathrm{d}t} J^{KL}.$$

In other words

$$\frac{\mathrm{d}p^i}{\mathrm{d}t} = F^i, \qquad \frac{\mathrm{d}K^{ij}}{\mathrm{d}t} = \Omega^i{}_m K^{mj}$$

where $\Omega^i{}_m$ is an affine speed, called also **Eringen's gyration**. We define it as follows

$$\Omega^i{}_j = \frac{\mathrm{d}\varphi^i{}_A}{\mathrm{d}t}\varphi^{-1A}{}_j$$

or in the co-moving representation

$$\widehat{\Omega}^A{}_B = \varphi^{-1A}{}_i \Omega^i{}_j \varphi^j{}_B.$$

The kinetic energy is given by the sum of translational T_{tr} and internal T_{int} kinetic energies:

$$T = T_{\mathrm{tr}} + T_{\mathrm{int}} = \frac{M}{2} g_{ij} \frac{\mathrm{d}r^i}{\mathrm{d}t}\frac{\mathrm{d}r^j}{\mathrm{d}t} + \frac{1}{2} g_{ij} \frac{\mathrm{d}\varphi^i{}_K}{\mathrm{d}t}\frac{\mathrm{d}\varphi^j{}_L}{\mathrm{d}t} J^{KL}.$$

So let us quote the following formula

$$\frac{\mathrm{d}K^{ij}}{\mathrm{d}t} = N^{ij} + 2\frac{\partial T_{int}}{\partial g_{ij}}.$$

If Lagrangian is given by

$$L = T - V\left(r^i, \varphi^i{}_K\right)$$

then p_i is a generator of spatial translations and $K^i{}_j$ is a generator of affine rotations about the centre of mass. The angular momentum S^{ij} given by

$$S^{ij} = K^{ij} - K^{ji}$$

thus, if N^{ij} is symmetric, then spin is conserved, i.e.,

$$\frac{\mathrm{d}S^{ij}}{\mathrm{d}t} = N^{ij} - N^{ji}.$$

We deal here with the system of $n^2 + n = n(n+1)$ degrees of freedom (in the n-dimensional space; physically it is 12, when $n = 3$) and this is just half the dimensionality of the general solution.

The forces are potential and given by

$$F^i = -g^{ij}\frac{\partial V}{\partial r^j}$$

and the momentum of forces are

$$N^{ij} = -\varphi^i{}_A \frac{\partial V}{\partial \varphi^k{}_A} g^{kj}.$$

When there exist dissipative forces non-derivable from Lagrangian or Hamiltonian, then there appear some additional terms. In the simplest case, we choose them just linear or quadratic in generalized velocities $\mathrm{d}r^i/\mathrm{d}t$ and $\mathrm{d}\varphi^i{}_k/\mathrm{d}t$.

Let us pay attention for the point that there are some additional geometric, namely group-implied, forces imposed on the system. Gyroscopic constraints, or rather pseudo-holonomic constraints of rigid motion, consist of the first equation. It means that $\Omega^i{}_j$, $\widehat{\Omega}^A{}_B$ are respectively g-skew-symmetric and η-skew-symmetric angular velocities in spatial and co-moving representations,

$$\Omega^i{}_j = -\Omega_j{}^i = -g_{jk}\Omega^k{}_l g^{li}, \qquad \widehat{\Omega}^A{}_B = -\widehat{\Omega}_B{}^A = -\eta_{BC}\widehat{\Omega}^C{}_D\eta^{DA}$$

where g is the metric tensor of the physical space and η is the material metric.

It is easy to see that the above conditions are holonomic and may be written down as the conditions of isometry,

$$g_{ij}\varphi^i{}_A\varphi^j{}_B = \eta_{AB}.$$

Then the reaction moments N_R are symmetric

$$N_{Rij} = N_{Rji}$$

and our equations are independent of explicitly non-specified reactions. Of course, gyroscopic reactions do not vanish, but their full tensor contractions with skew-symmetric affine virtual velocities (angular velocities) are vanishing in virtue of constraints. So, if we are taking the skew-symmetric part of original equations, we can eliminate reaction moments and then obtain the effective equations of motion. Interesting is the case of incompressible body, i.e., when we consider isochoric constraints. It is traditionally very familiar and important in continuum mechanics, first of all in fluids. The traces of affine velocities do vanish then

$$\mathrm{Tr}\,\Omega = \Omega^i{}_i = 0.$$

The total contractions of such virtual Ω-s with the reaction affine moment N_R must vanish:

$$N_R{}^{ij}\Omega_{ji} = N_R{}^{ij}\Omega^k{}_i g_{jk} = 0.$$

It is easy to see that then reactions are pure traces

$$N_R{}^i{}_j = \lambda\delta^i{}_j, \qquad N_R{}^{ij} = \lambda g^{ij}$$

where

$$\lambda = \frac{1}{n}\ \mathrm{Tr}\ N_R = \frac{1}{n}\ g_{ij}N_R{}^{ij}.$$

So, to eliminate the Lagrange multiplier λ, we must take the constraints condition (i.e., $\det\varphi = \mathrm{const}$) jointly with the g-traceless part of the initial equation itself, i.e., explicitly

$$\varphi^i{}_A\frac{\mathrm{d}^2\varphi^j{}_B}{\mathrm{d}t^2}J^{AB} - \frac{1}{n}g_{ab}\varphi^a{}_A\frac{\mathrm{d}^2\varphi^b{}_B}{\mathrm{d}t^2}J^{AB}g^{ij} = N^{ij} - \frac{1}{n}g_{ab}N^{ab}g^{ij}.$$

We can discuss constraints implied by the linear conformal group, the group generated by rotations and dilatations. In such a case an affine velocity (gyration) has the form

$$\Omega^i{}_j = \omega^i{}_j + \alpha\delta^i{}_j$$

where $\omega^i{}_j$ is the g-skew-symmetric angular velocity, and α is an arbitrary real, dilatational parameter, so that

$$g_{ij}\varphi^i{}_A\varphi^j{}_B = \lambda\eta_{AB}, \qquad \lambda > 0.$$

The reaction-free equations of motion consist of the skew-symmetric part of the original equation and of the g-trace of that equation, and reaction moments $N_R{}^{ij}$ are symmetric and g-traceless

$$\begin{aligned}\varphi^i{}_A\frac{\mathrm{d}^2\varphi^j{}_B}{\mathrm{d}t^2}J^{AB} - \varphi^j{}_A\frac{\mathrm{d}^2\varphi^i{}_B}{\mathrm{d}t^2}J^{AB} &= N^{ij} - N^{ji}\\ g_{ij}\varphi^i{}_A\frac{\mathrm{d}^2\varphi^j{}_B}{\mathrm{d}t^2}J^{AB} &= g_{ij}N^{ij}.\end{aligned}$$

At the end, let us quote some very interesting example of non-holonomic constraints, when Ω is g-symmetric, the purely rotation-free motion (of course, the only geometrically correct definition)

$$\Omega^i{}_j - \Omega_j{}^i = \Omega^i{}_j - g_{jk}g^{il}\Omega^k{}_l = 0.$$

Then the reactions forces are anti-symmetric. So, the above equation must be joined with the symmetric part of equations of motion as balance laws

$$\varphi^i{}_A\frac{\mathrm{d}^2\varphi^j{}_B}{\mathrm{d}t^2}J^{AB} + \varphi^j{}_A\frac{\mathrm{d}^2\varphi^i{}_B}{\mathrm{d}t^2}J^{AB} = N^{ij} + N^{ji}.$$

2. Vakonomic Constraints

Let Lagrangian of the dynamical system be $L\left(q,\dot{q}\right)$, i.e., it is a function of generalized coordinates $q^1,\ldots,q^n$ and their velocities, but we can also take the time into a consideration explicitly, i.e., $L\left(t,q,\dot{q}\right)$.

Then the constraints are given by the following expressions

$$F_a\left(q,\dot{q}\right)=0, \qquad a=1,\ldots,m$$

or in the second case it may be given by

$$F_a\left(t,q,\dot{q}\right)=0.$$

First of all, if the constraints are linear in velocities, then

$$F_a\left(q,\dot{q}\right)=\omega_{ai}\left(q\right)\dot{q}^i=\omega_{ai}\left(q\right)\frac{\mathrm{d}q^i}{\mathrm{d}t}$$

(the summation convention is supposed here).

The variational principle constrained by $F_a=0$ is given by the following expressions

$$\delta\int L\left(q(t),\dot{q}(t)\right)\mathrm{d}t=0, \qquad F_a\left(q(t),\dot{q}(t)\right)=0.$$

Remark 1. *The variations $\delta q^i\left(t\right)$ are subject to constraints.*

The Lusternik theorem give us that the last variational principle is equivalent to the corresponding non-restricted principle

$$\delta\int L\left[\mu\right]\left(q(t),\dot{q}(t)\right)\mathrm{d}t=0$$

where μ is the Lagrange multiplier and $L\left[\mu\right]$ is given by the expression

$$L\left[\mu\right]\left(q(t),\dot{q}(t)\right)=L(q(t),\dot{q}(t))-\mu^aF_a(q(t),\dot{q}(t)).$$

Mathematically here μ^a are some a priori unknown functions of time.

The variational principle for $L\left[\mu\right]$ implies that for constraints which are linear in velocities

$$F_a(q(t),\dot{q}(t))=\omega_{ai}(q(t))\dot{q}^i(t)$$

we can write the following equations of motion

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot{q}^i}-\frac{\partial L}{\partial q^i}&=\frac{\mathrm{d}\mu^a}{\mathrm{d}t}\omega_{ai}-\mu^a\left(\frac{\partial\omega_{aj}}{\partial q^i}-\frac{\partial\omega_{ai}}{\partial q^j}\right)\dot{q}^j\\ F_a(q(t),\dot{q}(t))&=\omega_{ai}(q(t))\dot{q}^i(t)=0.\end{aligned}$$

This is the system of $(n+m)$ differential equations for the $(n+m)$ variables $q^i\left(t\right)$ and $\mu^a\left(t\right)$ as functions of time.

Correspondingly the constraints reactions are given as follows

$$R_i = \frac{\mathrm{d}\mu^a}{\mathrm{d}t}\omega_{ai} + \mu^a \left(\frac{\partial \omega_{ai}}{\partial q^j} - \frac{\partial \omega_{aj}}{\partial q^i} \right) \frac{\mathrm{d}q^j}{\mathrm{d}t}.$$

For the holonomic constraints

$$F_a(q) = 0, \qquad a = 1, \ldots, m$$

in the reaction forces survives only the first term and then they are given by the d'Alembert expression

$$R_i = \lambda^a \omega_{ai}$$

with the multiplier $\lambda^a = \mathrm{d}\mu^a/\mathrm{d}t$. We see that for the holonomic constraints the variational procedure and d'Alembert principle are identical.

This variational procedure works smoothly also for constraints nonlinear in velocities and constraints imposed on higher-order time derivatives.

3. Non-variational Non-holonomic Constraints

Let Lagrangian be $L(q, \dot{q})$ and the constraints

$$F_a(q, \dot{q}) = 0, \qquad a = 1, \ldots, m$$

which in applications mostly often are linear in velocities

$$F_a(q, \dot{q}) = \omega_{ai}(q)\, \dot{q}^i.$$

Then the d'Alembert principle give us the following equations of motion

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = R_i$$

where R_i are reaction forces, which vanish on velocities compatible with constraints

$$\omega_{ai}(q)\, \dot{q}^i = 0, \qquad \text{i.e.,} \qquad R_i \dot{q}^i = 0.$$

This implies that

$$R_i = \lambda^a \omega_{ai}$$

but without an additional term vanishing on all generalized velocities.

By analogy the similar expressions can be written also for systems with dissipative forces. The non-constrained dynamics is given by the following equations of motion

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = D_i$$

where D_i are covariant vectors of non-variational, e.g., **friction forces**.

The corresponding constrained systems is given by the expressions

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = D_i + R_i$$
$$F_a\left(q(t), \dot{q}(t)\right) = \omega_{ai}(q)\dot{q}^i = 0$$

where R_i are the **reaction forces**.

There are two prescriptions for calculating R_i, namely

1. The d'Alembert prescription

$$R_i = \lambda^a \omega_{ai} = 0, \qquad \text{i.e.,} \qquad R_i \dot{q}^i = 0$$

   for every virtual velocity satisfying the constraints

2. The vaconomic prescription

$$R_i = \frac{\mathrm{d}\mu^a}{\mathrm{d}t}\omega_{ai} + \mu^a\left(\frac{\partial \omega_{ai}}{\partial q^j} - \frac{\partial \omega_{aj}}{\partial q^i}\right)\frac{\mathrm{d}q^j}{\mathrm{d}t}.$$

4. Non-holonomic Constraints of Rotation-less Affine Motion

The affine motion is defined as follows

$$\xi^i(t) = r^i(t) + \varphi^i{}_A(t)a^A$$

where ξ^i are **Euler coordinates** and a^K are **Lagrange coordinates**.

Then the affine velocity is given by the expression

$$\Omega^i{}_j = \frac{\mathrm{d}\varphi^i{}_A}{\mathrm{d}t}\varphi^{-1A}{}_j$$

and its co-moving counterpart is as follows

$$\widehat{\Omega}^A{}_B = \varphi^{-1A}{}_i\frac{\mathrm{d}\varphi^i{}_B}{\mathrm{d}t} = \varphi^{-1A}{}_i\Omega^i{}_j\varphi^j{}_B.$$

For the gyroscopic (metrically rigid) motion we have that

$$\Omega^i{}_j + \Omega_j{}^i = \Omega^i{}_j + g_{ja}\Omega^a{}_b g^{bi} = 0$$

i.e., they are g-antisymmetric. This is non-holonomic description of holonomic constraints. Skew-symmetric matrices form a Lie algebra and those equations are integrated to the orthogonal group.

By analogy, the rotation-less motion is primarily described by

$$\Omega^i{}_j - \Omega_j{}^i = \Omega^i{}_j - g_{jk}g^{il}\Omega^k{}_l = 0$$

i.e., by the g-symmetry. But symmetric matrices do not form a Lie algebra. Moreover thoe are non-holonomic constraints and they are not integrated to any submanifold.

The polar decomposition of φ can be written as follows

$$\varphi = UA$$

where U is an orthogonal (isometric) matrix and A is an η-symmetric one

$$U \in \mathrm{O}(U, \eta; V, g), \qquad A \in \mathrm{Symm}(U, \eta)$$

i.e.,

$$\eta_{AB} = g_{ij}\varphi^i{}_A\varphi^j{}_B, \qquad \eta_{AC}A^C{}_B = \eta_{BC}A^C{}_A.$$

The co-moving angular velocity $\widehat{\omega}$ of the U-rotator is given by

$$\widehat{\omega} = U^{-1}\frac{\mathrm{d}U}{\mathrm{d}t}.$$

The kinetic energy can be written as the sum of the translational and internal (relative) terms as follows

$$T = T_{\mathrm{tr}} + T_{\mathrm{int}} = \frac{M}{2}g_{ij}\frac{\mathrm{d}r^i}{\mathrm{d}t}\frac{\mathrm{d}r^j}{\mathrm{d}t} + \frac{1}{2}g_{ij}\frac{\mathrm{d}\varphi^i{}_A}{\mathrm{d}t}\frac{\mathrm{d}\varphi^j{}_B}{\mathrm{d}t}J^{AB}$$

where m is the total mass and J^{AB} is the co-moving tensor of inertia, i.e.,

$$m = \int \mathrm{d}\mu(a), \qquad J^{AB} = \int a^A a^B \mathrm{d}\mu(a).$$

In the polar decomposition the internal kinetic energy T_{int} becomes as follows

$$\begin{aligned} T_{\mathrm{int}} = {} & \frac{1}{2}\eta_{KL}\frac{\mathrm{d}A^K{}_A}{\mathrm{d}t}\frac{\mathrm{d}A^L{}_B}{\mathrm{d}t}J^{AB} + \eta_{KL}\widehat{\omega}^K{}_C A^C{}_A\frac{\mathrm{d}A^L{}_B}{\mathrm{d}t}J^{AB} \\ & + \frac{1}{2}\eta_{KL}\widehat{\omega}^K{}_C\widehat{\omega}^L{}_D A^C{}_A A^D{}_B J^{AB}. \end{aligned}$$

Obviously, $\widehat{\omega}$ is η-skew-symmetric

$$\eta_{AC}\widehat{\omega}^C{}_B = -\eta_{BC}\widehat{\omega}^C{}_A.$$

The g-symmetry constraints on Ω imply that

$$\widehat{\omega} = \frac{1}{2}\left[A^{-1}, \frac{\mathrm{d}A}{\mathrm{d}t}\right] = \frac{1}{2}\left(A^{-1}\frac{\mathrm{d}A}{\mathrm{d}t} - \frac{\mathrm{d}A}{\mathrm{d}t}A^{-1}\right).$$

Substituting this to the expression for the internal kinetic energy T_{int}, we obtain that

$$\begin{aligned} T_{\mathrm{int}}^{\mathrm{Vak}} = {} & \frac{1}{2}\eta_{KL}\frac{\mathrm{d}A^K{}_A}{\mathrm{d}t}\frac{\mathrm{d}A^L{}_B}{\mathrm{d}t}J^{AB} \\ & + \frac{1}{4}\eta_{KL}A^{-1K}{}_D\frac{\mathrm{d}A^D{}_C}{\mathrm{d}t}A^C{}_A\frac{\mathrm{d}A^L{}_B}{\mathrm{d}t}J^{AB} \\ & + \frac{1}{8}\eta_{KL}A^{-1K}{}_E\frac{\mathrm{d}A^E{}_C}{\mathrm{d}t}A^C{}_A A^{-1L}{}_F\frac{\mathrm{d}A^F{}_D}{\mathrm{d}t}A^D{}_B J^{AB}. \end{aligned}$$

The simplest vakonomic Lagrangian is obtained by putting

$$L_{\text{int}}^{\text{Vak}} = T_{\text{int}}^{\text{Vak}} + \mathcal{V}(G)$$

where the potential $\mathcal{V}$ depends on the Green deformation tensor

$$G_{AB} = g_{ij}\varphi^{i}{}_{A}\varphi^{j}{}_{B} = \eta_{CD}A^{C}{}_{A}A^{D}{}_{B}.$$

As the Lagrangian is expressed through A and its time derivative $\mathrm{d}A/\mathrm{d}t$, we can subject it directly to the variational procedure by the substitution

$$A \to A + \delta A$$

and developing the resulting δL up to the first-order terms in δA.

Remark 2. *The matrix A is η-symmetric and so must be also $\lambda^{K}{}_{L}$ in the expansion*

$$\delta \int L^{\text{Vak}}\mathrm{d}t = \int \lambda^{K}{}_{L}\left(A(t), \dot{A}(t), \ddot{A}(t)\right)\delta A^{L}{}_{K}(t)\mathrm{d}t.$$

But it needs not be so from the very calculation because the summation convention removes the skew-symmetric part. However, the true final equations of motion must be η-symmetric

$$\mathrm{Sym}_{\eta}\lambda^{K}{}_{L} = \lambda^{K}{}_{L} + \eta^{KA}\eta_{LB}\lambda^{A}{}_{B} = 0$$

i.e,

$$\lambda_{KL} + \lambda_{LK} = 0$$

where

$$\lambda_{KL} = \eta_{KC}\lambda^{C}{}_{L}.$$

One can show that for the usual (non-vakonomic) constraints of the rotation-less motion the evolution of the system is given by the symmetric part of the following tensor equation

$$AJ_{\eta}\frac{\mathrm{d}^2A}{\mathrm{d}t^2} - \frac{1}{2}AJ_{\eta}A\frac{\mathrm{d}}{\mathrm{d}t}\left[A^{-1}, \frac{\mathrm{d}A}{\mathrm{d}t}\right] - AJ_{\eta}\frac{\mathrm{d}}{\mathrm{d}t}\left[A^{-1}, \frac{\mathrm{d}A}{\mathrm{d}t}\right] + \frac{1}{4}AJ_{\eta}A\frac{\mathrm{d}}{\mathrm{d}t}\left[A^{-1}, \frac{\mathrm{d}A}{\mathrm{d}t}\right]^2 = \overline{N}$$

where

$$J_{\eta}{}^{K}{}_{L} = J^{KM}\eta_{ML}, \qquad \overline{N}^{KL} = A^{K}{}_{M}A^{L}{}_{N}\widehat{N}^{MN}$$

$$\widehat{N}^{AB} = \varphi^{-1A}{}_{i}\varphi^{-1B}{}_{j}N^{ij}, \qquad N^{ij} = -\,g^{jk}\varphi^{i}{}_{M}\frac{\partial \mathcal{V}}{\partial \varphi^{k}{}_{M}}.$$

Solving the symmetric part of our equations for A, we find $A(t)$, then we substitute it to $\widehat{\omega}$ and solving equation

$$\frac{\mathrm{d}U}{\mathrm{d}t} = U\widehat{\omega}$$

we find $U(t)$. Finally, substituting it to

$$\varphi(t) = U(t)A(t)$$

we solve the problem, at least in principle.

The structures of vakonomic and d'Alembert equations are evidently different.

The variational derivative of $T_{\rm int}^{\rm Vak}$ with respect to the symmetric tensor

$$A_{AB} = \eta_{AC}A^{C}{}_{B} = A_{BA}$$

is given by

$$\begin{aligned}
\left.\frac{\delta T_{\rm int}^{\rm Vak}}{\delta A_{AB}}\right|_{\rm symm} &= -\frac{1}{4}\frac{{\rm d}^2}{{\rm d}t^2}A^{(A}{}_{L}J^{B)L} - \frac{1}{4}\frac{\rm d}{{\rm d}t}\left(\left(A^{-1}\right)^{(A}{}_{E}J^{B)L}\frac{{\rm d}A^{E}{}_{C}}{{\rm d}t}A^{C}{}_{L}\right)\\
&-\frac{1}{4}\eta_{KL}\frac{\rm d}{{\rm d}t}\left(\frac{{\rm d}A^{K}{}_{E}}{{\rm d}t}\left(A^{-1}\right)^{L(A}A^{B)}{}_{D}\right)J^{ED}\\
&-\frac{1}{4}\eta_{KL}\frac{\rm d}{{\rm d}t}\left(\left(A^{-1}\right)^{K}{}_{E}\frac{{\rm d}A^{E}{}_{C}}{{\rm d}t}A^{C}{}_{F}\left(A^{-1}\right)^{L(A}A^{B)}{}_{D}\right)J^{FD}\\
&-\frac{1}{4}\eta_{KL}\frac{{\rm d}A^{K}{}_{E}}{{\rm d}t}\frac{{\rm d}A^{F}{}_{D}}{{\rm d}t}A^{D}{}_{G}\left(A^{-1}\right)^{L(A}\left(A^{-1}\right)^{B)}{}_{F}J^{EG}\\
&-\frac{1}{4}\eta_{KL}\left(A^{-1}\right)^{K}{}_{E}\frac{{\rm d}A^{E}{}_{C}}{{\rm d}t}A^{C}{}_{M}\frac{{\rm d}A^{F}{}_{D}}{{\rm d}t}A^{D}{}_{N}\left(A^{-1}\right)^{L(A}\left(A^{-1}\right)^{B)}{}_{F}J^{MN}\\
&+\frac{1}{4}\eta_{KL}\frac{{\rm d}A^{K}{}_{D}}{{\rm d}t}\left(A^{-1}\right)^{L}{}_{E}\frac{{\rm d}A^{E(A}}{{\rm d}t}J^{B)D}\\
&+\frac{1}{4}\eta_{KL}\left(A^{-1}\right)^{K}{}_{E}\frac{{\rm d}A^{E}{}_{C}}{{\rm d}t}A^{C}{}_{D}\left(A^{-1}\right)^{L}{}_{F}\frac{{\rm d}A^{F(A}}{{\rm d}t}J^{B)D}.
\end{aligned}$$

When there are hyperelastic forces derivable from the potential $\mathcal{V}$ depending only on the Green deformation tensor, then equations of motion have the following form

$$\left.\frac{\delta T_{\rm int}^{\rm Vak}}{\delta A_{AB}}\right|_{\rm symm} = -A_{KC}\eta^{K(A}\widehat{N}^{B)C}$$

where

$$\widehat{N}^{BC} = -\left(D\mathcal{V}\right)^{BC}.$$

In spite of their apparently complicated structure, the above equations are readable. And having them solved for the time dependence of A_{AB}, we obtain the time dependence of $\widehat{\omega}$, and then, solving (in principle) equation defining $\widehat{\omega}$ for dependence $t \to U(t)$, we finally obtain (in principle) $\varphi = UA$.

Let us mention that all tensor indices are shifted from their natural position with the help of η.

The usual d'Alembert procedure leads to the following less readable form

$$J^{AB}\frac{\mathrm{d}^2 A^{B(C}}{\mathrm{d}t^2}A^{D)}{}_A - J^A{}_B A^B{}_E \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\left(\left(A^{-1}\right)^E{}_F \frac{\mathrm{d}}{\mathrm{d}t}\left(A^{F(C}\right)A^{D)}{}_A \right.$$
$$\left. -\frac{\mathrm{d}}{\mathrm{d}t}\left(A^E{}_F\right)\left(A^{-1}\right)^{F(C}A^{D)}{}_A\right) - J^A{}_B\frac{\mathrm{d}A^B{}_E}{\mathrm{d}t}\left(\left(A^{-1}\right)^E{}_F\frac{\mathrm{d}}{\mathrm{d}t}\left(A^{F(C}\right)A^{D)}{}_A\right.$$
$$\left. -\frac{\mathrm{d}}{\mathrm{d}t}\left(A^E{}_F\right)\left(A^{-1}\right)^{F(C}A^{D)}{}_A\right) + \frac{1}{4}J^A{}_B A^B{}_E\left(\left(A^{-1}\right)^E{}_G\frac{\mathrm{d}}{\mathrm{d}t}\left(A^G{}_F\right)\right.$$
$$\left. -\frac{\mathrm{d}}{\mathrm{d}t}\left(A^E{}_G\right)\left(A^{-1}\right)^G{}_F\right)\left(\left(A^{-1}\right)^F{}_H\frac{\mathrm{d}}{\mathrm{d}t}\left(A^{H(C}\right)A^{D)A}\right.$$
$$\left. -\frac{\mathrm{d}}{\mathrm{d}t}\left(A^F{}_H\right)\left(A^{-1}\right)^{H(C}A^{D)}{}_A\right) \quad = \quad \overline{N}^{(CD)}.$$

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