Fourteenth International Conference on
Geometry, Integrability and Quantization
June 8–13, 2012, Varna, Bulgaria
Ivaïlo M. Mladenov, Andrei Ludu
and Akira Yoshioka, Editors
Avangard Prima, Sofia 2013, pp 160–175
doi: 10.7546/giq-14-2013-160-175

SOME REMARKS ON THE EXPONENTIAL MAP ON THE GROUPS $\mathrm{SO}(n)$ AND $\mathrm{SE}(n)$

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Abstract. The problem of describing or determining the image of the exponential map $\exp : \mathfrak{g} \to G$ of a Lie group G is important and it has many applications. If the group G is compact, then it is well-known that the exponential map is surjective, hence the exponential image is G. In this case the problem is reduced to the computation of the exponential and the formulas strongly depend on the group G. In this paper we discuss the generalization of Rodrigues formulas for computing the exponential map of the special orthogonal group $\mathrm{SO}(n)$, which is compact, and of the special Euclidean group $\mathrm{SE}(n)$, which is not compact but its exponential map is surjective, in the case $n \geq 4$.

1. Introduction. Lie Groups and the Exponential Map

Let G be a Lie group with its Lie algebra $\mathfrak{g}$. The exponential map $\exp : \mathfrak{g} \to G$ is defined by $\exp(X) = \gamma_X(1)$, where $X \in \mathfrak{g}$ and γ_X is the one-parameter subgroup of G induced by X. Recall the following general properties of the exponential map:

1. For every $t \in \mathbb{R}$ and for every $X \in \mathfrak{g}$, we have $\exp(tX) = \gamma_X(t)$
2. For every $s, t \in \mathbb{R}$ and for every $X \in \mathfrak{g}$, we have
$$\exp(sX)\exp(tX) = \exp(s+t)X$$
3. For every $t \in \mathbb{R}$ and for every $X \in \mathfrak{g}$, we have $\exp(-tX) = \exp(tX)^{-1}$
4. $\exp\colon \mathfrak{g} \to G$ is a smooth mapping, it is a local diffeomorphism at $0 \in \mathfrak{g}$ and $\exp(0) = e$, where e is the unity element of the group G
5. The image $\exp(\mathfrak{g})$ of the exponential map generates the connected component G_e of the unity $e \in G$

6. If $f : G_1 \to G_2$ is a morphism of Lie groups and $f_* : \mathfrak{g}_1 \to \mathfrak{g}_1$ is the induced morphism of Lie algebras, then $f \circ \exp_1 = \exp_2 \circ f$.

As we can note from the previous Property 5 (see also [2]), the following problems are of special importance

Problem 1. *Find the conditions on the group G such that the exponential is surjective.*

Problem 2. *Determine the image $E(G)$ of the exponential map.*

J. Dixmier has suggested to study Problem 2 for resoluble Lie groups. Concerning Problem 1, only in few special situations we have $G = E(G)$, i.e., the surjectivity of the exponential map. A Lie group satisfying this property is called **exponential**. Every compact and connected Lie group is exponential (see [1]), but there are exponential Lie groups which are not compact.

Even if we know that the exponential map is surjective, to get closed formulas for the exponential map for different Lie groups is an interesting problem. Such formulas are well-known for the **special orthogonal group** $\mathrm{SO}(n)$ and for the **special Euclidean group** $\mathrm{SE}(n)$, when $n = 2, 3$, as Rodrigues' formulas. One of the main goal of our presentation is to discuss the possibility to extend the Rodrigues' formulas for these two Lie groups in dimensions $n \geq 4$.

2. The Rodrigues Formula for the Group $\mathrm{SO}(n)$

The exponential map $\exp : \mathfrak{gl}(n, \mathbb{R}) = M_n(\mathbb{R}) \to \mathrm{GL}(n, \mathbb{R})$, where $\mathrm{GL}(n, \mathbb{R})$ denotes the Lie group of real invertible $m \times n$ matrices, is defined by (see for instance Chevalley [6], Marsden and Ratiu [13], or Warner [24])

$$\exp(X) = \sum_{k=0}^{\infty} \frac{1}{k!} X^k. \tag{1}$$

Moler and van Loan [16] discussed in details with numerous numerical examples the principal methods to compute the exponential of a matrix.

According to the well-known Hamilton-Cayley theorem, it follows that

$$\exp(X) = \sum_{k=0}^{n-1} a_k(X) X^k \tag{2}$$

where the real coefficients $a_0(X), \cdots, a_{n-1}(X)$ depend on the matrix X. More precisely, $a_0, \cdots, a_{n-1}$ are functions of the eigenvalues $\lambda_1, \ldots, \lambda_n$ of the matrix X, i.e., we have $a_j = a_j(\lambda_1, \ldots, \lambda_n), j = 0, \cdots, n-1$. From this formula, it follows that $\exp(X)$ is a polynomial of X. The problem to find a reasonable formula for $\exp(X)$ is reduced to the problem to determinate the coefficients $a_0, \cdots, a_{n-1}$. Because the historical argument, we will call these coefficients the **Rodrigues coefficients** of the matrix X.

The following general result is proved in the paper Andrica and Rohan [3, 4] for matrices of the general linear group $\mathrm{GL}(n, \mathbb{R})$.

Theorem 1. 1) *The Rodrigues coefficients of the matrix are solutions to the system*

$$\sum_{k=0}^{n-1} S_{k+j} a_k = \sum_{s=1}^{n} \lambda_s^j \mathrm{e}^{\lambda_s}, \qquad j = 0, \dots, n-1 \tag{3}$$

where $\lambda_1, \dots, \lambda_n$ *are the eigenvalues of* X*, and* $S_j = \lambda_1^j + \dots + \lambda_n^j$.
2) *If the eigenvalues* $\lambda_1, \dots, \lambda_n$ *of the matrix* X *are pairwise distinct, then the Rodrigues coefficients* $a_0, \dots, a_{n-1}$ *are perfectly determined by the system and they are linear combinations of* $\mathrm{e}^{\lambda_1}, \dots, \mathrm{e}^{\lambda_n}$ *with some coefficients which are rational functions of* $\lambda_1, \dots, \lambda_n$*, i.e., we have*

$$a_k = A_k^{(1)} \mathrm{e}^{\lambda_1} + \dots + A_k^{(n)} \mathrm{e}^{\lambda_n}, \qquad k = 0, \dots, n-1.$$

It is well-known that the Lie algebra $\mathfrak{so}(n)$ of the special orthogonal group $\mathrm{SO}(n)$ consists in all skew-symmetric matrices in $M_n(\mathbb{R})$ and that the Lie bracket is the standard commutator $[A, B] = AB - BA$. Due to geometric reasons, the matrices in $\mathrm{SO}(n)$ are also called **rotation matrices**.

The exponential map $\exp : \mathfrak{so}(n) \to \mathrm{SO}(n)$ is defined by the same formula (1) because it is given by the restriction $\exp|_{\mathfrak{so}(n)}$ of the exponential map $\exp : \mathfrak{gl}(n, \mathbb{R}) \to \mathrm{GL}(n, \mathbb{R})$. The matrices in $\mathfrak{so}(n)$ have two essential properties which simplify the computation of the Rodrigues coefficients

- If n is odd, then they are singular, i.e., they have one eigenvalue equal to 0 (possible with a multiplicity)
- The non-zero eigenvalues are purely imaginary and, of course, conjugated.

According to the well-known Euler formula $\mathrm{e}^{\mathrm{i}\alpha} = \cos\alpha + \mathrm{i}\sin\alpha$, we obtain the following consequence of Theorem 1 which is useful in concrete applications.

Corollary 2. *If the eigenvalues* $\lambda_1, \dots, \lambda_n$ *of the matrix* X *are pairwise distinct, then the Rodrigues coefficients* $a_0, \dots, a_{n-1}$ *are perfectly determined by the system and they are linear combinations of* $\cos\alpha_1, \cdots, \cos\alpha_m, \sin\alpha_1, \cdots, \sin\alpha_m$ *having as coefficients rational functions of* $\alpha_1, \dots, \alpha_m$*, where* $\pm\mathrm{i}\alpha_1, \dots, \pm\mathrm{i}\alpha_m$, $m = \lfloor \frac{n}{2} \rfloor$*, are the eigenvalues of matrix* X*. That is, we have*

$$a_k = b_k^{(1)} \cos\alpha_1 + \dots + b_k^{(m)} \cos\alpha_m + c_k^{(1)} \sin\alpha_1 + \dots + c_k^{(m)} \sin\alpha_m, \; k = 0, \dots, n-1.$$

3. Illustrating Some Concrete Cases

3.1. The Classical Cases $n = 2$ and $n = 3$

Clearly, when $X = O_n$, we have $\exp(X) = I_n$ hence, in this situation we have $a_0 = 1, a_1 = \dots = a_{n-1} = 0$.

When $n = 2$, a skew-symmetric matrix $X \neq O_2$ can be written as

$$X = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}, \qquad a \in \mathbb{R}^*$$

having the eigenvalues $\lambda_1 = a\mathrm{i}$, $\lambda_2 = -a\mathrm{i}$.
The system (3) in this case is

$$\begin{aligned} 2a_0 + (\lambda_1 + \lambda_2)a_1 &= \mathrm{e}^{\lambda_1} + \mathrm{e}^{\lambda_2} \\ (\lambda_1 + \lambda_2)a_0 + (\lambda_1^2 + \lambda_2^2)a_1 &= \lambda_1\mathrm{e}^{\lambda_1} + \lambda_2\mathrm{e}^{\lambda_2} \end{aligned}$$

and hence we immediately obtain

$$a_0 = \frac{1}{2}\left(\mathrm{e}^{a\mathrm{i}} + \mathrm{e}^{-a\mathrm{i}}\right) = \cos a$$

$$a_1 = \frac{\lambda_1\mathrm{e}^{\lambda_1} + \lambda_2\mathrm{e}^{\lambda_2}}{\lambda_1^2 + \lambda_2^2} = \frac{\mathrm{e}^{a\mathrm{i}} - \mathrm{e}^{-a\mathrm{i}}}{2a} = \frac{\sin a}{a}$$

and then

$$\exp(X) = (\cos a)I_2 + \frac{\sin a}{a}X. \tag{4}$$

It follows also that

$$a_0(X) = \cos a, \qquad a_1(X) = \frac{\sin a}{a}.$$

When $n = 3$, a real skew-symmetric matrix X is of the form

$$X = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix}$$

having the characteristic polynomial $p_X(t) = t^3 + (a^2 + b^2 + c^2)t = t^3 + \theta^2 t$, where $\theta = \sqrt{a^2 + b^2 + c^2}$. The eigenvalues of X are $\lambda_1 = \mathrm{i}\theta$, $\lambda_2 = -\mathrm{i}\theta$, $\lambda_3 = 0$. It is clear that $X = O_3$ if and only if $\theta = 0$, hence it suffices to consider only the situation $\theta \neq 0$. The system (3) is equivalent to

$$\begin{aligned} 3a_0 - 2\theta^2 a_2 &= 1 + \mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{-\mathrm{i}\theta} \\ -2\theta^2 a_1 &= \mathrm{i}\theta(\mathrm{e}^{\mathrm{i}\theta} - \mathrm{e}^{-\mathrm{i}\theta}) \\ -2\theta^2 a_0 + 2\theta^4 a_2 &= -\theta^2(\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{-\mathrm{i}\theta}) \end{aligned}$$

Because $\theta \neq 0$, it follows that

$$a_0 = 1, \qquad a_1 = \frac{\sin\theta}{\theta}, \qquad a_2 = \frac{1 - \cos\theta}{\theta^2}$$

giving the well-known classical formula due to Rodrigues.

$$\exp(X) = I_3 + \frac{\sin\theta}{\theta}X + \frac{1 - \cos\theta}{\theta^2}X^2. \tag{5}$$

The Lie algebra $(\mathrm{SO}(3), [.\,,.])$ is canonical isomorphic to the Lie algebra $(\mathbb{R}^3, \times)$, where "$\times$" denotes the classical cross product, and the isomorphism is given by $v \in \mathbb{R}^3 \mapsto \widehat{v} \in \mathrm{SO}(3)$, where

$$v = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \qquad \text{and} \qquad \widehat{v} = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix}.$$

According to this isomorphism, the Rodrigues formula (5) can be written in the following equivalent form ([1, Proposition 6.1.6])

$$\exp(\widehat{v}) = I_3 + \frac{\sin ||v||}{||v||}\widehat{v} + \frac{1}{2}\left(\frac{\sin \frac{||v||}{2}}{\frac{||v||}{2}}\right)^2 \widehat{v}^2. \tag{6}$$

3.2. The Case $n = 4$

The general skew-symmetric matrix $X \in \mathfrak{so}(4)$ is

$$X = \begin{pmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{pmatrix}$$

and the corresponding characteristic polynomial is given by

$$p_X(t) = t^4 + (a^2 + b^2 + c^2 + d^2 + \mathrm{e}^2 + f^2)t^2 + (af - be + cd)^2.$$

Let $\lambda_{1,2} = \pm\alpha\mathrm{i}$, $\lambda_{3,4} = \pm\beta\mathrm{i}$ be the eigenvalues of the matrix X, where $\alpha, \beta \in \mathbb{R}$. After simple algebraic manipulations, the system (3) becomes

$$\begin{aligned} 2a_0 - (\alpha^2 + \beta^2)a_2 &= \cos\alpha + \cos\beta \\ -(\alpha^2 + \beta^2)a_1 + (\alpha^4 + \beta^4)a_3 &= -\alpha\sin\alpha - \beta\sin\beta \\ -(\alpha^2 + \beta^2)a_0 + (\alpha^4 + \beta^4)a_2 &= -\alpha^2\sin\alpha - \beta^2\sin\beta \\ (\alpha^4 + \beta^4)a_1 - (\alpha^6 + \beta^6)a_3 &= \alpha^3\sin\alpha + \beta^3\sin\beta \end{aligned}$$

We consider the following three cases

Case 1. If $\alpha \neq \beta$, $\alpha, \beta \in \mathbb{R}^*$, then by grouping the first equation with the third one, and the second equation with the last one, we obtain the Rodrigues coefficients

$$a_0 = \frac{\beta^2\cos\alpha - \alpha^2\cos\beta}{\beta^2 - \alpha^2}, \qquad a_1 = \frac{\beta^3\sin\alpha - \alpha^3\sin\beta}{\alpha\beta(\beta^2 - \alpha^2)}$$

$$a_2 = \frac{\cos\alpha - \cos\beta}{\beta^2 - \alpha^2}, \qquad a_3 = \frac{\beta\sin\alpha - \alpha\sin\beta}{\alpha\beta(\beta^2 - \alpha^2)}.$$

In this case it follows the corresponding Rodrigues formula in the form

$$\exp(X) = \frac{\beta^2\cos\alpha - \alpha^2\cos\beta}{\beta^2-\alpha^2} I_4 + \frac{\beta^3\sin\alpha - \alpha^3\sin\beta}{\alpha\beta(\beta^2-\alpha^2)} X + \frac{\cos\alpha-\cos\beta}{\beta^2-\alpha^2} X^2 + \frac{\beta\sin\alpha - \alpha\sin\beta}{\alpha\beta(\beta^2-\alpha^2)} X^3. \quad (7)$$

Case 2. If $\alpha \neq 0$ and $\beta = 0$, then we will use the idea of the paper Andrica and Rohan [4], i.e., we can find the Rodrigues coefficients from the formulas in Case 1 by considering the limits when $\beta \to 0$. After easy computations we find the corresponding Rodrigues formula to this case is

$$\exp(X) = I_4 + X + \frac{1-\cos\alpha}{\alpha^2} X^2 + \frac{\alpha - \sin\alpha}{\alpha^3} X^3. \quad (8)$$

Case 3. If $\alpha = \beta \neq 0$, then we will use again the same method by considering the limits $\beta \to \alpha$, and we obtain the following Rodrigues formula

$$\exp(X) = \frac{\alpha\sin\alpha + 2\cos\alpha}{2} I_4 + \frac{3\sin\alpha - \alpha\cos\alpha}{2\alpha} X + \frac{\sin\alpha}{2\alpha} X^2 + \frac{\sin\alpha - \alpha\cos\alpha}{2\alpha^3} X^3. \quad (9)$$

Let us note that in the paper of Andrica and Rohan [3] the results in the singular situations contained in Cases 2 and 3 are obtained by using the Putzer method (see the original paper [21]). In the paper of Politi [20] it is obtained, by using a different method, the same result as in Case 1, but the singular cases are not considered in a concrete way.

Going back to the classical Rodrigues formula (5), it turns out that it is more convenient to normalize X, that is, to write $X = \theta X_1$ (where $X_1 = X/\theta$, assuming that $\theta \neq 0$), in which case the formula becomes

$$\exp(\theta X_1) = I_3 + \sin\theta X_1 + (1-\cos\theta) X_1^2.$$

Also, given $R \in \mathrm{SO}(3)$, we can find $\cos\theta$ because $\mathrm{tr}(R) = 1 + 2\cos\theta$, and we can find X_1 by observing that

$$\frac{1}{2}(R - R^\top) = \sin\theta X_1.$$

Actually, the above formula cannot be used when $\theta = 0$ or $\theta = \pi$, as $\sin\theta = 0$ in these cases. When $\theta = 0$, we have $R = I_3$ and $X_1 = 0$, and when $\theta = \pi$, we need to find X_1 such that

$$X_1^2 = \frac{1}{2}(R - I_3).$$

As X_1 is a skew-symmetric 3×3 matrix, this amounts to solving some simple equations with three unknowns. Again, the problem is completely solved.

What about the cases when $n \geq 4$? The reason why Rodrigues' formula can be derived is that

$$X^3 = -\theta^2 X$$

or, equivalently, $X_1^3 = -X_1$. Unfortunately, for $n \geq 4$, given any non-null skew-symmetric $n \times n$ matrix X, it is generally false that $X^3 = -\theta^2 X$, and the reasoning used in the three dimentional case does not apply.

In the paper of Gallier and Xu [7], it is shown that there is an implicit generalization of Rodrigues' formula for computing the exponential map $\exp : \mathfrak{so}(n) \to \mathrm{SO}(n)$, when $n \geq 4$.

Theorem 3. *Given any non-null skew-symmetric $n \times n$ matrix X, where $n \geq 3$, and if*

$$\{\mathrm{i}\theta_1, -\mathrm{i}\theta_1, \ldots, \mathrm{i}\theta_p, -\mathrm{i}\theta_p\}$$

is the set of distinct eigenvalues of X, where $\theta_j > 0$ and each $\mathrm{i}\theta_j$ (and $-\mathrm{i}\theta_j$) has multiplicity $k_j \geq 1$, there are p unique skew-symmetric matrices $X_1, \ldots, X_p$ such that the following relations hold

$$X = \theta_1 X_1 + \ldots + \theta_p X_p, \qquad X_i X_j = X_j X_i = O_n \ (i \neq j), \qquad X_i^3 = -X_i$$

for all i, j with $1 \leq i, j \leq p$, and $2p \leq n$. Furthermore, we have

$$\exp X = \mathrm{e}^{\theta_1 X_1 + \ldots + \theta_p X_p} = I_n + \sum_{i=1}^{p} \left((\sin \theta_i) X_i + (1 - \cos \theta_i) X_i^2 \right)$$

and $\{\theta_1, \ldots, \theta_p\}$ is the set of the distinct positive square roots of the $2m$ positive eigenvalues of the symmetric matrix $-1/4(X - X^\top)^2$, where $m = k_1 + \ldots + k_p$.

Unfortunately, this result is an implicit one because we are not able to determine the matrices $X_1, \ldots, X_p$.

4. Surjectivity of the Exponential Map on $\mathrm{SO}(n)$

It is well known that for every compact connected Lie group the exponential map is surjective (see Bröcker and Dieck [5], Andrica and Caşu [1] for the standard proof, or Rohan [22] for a new idea of the proof given by Tao), that every compact connected Lie group is exponential (see also the monograph of Wüstner [25] for details about the exponential groups). Because the group $\mathrm{SO}(n)$ is compact it follows that the exponential map $\exp : \mathfrak{so}(n) \to \mathrm{SO}(n)$ is surjective. The surjectivity of exp for the group $\mathrm{SO}(n)$ is an important property. Indeed, it implies the existence of a locally inverse function $\log : \mathrm{SO}(n) \to \mathfrak{so}(n)$, and this has interesting applications. Gallier and Xu [7] have mentioned that the functions exp and log for the group $\mathrm{SO}(n)$ can be used for motion interpolation (see Kim *et al* [11, 12], [14, 15, 17] and Park and Ravani [18, 19]). Motion interpolation and rational motions have also been investigated by Jüttler [9, 10]. Also, the surjectivity of the exponential map for the group $\mathrm{SO}(n)$ gives the possibility to describe the rotations of the Euclidean space $\mathbb{R}^n$ (see the recent paper of Rohan [22]).

Even if the following result is clear because for every $n \geq 1$, the group $\mathrm{SO}(n)$ is compact, we prefer to present the alternative proof because it gives an explicit way to find solutions to the matrix equation $\exp(X) = R$.

Proposition 4. *The exponential map*

$$\exp : \mathfrak{so}(3) \to \mathrm{SO}(3)$$

is surjective.

Proof: We show that for any rotation matrix $R \in \mathrm{SO}(3)$

$$R = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix}$$

there is $\widehat{\omega} \in \mathfrak{so}(3)$ so that

$$\exp(\widehat{\omega}) = R$$

or, via the Rodrigues formula, equivalent to

$$I_3 + \frac{\sin||\omega||}{||\omega||}\widehat{\omega} + \frac{1-\cos||\omega||}{||\omega||^2}\widehat{\omega}^2 = R.$$

From the above relation we obtain that

$$1 + 2\cos||\omega|| = \mathrm{tr}(R).$$

Because

$$-1 \leq \mathrm{tr}(R) \leq 3$$

we can conclude also that

$$||\omega|| = \arccos\frac{\mathrm{tr}(R)-1}{2}.$$

On the other hand we obtain

$$r_{32}-r_{23} = 2\omega_1\frac{\sin||\omega||}{||\omega||}, \qquad r_{13}-r_{31} = 2\omega_2\frac{\sin||\omega||}{||\omega||}, \qquad r_{21}-r_{12} = 2\omega_3\frac{\sin||\omega||}{||\omega||}.$$

So, we can consider

$$\omega = \frac{||\omega||}{2\sin||\omega||}\begin{pmatrix} r_{32}-r_{23} \\ r_{13}-r_{31} \\ r_{21}-r_{12} \end{pmatrix}$$

in order to obtain

$$\exp(\widehat{\omega}) = R.$$

■

Using the surjectivity of the exponential map $\exp : \mathfrak{so}(n) \to \mathrm{SO}(n)$ and the fact that if

$$E_i = \begin{pmatrix} 0 & -\theta_i \\ \theta_i & 0 \end{pmatrix}$$

then

$$\exp(E_i) = \begin{pmatrix} \cos\theta_i & -\sin\theta_i \\ \sin\theta_i & \cos\theta_i \end{pmatrix}$$

using Theorem 3, we obtain the following characterization of rotations in $\mathrm{SO}(n)$ for $n \geq 3$

Theorem 5. *Given any rotation matrix $R \in \mathrm{SO}(n)$, where $n \geq 3$, if*

$$\left\{ \mathrm{e}^{\mathrm{i}\theta_1}, \mathrm{e}^{-\mathrm{i}\theta_1}, \ldots, \mathrm{e}^{\mathrm{i}\theta_p}, \mathrm{e}^{-\mathrm{i}\theta_p} \right\}$$

is the set of distinct eigenvalues of R different from 1, where $0 < \theta_i \leq \pi$, there are p skew-symmetric matrices $X_1, \ldots, X_p$ such that

$$X_i X_j = X_j X_i = O_n, \qquad i \neq j$$

$$X_i^3 = -X_i$$

for all i, j with $1 < i, j \leq p$, and $2p \leq n$, and furthermore

$$R = \exp(\theta_1 B_1 + \ldots + \theta_p B_p) = I_n + \sum_{i=1}^{p} \left(\sin\theta_i X_i + (1 - \cos\theta_i) X_i^2 \right).$$

5. The Special Euclidean Group $\mathrm{SE}(n)$

The Euclidean group $\mathrm{E}(n)$ is the group of all isometries of the Euclidean space $\mathbb{R}^n$. When $n = 2, \mathrm{E}(2)$ consists in all plane translations, rotations and reflections.

The group of isometries of $\mathbb{R}^n$ can be represented by the matrix group denoted by $\mathrm{E}(n)$

$$\mathrm{E}(n) := \left\{ \begin{pmatrix} R & \mathbf{v} \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}(n+1, \mathbb{R});\ R \in \mathrm{SO}\,(n)\,, \quad \mathbf{v} \in \mathbb{R}^{n \times 1} \right\}$$

in terms of $(n+1) \times (n+1)$ matrices. The set of affine maps ρ of $\mathbb{R}^n$ defined by

$$\rho(\mathbf{x}) = R\mathbf{x} + \mathbf{u}$$

$R \in \mathrm{SO}\,(n)$ is a group under composition, called the group of direct affine isometries, or rigid motions, denotes as $\mathrm{SE}\,(n)$.

The vector space of real $(n+1) \times (n+1)$ matrices of the form

$$\Omega = \begin{pmatrix} B & \mathbf{u} \\ 0 & 0 \end{pmatrix}$$

where B is a skew-symmetric matrix and $\mathbf{u}$ is a vector in $\mathbb{R}^n$ is denoted by $\mathfrak{se}\,(n)$. The group SE (n) is a Lie group, called the special Euclidean group, and $\mathfrak{se}\,(n)$ is its Lie algebra. In what follows we will concentrate on some topological properties of the group SE (n).

It turns out that the group $\mathrm{E}(n)$ is not a connected Lie group. The special Euclidean group $\mathrm{SE}(n)$ is in fact the connected component of the identity of $\mathrm{E}(n)$. The Lie subgroup $\mathrm{SE}(n)$ corresponds to the group of all orientation-preserving isometries R with the property, $\det R = 1$.

For $n = 2$, we have

$$\mathrm{SE}(2) := \left\{ \begin{pmatrix} R_\theta & \mathbf{v} \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}(3,\mathbb{R});\ R_\theta \in \mathrm{SO}(2) \text{ and } \mathbf{v} \in \mathbb{R}^{2\times 1} \right\}$$

where

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \text{ and } \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

The group $\mathrm{SE}(n)$ is closed in $\mathrm{GL}(n+1,\mathbb{R})$, where the topology in $\mathrm{GL}(n+1,\mathbb{R})$ is defined by the Frobenius norm. Indeed, let $(A_m)_{m>0}$ be any sequence of elements in $\mathrm{SE}(n)$, and let $A_m \to A$ as $m \to \infty$. Since $\mathbb{R}^n$ is a complete space we have that $\mathbf{v} \in \mathbb{R}^n$. Since $\mathrm{SO}(n)$ is a closed subgroup of $\mathrm{GL}(n,\mathbb{R})$ it follows that there are two possible cases: $R \in \mathrm{SO}(n)$ and $R \notin \mathrm{GL}(n,\mathbb{R})$. In the first case A clearly satisfies all the properties of being an element of $\mathrm{SE}(n)$. If $R \notin \mathrm{GL}(n,\mathbb{R})$ then $\det R = 0$. If this is the case we have $\det A = 0$. Since $\det A = 0$ it follows that $A \notin \mathrm{GL}(n+1,\mathbb{R})$, which is not possible.

Therefore $\mathrm{SE}(n)$ is closed in $\mathrm{GL}(n+1,\mathbb{R})$. Hence it is a matrix Lie group (cf also [8] and [24]).

The group $\mathrm{SE}(n)$ is not bounded, hence it is not compact. To see this property, we have just to consider the sequence of matrices

$$A_m = \begin{pmatrix} I_n & \mathbf{v}_m \\ 0 & 1 \end{pmatrix}$$

where the vector $\mathbf{v}_m \in \mathbb{R}^n$ has the first component m and the other components equal to 0. The Frobenius norm of A_m is $||A_m|| = \sqrt{m+2}$, hence the sequence A_m is not bounded. Therefore $\mathrm{SE}(n)$ is not bounded, hence it is not compact.

5.1. The Rodrigues Formula for SE (n)

Let Ω be a matrix in $\mathfrak{se}\,(n)$

$$\Omega = \begin{pmatrix} X & \mathbf{u} \\ 0 & 0 \end{pmatrix}$$

where X is a skew–symmetric square matrix with real entries. The following simple observation is useful in determining a Rodrigues formula for the group SE (n). The characteristic polynomial p_Ω of the matrix Ω satisfies the following relation

$$p_\Omega(t) = tp_X(t).$$

Indeed, we have

$$p_\Omega(t) = \det(tI_{n+1} - \Omega) = \det \begin{pmatrix} tI_n - X & -\mathbf{u} \\ 0 & t \end{pmatrix} = t\det(tI_n - X) = tp_X(t).$$

When $n = 2$, consider a skew-symmetric matrix $X \neq O_2$

$$X = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}, \qquad a \in \mathbb{R}^*.$$

According to the observation above, the matrix $\Omega \in \mathfrak{se}(2)$ has the eigenvalues $\lambda_1 = a\mathrm{i}, \lambda_2 = -a\mathrm{i}, \lambda_3 = 0$. The Rodrigues formula is of the form

$$\exp(\Omega) = A_0 I_3 + A_1\Omega + A_2\Omega^2$$

and from Theorem 1, the Rodrigues coefficients A_0, A_1, A_2 satisfy the system (3) which simplifies exactly to the system giving the classical Rodrigues coefficients in Subsection 3.1. We obtain the formula

$$\exp(\Omega) = I_3 + \frac{\sin a}{a}\Omega + \frac{1 - \cos a}{a^2}\Omega^2. \tag{10}$$

The formula (10) helps us to prove after easy computations that $\exp A \in \mathrm{SE}(2)$ for all $A \in \mathfrak{se}(2)$.

When $n = 3$, consider a skew-symmetric matrix $X \neq O_3$

$$X = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix}$$

having the characteristic polynomial $p_X(t) = t^3 + (a^2 + b^2 + c^2)t = t^3 + \theta^2 t$, where $\theta = \sqrt{a^2 + b^2 + c^2}$.

The characteristic polynomial of $\Omega \in \mathfrak{se}(3)$ is $p_\Omega(t) = tp_X(t) = t^4 + \theta^2 t^2$ and hence, the eigenvalues of Ω are $\lambda_1 = \mathrm{i}\theta, \lambda_2 = -\mathrm{i}\theta$, $\lambda_3 = \lambda_4 = 0$. The Rodrigues formula is of the form

$$\exp(\Omega) = A_0 I_4 + A_1\Omega + A_2\Omega^2 + A_3\Omega^3.$$

Because we have a double eigenvalue $\lambda_3 = \lambda_4 = 0$, we will use the Putzer method (see the original paper [21] or Andrica and Rohan [3]). The Putzer matrix is

$$C = \begin{pmatrix} 0 & \theta^2 & 0 & 1 \\ \theta^2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

and after simple computations (details are given in the paper of Andrica and Rohan [3]), we obtain the following Rodrigues formula

$$\exp(\Omega) = I_4 + \Omega + \frac{1-\cos\theta}{\theta^2}\Omega^2 + \frac{\alpha - \sin\theta}{\theta^3}\Omega^3. \tag{11}$$

This formula is mentioned in the book by Selig [23, Chapter 4, pp 51-83] where it is obtained by a different method. Note that it has exactly the form as formula obtained in Case 2 of Subsection 3.2.

According to the isomorphism of Lie algebras $\mathbb{R}^3 \to \mathfrak{so}(3), \omega \to \widehat{\omega}$, mentioned in Subsection 3.1, formula (11) can be written in the form ([1, Proposition 7.1.8])

$$\exp\begin{pmatrix} \widehat{\omega} & \mathbf{v} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \exp(\widehat{\omega}) & \frac{1}{\|\mathbf{v}\|^2}((1+\omega.\omega^t)I_3 - \exp(\widehat{\omega})\widehat{\omega})\mathbf{v} \\ 0 & 1 \end{pmatrix}$$

when $\omega \neq 0$.

Proposition 6. *The map* $\exp : \mathfrak{se}(2) \to \mathrm{SE}(2)$ *is surjective and it is not injective.*

Proof: Let

$$(v, R_\theta) = \begin{pmatrix} R_\theta & \mathbf{v} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & v_1 \\ \sin\theta & \cos\theta & v_2 \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{SE}(2).$$

Using formula (10), the relation $\exp(\Omega) = (v, R_\theta)$, where

$$\Omega = \begin{pmatrix} 0 & -\theta & x_1 \\ \theta & 0 & x_2 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \theta \neq 0$$

is equivalent to

$$I_3 + \frac{\sin\theta}{\theta}\Omega + \frac{1-\cos\theta}{\theta^2}\Omega^2 = \begin{pmatrix} R_\theta & \mathbf{v} \\ 0 & 1 \end{pmatrix}.$$

Then, solving a simple linear system in x_1, x_2, we obtain that for

$$x_1 = \frac{\theta\sin\theta v_1}{2(1-\cos\theta)} + \frac{\theta v_2}{2}, \qquad x_2 = \frac{\theta\sin\theta v_2}{2(1-\cos\theta)} - \frac{\theta v_1}{2}$$

we have $\exp(\Omega) = (v, R_\theta)$.

Consider the following two matrices $\Omega_1, \Omega_2 \in \mathfrak{se}(2)$, where

$$\Omega_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \Omega_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then, we have

$$\exp(\Omega_1) = \exp(\Omega_2) = I_3$$

therefore the map $\exp : \mathfrak{se}(2) \to \mathrm{SE}(2)$ is not injective. ■

Following the paper of Gallier and Xu [7], we present a Rodrigues-like formula showing how to compute the exponential $\exp \Omega$ of an element Ω of the Lie algebra $\mathfrak{se}(n)$, where $n \geq 3$. We need the following key lemma.

Lemma 7. *Given any $(n+1) \times (n+1)$ matrix of the form $\Omega = \begin{pmatrix} X & \mathbf{v} \\ 0 & 0 \end{pmatrix}$ then*

$$\exp \Omega = \begin{pmatrix} \exp X & A\mathbf{v} \\ 0 & 1 \end{pmatrix}$$

where

$$A = I_n + \sum_{k \geq 1} \frac{X^k}{(k+1)!}.$$

The proof is immediate by induction on k.
Observing that

$$A = I_n + \sum_{k \geq 1} \frac{X^k}{(k+1)!} = \int_0^1 \exp(tX)\mathrm{d}t$$

we can now prove the following result

Theorem 8. *Let Ω be a $(n+1) \times (n+1)$ matrix in the form given above where X is a non-null skew-symmetric matrix and $\mathbf{v} \in \mathbb{R}^n$, with $n \geq 3$. If $\{\mathrm{i}\theta_1, -\mathrm{i}\theta_1, \ldots, \mathrm{i}\theta_p, -\mathrm{i}\theta_p\}$ is the set of distinct eigenvalues of X, where $\theta_i > 0$, there are p unique skew-symmetric matrices $X_1, \ldots, X_p$ such that the conditions in* Theorem 3 *hold. Furthermore we have*

$$\exp(\Omega) = \begin{pmatrix} \exp(X) & A\mathbf{v} \\ 0 & 1 \end{pmatrix}$$

where

$$\exp(X) = I_n + \sum_{i=1}^{p} \left(\sin\theta_i X_i + (1 - \cos\theta_i) X_i^2 \right)$$

and

$$A = I_n + \sum_{i=1}^{p} \left(\frac{1 - \cos\theta_i}{\theta_i} X_i + \frac{\theta_i - \sin\theta_i}{\theta_i^2} X_i^2 \right).$$

Proof: The existence and uniqueness of $X_1, \ldots, X_p$ and the formula for $\exp B$ come from Theorem 3. Since

$$V = I_n + \sum_{k\geq 1} \frac{X^k}{(k+1)!} = \int_0^1 \exp(tX)\mathrm{d}t$$

we have

$$\begin{aligned} V &= \int_0^1 \left[I_n + \sum_{i=1}^{p} \left(\sin t\theta_i X_i + (1 - \cos t\theta_i) X_i^2 \right) \right] \mathrm{d}t \\ &= \left[tI_n + \sum_{i=1}^{p} \left(-\frac{\cos t\theta_i}{\theta_i} X_i + \left(t - \frac{\sin t\theta_i}{\theta_i} \right) X_i^2 \right) \right]_0^1 \\ &= I_n + \sum_{i=1}^{p} \left(\frac{1 - \cos\theta_i}{\theta_i} X_i + \frac{\theta_i - \sin\theta_i}{\theta_i} X_i^2 \right). \end{aligned}$$

∎

Remark 9. *Given* $\Omega = \begin{pmatrix} X & \mathbf{v} \\ 0 & 0 \end{pmatrix}$ *where* $X = \theta_1 X_1 + \ldots + \theta_p X_p$, *if we let* $\Omega_i = \begin{pmatrix} X_i & \mathbf{v}/\theta_i \\ 0 & 0 \end{pmatrix}$ *using the fact that* $X_i^3 = -X_i$ *and the relation*

$$\Omega_i^k = \begin{pmatrix} X_i^k & X_i^{k-1}\mathbf{v}/\theta_i \\ 0 & 0 \end{pmatrix}$$

it is easily verified that

$$\exp(\Omega) = I_{n+1} + \Omega + \sum_{i=1}^{p} \left((1 - \cos\theta_i)\Omega_i^2 + (\theta_i - \sin\theta_i)\Omega_i^3 \right).$$

Acknowledgements

The author would like to thank to the organizers of the Fourteenth International Conference Geometry, Integrability and Quantization, June 2012, Varna, Bulgaria for the invitation and for giving the opportunity to present her results. This work is supported by Contract # POSDRU/88/1.5/S/60185 – “Innovative Doctoral Studies in a Knowledge Based Society”, Babeş-Bolyai University, Cluj-Napoca, Romania.

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