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HARMONIC SPHERES AND YANG–MILLS FIELDS

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Abstract. We study a relation between harmonic spheres in loop spaces of compact Lie groups and Yang–Mills fields on the Euclidean four-space $\mathbb{R}^4$.

CONTENTS

1. Introduction

In the paper we study a relation between two classes of objects, arising in theoretical physics, which from the first glance seem to be very far from each other. The first class is formed by harmonic spheres, i.e., harmonic maps of the Riemann sphere into Riemannian manifolds, coinciding with the classical solutions of the sigma-model theory in theoretical physics. The second class consists of Yang–Mills fields on the Euclidean four-space $\mathbb{R}^4$.

Harmonic spheres in a given oriented Riemannian manifold are the smooth maps of the Riemann sphere into this manifolds which are the extremals of the energy functional given by the Dirichlet integral. They satisfy nonlinear second order elliptic equations, generalizing Laplace–Beltrami equation. If the target Riemannian manifold is Kähler then holomorphic and anti-holomorphic spheres realize local minima of the energy. However, this functional usually have also non-minimal critical points.

On the other hand, Yang–Mills fields are the extremals of Yang–Mills action functional. Local minima of this functional are called instantons and anti-instantons. It was believed that they exhaust all critical points of Yang–Mills action on $\mathbb{R}^4$, until examples of non-minimal Yang–Mills fields were constructed.

There is an evident formal similarity between Yang–Mills fields and harmonic maps and after Atiyah's paper [2] it became clear that there is a deep reason for such a similarity. Namely, Atiyah has proved that the moduli space of G-instantons on $\mathbb{R}^4$ can be identified with the space of based holomorphic spheres in the loop space ΩG of a compact Lie group G. Generalizing this theorem, we formulate a conjecture stating that it should exist a bijective correspondence between the moduli space of Yang–Mills G-fields on $\mathbb{R}^4$ and the space of based harmonic spheres in the loop space ΩG. In our lectures we discuss this conjecture and propose an idea of its proof.

2. Harmonic Maps

2.1. Harmonic Self-maps of the Riemann Sphere

Consider the following problem, arising in the ferromagnetic theory. Suppose that at any point $x = (x_1, x_2)$ of the Euclidean plane $\mathbb{R}^2$ it is given a unit vector $\varphi(x) \in \mathbb{R}^3$, depending smoothly on x. In other words, we have a smooth map

$$\varphi : \mathbb{R}^2 \to \mathbb{S}^2, \qquad x \longmapsto \varphi(x)$$

of the Euclidean plane $\mathbb{R}^2$ into the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$.
Define the **energy** of φ by the following **Dirichlet integral**

$$E(\varphi) = \frac{1}{2} \int_{\mathbb{R}^2} |\mathrm{d}\varphi|^2 \mathrm{d}x_1 \mathrm{d}x_2$$

where

$$|\mathrm{d}\varphi|^2 = \left| \frac{\partial \varphi}{\partial x_1} \right|^2 + \left| \frac{\partial \varphi}{\partial x_2} \right|^2 .$$

Problem 1. *Find all smooth maps $\varphi : \mathbb{R}^2 \to \mathbb{S}^2$ with a finite energy $E(\varphi) < \infty$ which are extremal with respect to $E(\varphi)$.*

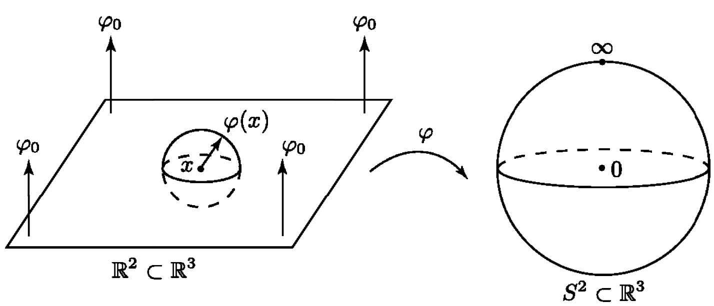

Figure 1. Smooth map from the Euclidean plane $\mathbb{R}^2$ to the two-sphere $\mathbb{S}^2$.

Due to the finite energy condition it is natural to impose on maps φ the following **asymptotic condition**

$$\varphi(x) \longrightarrow \varphi_0 \quad \text{uniformly for } |x| \to \infty$$

where φ_0 is a fixed point of $\mathbb{S}^2$. Under this condition the maps $\varphi : \mathbb{R}^2 \to \mathbb{S}^2$ extend to continuous maps

$$\varphi : \mathbb{S}^2 = \mathbb{R}^2 \cup \{\infty\} \longrightarrow \mathbb{S}^2.$$

It is well known that continuous maps $\varphi : \mathbb{S}^2 \to \mathbb{S}^2$ have a topological invariant, called the **degree** of the map. This invariant counts how many times (counted with respect to orientation) the image of φ covers the sphere $\mathbb{S}^2$ in the target space. It can be computed by the formula

$$\deg \varphi = \int_{\mathbb{R}^2} \varphi^* \omega$$

where ω is the normalized volume form on the sphere $\mathbb{S}^2$, satisfying $\int_{\mathbb{S}^2} \omega = 1$, and $\varphi^*\omega$ is the preimage of ω under the map φ.

Taking into account this invariant, we can reformulate our original problem as follows

Problem 2. *Find all extremals of the energy $E(\varphi)$ in the class of smooth maps $\varphi : \mathbb{R}^2 \to \mathbb{S}^2$ with $E(\varphi) < \infty$ of a given degree $k = \deg \varphi$.*

To solve this problem, we introduce the complex coordinates. Namely, denote by $z = x_1 + \mathrm{i}x_2$ the complex coordinate in the definition domain $\mathbb{R}^2 \approx \mathbb{C}$ and by w the stereographic complex coordinate in the image $\mathbb{S}^2 \setminus \{\infty\}$. In these coordinates the expression for the energy of the map $\varphi = w(z)$ takes the form

$$E(\varphi) = 2 \int_{\mathbb{C}} \frac{|w_z|^2 + |w_{\bar{z}}|^2}{(1 + |w|^2)^2} |\mathrm{d}z \wedge \mathrm{d}\bar{z}|$$

where $w_z = \frac{\partial w}{\partial z}$, $w_{\bar{z}} = \frac{\partial w}{\partial \bar{z}}$. The formula for the degree of φ is rewritten as

$$\deg \varphi = \frac{1}{2\pi} \int_{\mathbb{C}} \frac{|w_z|^2 - |w_{\bar{z}}|^2}{(1 + |w|^2)^2} |\mathrm{d}z \wedge \mathrm{d}\bar{z}|.$$

Comparing these two formulae, we arrive at inequality

$$E(\varphi) \geq 4\pi |\deg \varphi|.$$

Moreover, the equality here can be attained only by

- holomorphic functions $\varphi = w(z)$ for $k = \deg \varphi \geq 0$, satisfying $w_{\bar{z}} \equiv 0$;
- anti-holomorphic functions $\varphi = w(z)$ for $k < 0$, satisfying $w_z \equiv 0$.

In other words, holomorphic maps $\varphi = w(z)$ realize minima of $E(\varphi)$ in topological classes with $k \geq 0$, while anti-holomorphic functions $\varphi = w(z)$ realize minima of $E(\varphi)$ in topological classes with $k < 0$. For minimizing maps in these classes the value of $E(\varphi)$ is equal to $4\pi|k|$, i.e., it is an integer modulo 4π. Hence, the energy in our problem is "quantized" which sometimes happens in nonlinear classical physical systems.

To find explicit formulas for the minimizing maps, we suppose, for definiteness, that $k > 0$. We also note that the value of $E(\varphi)$ does not change under rotations of the sphere $\mathbb{S}^2$ in the target space (by this reason this model is often called the "SO(3)-model"). Due to this $\mathrm{SO}(3)$-invariance of the problem we can fix the asymptotic value φ_0 by setting it equal to $w_0 = 1$. So we have to describe holomorphic maps of the Riemann sphere $\mathbb{S}^2 = \mathbb{R}^2 \cup \{\infty\}$ into itself of degree k which are equal to one at infinity. Such maps are obligatory rational and, having degree k, they should have the form

$$\varphi = w(z) = \prod_{j=1}^{k} \frac{z - a_j}{z - b_j}$$

where $a_j \neq b_j$ are arbitrary complex numbers.
Note that the space of solutions of our problem depends on $4k$ real parameters (or $4k + 2$ real parameters if we add rotations of S^2 in the image).

Remark 1. *We have described all local minima of $E(\varphi)$. It can be proved that this functional has no other critical points apart from the local minima (which is an effect of two-dimensionality of the target manifold $\mathbb{S}^2$).*

2.2. General Definition of Harmonic Maps

Let M be an oriented Riemannian manifold of dimension m, provided with a Riemannian metric g with metric tensor (g_{ij}), and N is an oriented Riemannian manifold of dimension n, provided with a Riemannian metric h with metric tensor $h_{\alpha\beta}$.

Definition 2. *Let $\varphi : (M, g) \to (N, h)$ be a smooth map. Its energy is given by the Dirichlet integral*

$$E(\varphi) = \frac{1}{2} \int_M |\mathrm{d}\varphi(p)|^2 \mathrm{vol}_g$$

where $\mathrm{d}\varphi$ is the differential of φ and vol_g is the volume element of metric g.

The squared norm of the differential can be computed in local coordinates as follows. Choose local coordinates x^i at $p \in M$ and u^α at $q = \varphi(p) \in N$. Then

$$|\mathrm{d}\varphi(p)|^2 = \sum_{i,j} \sum_{\alpha,\beta} g^{ij} \frac{\partial \varphi^\alpha}{\partial x^i} \frac{\partial \varphi^\beta}{\partial x^j} h_{\alpha\beta}$$

where $\varphi^\alpha = \varphi^\alpha(x)$ are the components of φ, $g^{ij} = (g^{-1})_{ij}$ are the entries of the inverse matrix of (g_{ij}), vol_g is the volume element of g, given in local coordinates by the formula

$$\mathrm{vol}_g = \sqrt{|\det(g_{ij})|} \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^m.$$

Remark 3. *There is also an invariant description of the differential* $\mathrm{d}\varphi$*. Namely, the map* $\varphi : M \to N$ *generates the tangent map* $\varphi_* : TM \to TN$ *which may be identified with a section* $\mathrm{d}\varphi$ *of the bundle*

$$T^*M \otimes \varphi^{-1}(TN) \longrightarrow N$$

where $\varphi^{-1}(TN)$ *is the inverse image of* TN *under the map* φ *whose fibre at* $p \in M$ *coincides with the fibre* T_qN *at* $q = \varphi(p)$*. The bundle* $T^*M \otimes \varphi^{-1}(TN)$ *is provided with a natural Riemannian metric, induced by Riemannian metrics* g *and* h*. (The local expression for this metric can be read from the local formula for* $|\mathrm{d}\varphi(p)|^2$*.)*

Example 4. *Let* M *be an open subset in* $\mathbb{R}^m$ *and* N *be an open subset in* $\mathbb{R}^n$*. Then the squared norm of the differential of a smooth map* $\varphi = (\varphi^1, \dots, \varphi^n) : M \to N$ *is given by*

$$|\mathrm{d}\varphi(x)|^2 = \sum_{i=1}^{m}\sum_{\alpha=1}^{n}\left|\frac{\partial\varphi^\alpha}{\partial x^i}\right|^2 = \sum_{i=1}^{m}\left|\frac{\partial\varphi}{\partial x^i}\right|^2$$

and the energy is equal to

$$E(\varphi) = \frac{1}{2}\int_M \sum_{i=1}^{m}\left|\frac{\partial\varphi}{\partial x^i}\right|^2 \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^m .$$

Extremals of $E(\varphi)$ *are given by the maps* $\varphi = (\varphi^\alpha)$ *with components* φ^α *being harmonic functions.*

Definition 5. *A smooth map* $\varphi : M \to N$ *is called harmonic if it is extremal for the energy functional* $E(\varphi)$ *with respect to all smooth variations of* φ *with compact supports.*

Let us write down the Euler–Lagrange equations for $E(\varphi)$ in local coordinates x^i on M and (u^α) on N. Denote by ${}^M\nabla$ the **Levi-Civita connection** of M, represented locally by the **Christoffel symbol** ${}^M\Gamma^k_{ij}$, and by ${}^N\nabla$ the Levi-Civita connection of N, represented locally by the Christoffel symbol ${}^N\Gamma^\gamma_{\alpha\beta}$. In these coordinates the Euler–Lagrange equations take the form

$$\sum_{i,j} g^{ij}\left\{\frac{\partial^2\varphi^\gamma}{\partial x_i \partial x_j} - \sum_k {}^M\Gamma^k_{ij}\frac{\partial\varphi^\gamma}{\partial x_k} + \sum_{\alpha,\beta} {}^N\Gamma^\gamma_{\alpha\beta}(\varphi)\frac{\partial\varphi^\alpha}{\partial x_i}\frac{\partial\varphi^\beta}{\partial x_j}\right\}$$
$$= \Delta_M\varphi^\gamma + \sum_{i,j} g^{ij}\sum_{\alpha,\beta} {}^N\Gamma^\gamma_{\alpha\beta}(\varphi)\frac{\partial\varphi^\alpha}{\partial x_i}\frac{\partial\varphi^\beta}{\partial x_j} = 0\ , \quad \gamma = 1, \dots, n.$$

The operator

$$\Delta_M \varphi^\gamma = \sum_{i,j} g^{ij} \left\{ \frac{\partial^2 \varphi^\gamma}{\partial x_i \partial x_j} - \sum_k {}^M\Gamma^k_{ij} \frac{\partial \varphi^\gamma}{\partial x_k} \right\}$$

is the standard **Laplace–Beltrami operator** of M, determined by metric g. Note that it is a *linear* differential operator of second order in φ^γ. The second term in Euler–Lagrange equations depends on the geometry of the target space N and is *quadratic* with respect to derivatives of φ^γ.

Example 6. *For $N = \mathbb{R}^n$ the Euler–Lagrange equations reduce to the Laplace–Beltrami equations on the components of φ. Their solutions are given by harmonic functions φ^γ. For $m = \dim M = 1$ harmonic maps $\varphi : M \to N$ coincide with geodesics of N, parameterized by the arc length.*

Remark 7. *One can write down the Euler–Lagrange equations for $E(\varphi)$ also in an invariant form. Recall that $\mathrm{d}\varphi$ may be identified with a section of the bundle $T^*M \otimes \varphi^{-1}(TN)$. As we pointed out above, this bundle can be provided with a natural connection ∇, generated by Levi-Civita connections ${}^M\nabla$ and ${}^N\nabla$. The Euler–Lagrange equations in terms of this connection are written in the form*

$$\mathrm{tr}(\nabla \mathrm{d}\varphi) = 0$$

where the vector field $\tau_\varphi = \mathrm{tr}(\nabla \mathrm{d}\varphi)$ is called the ***stress tensor*** *of φ.*

2.3. Harmonic Maps of Almost Complex Manifolds

Let M be an almost complex Riemannian manifold, provided with an almost complex structure ${}^MJ$, compatible with Riemannian metric g, and N be an almost complex Riemannian manifold, provided with an almost complex structure ${}^NJ$, compatible with Riemannian metric h.

Recall that an **almost complex structure** J on M is a smooth family $\{J_p\}_{p \in M}$ of endomorphisms $J_p : T_pM \to T_pM$ such that $J_p^2 = -I$. This structure J is *integrable* if it generates the $\bar{\partial}_J$-operator, satisfying the integrability condition $\bar{\partial}_J^2 = 0$. The *compatibility of J with Riemannian metric g* means that the two-form ω on M, defined by

$$\omega(X, Y) := g(X, JY)$$

is symplectic and the metric g is Hermitian. A manifold (M, g, J, ω) with such an almost complex structure is called **almost Kähler** and it is called **Kähler** if J is integrable.

Definition 8. *Let $\varphi : M \to N$ be a smooth map of almost Kähler manifolds. It is holomorphic if the tangent map $\varphi_* : TM \to TN$ commutes with almost complex* *structures* ${}^{M}J$ *and* ${}^{N}J$, *i.e.,*

$$\varphi_* \circ {}^{M}J = {}^{N}J \circ \varphi_* .$$

It is called ***anti-holomorphic*** *if* φ_* *anti-commutes with* ${}^{M}J$ *and* ${}^{N}J$.

Theorem 9 (Lichnerowicz). *Let* $\varphi : M \to N$ *be a smooth map of almost Kähler manifolds. Holomorphic and anti-holomorphic maps* φ *realize local minima of the energy functional* $E(\varphi)$ *in a given topological class.*

However, in general, the energy functional $E(\varphi)$ has also non-minimal critical points (harmonic maps).

In our course we shall be interested in the following problem.

Problem 3. *Describe all harmonic spheres* $\varphi : \mathbb{P}^1 \to N$, *i.e., harmonic maps of the Riemann sphere* $\mathbb{P}^1 = \mathbb{S}^2$ *to a given Riemannian manifold* N, *by reducing this problem to the description of holomorphic spheres in almost Kähler manifolds.*

3. Instantons and Yang–Mills Fields

This Section contains a brief introduction to Yang–Mills fields. A detailed exposition of this theory the reader may found in the books by Atiyah [1], Freed–Uhlenbeck [7] and Naber [8].

3.1. Yang–Mills Equations on $\mathbb{R}^4$

Let G be a compact Lie group (gauge group). A **gauge G-potential** on $\mathbb{R}^4$ is a connection in a principal G-bundle over $\mathbb{R}^4$, identified with a one-form A on $\mathbb{R}^4$ with values in the Lie algebra $\mathfrak{g}$ of G. If G coincides with the group $\mathrm{U}(n)$ of unitary $(n \times n)$-matrices then this form may be written as

$$A = \sum_{\mu=1}^{4} A_\mu(x)\mathrm{d}x_\mu$$

where $x = (x_1, x_2, x_3, x_4)$ are coordinates on $\mathbb{R}^4$, $A_\mu(x)$ are smooth functions on $\mathbb{R}^4$ with values in skew-Hermitian $(n \times n)$-matrices. For $n = 1$ the gauge potential is the Euclidean analogue of the electromagnetic four-potential. A **gauge G-field** F is the curvature of connection A, given by a two-form on $\mathbb{R}^4$ with values in $\mathfrak{g}$ of the form

$$F = DA = \mathrm{d}A + \frac{1}{2}[A, A]$$

where $D : \Omega^1(\mathbb{R}^4, \mathfrak{g}) \to \Omega^2(\mathbb{R}^4, \mathfrak{g})$ is the covariant exterior derivative, generated by the connection A. In the case $G = \mathrm{U}(n)$ this form is equal to

$$F = \sum_{\mu,\nu=1}^{4} F_{\mu\nu}(x)\mathrm{d}x_\mu \wedge \mathrm{d}x_\nu$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$$

with $\partial_\mu := \partial/\partial x_\mu$, $\mu = 1, 2, 3, 4$. For $n = 1$ the form $\{F_{\mu\nu}\}$ coincides with the Euclidean analogue of the Maxwell tensor of electromagnetic field.

A **gauge transform** is a smooth map $g : \mathbb{R}^4 \to G$, acting on gauge potentials and fields by the formula

$$A \longmapsto A_g := g^{-1}dg + g^{-1}Ag, \qquad g : F \longmapsto F_g := g^{-1}Fg$$

where G acts on $\mathfrak{g}$ by the adjoint representation. In the case $G = \mathrm{U}(1)$ the gauge transform coincides with the multiplication by the factor $g(x) = \mathrm{e}^{\mathrm{i}\theta(x)}$ so that $A \mapsto A - \mathrm{i}d\theta$ and F does not change under this map.

Define the **Yang–Mills action** functional by the formula

$$S(A) = \frac{1}{2}\int_{\mathbb{R}^4} \|F\|^2 \mathrm{d}^4x$$

where

$$\|F\|^2 = \sum_{\mu,\nu=1}^{4} \|F_{\mu\nu}\|^2$$

and the norm $\|F_{\mu\nu}\|$ is computed with the help of an invariant inner product on $\mathfrak{g}$. In the case $G = \mathrm{U}(n)$ one can take for such a product $\langle X, Y\rangle := \mathrm{tr}(XY)$. Then the formula for $S(A)$ will rewrite as

$$S(A) = \frac{1}{2}\int_{\mathbb{R}^4} \mathrm{tr}(F \wedge *F)$$

where $*$ is the Hodge star-operator on $\mathbb{R}^4$.

The functional $S(A)$ is invariant under gauge transformations so that $S(A)$ depends on the class of the connection A modulo gauge transformations rather than A itself.

Definition 10. *Yang–Mills fields are the gauge fields F with finite Yang–Mills action $S(A) < \infty$, realizing the extrema of $S(A)$. The corresponding gauge potentials A are called the Yang–Mills connections.*

Yang–Mills fields satisfy the Euler–Lagrange equations for $S(A)$ which have the form

$$D^*F = 0$$

where $D^* : \Omega^2(\mathbb{R}^4, \mathfrak{g}) \to \Omega^1(\mathbb{R}^4, \mathfrak{g})$ is the formal adjoint of D. It is equal to $D^* = - * D*$ so that the Euler–Lagrange equations for $S(A)$ may be rewritten as

$$D(*F) = 0.$$

This equation is called the **Yang–Mills equation** and is sometimes supplemented with the **Bianchi identity**

$$DF = 0$$

automatically satisfied for gauge fields F.

3.2. Instantons

A gauge field F is called **selfdual** (respectively **anti-selfdual**) if

$$*F = F, \qquad \text{(respectively } *F = -F).$$

It is an immediate corollary of Bianchi identity that solutions of **duality equations**

$$*F = \pm F$$

satisfy the Yang–Mills equations.

If we write down the form F as a sum

$$F = F_+ + F_-$$

with $F_\pm = \frac{1}{2}(*F \pm F)$ then the formula for the Yang–Mills action can be rewritten in the form

$$S(A) = \frac{1}{2} \int_{\mathbb{R}^4} \left(\|F_+\|^2 + \|F_-\|^2 \right) \mathrm{d}^4 x.$$

For gauge fields F with finite Yang–Mills action the quantity

$$k(A) = \frac{1}{8\pi^2} \int_{\mathbb{R}^4} \left(-\|F_+\|^2 + \|F_-\|^2 \right) \mathrm{d}^4 x = \frac{1}{8\pi^2} \int_{\mathbb{R}^4} \mathrm{tr}(F \wedge F)$$

is an integer-valued topological invariant, called the **topological charge** of F. Evidently,

$$S(A) \geq 4\pi^2 |k(A)|.$$

The minimum of $S(A)$, equal to $4\pi^2|k|$ in the topological class of gauge potentials with finite Yang–Mills action and fixed topological charge $k(A) = k$, may be attained for $k > 0$ only on anti-selfdual fields and for $k < 0$ only on selfdual ones.

Definition 11. *Anti-selfdual fields with finite action $S(A) < \infty$ are called the instantons while selfdual fields with finite action $S(A) < \infty$ are called the anti-instantons.*

Instantons and anti-instantons realize local minima of the action $S(A)$, however, there exist also non-minimal critical points of this functional.

One of the main objects in Yang–Mills theory is the **moduli space** of Yang–Mills fields which is the quotient of the space of all Yang–Mills fields modulo gauge transforms. The structure of this space is far from being understood and one of our goals is to approach this problem on the base of harmonic spheres conjecture. However, the analogous problem for instantons, i.e., the description of the moduli space of instantons on $\mathbb{R}^4$, was solved by Atiyah–Drinfeld–Hitchin–Manin with the help of the twistor approach, introduced in the next Section.

Comparing Yang-Mills fields with harmonic maps, introduced in Section 2, we observe the following evident analogy between:

$$\{\text{(anti-)holomorphic maps}\} \longleftrightarrow \{\text{(anti-)instantons}\}$$

and

$$\{\text{harmonic maps}\} \longleftrightarrow \{\text{Yang–Mills fields}\}\,.$$

As we shall see from the Atiyah theorem and harmonic spheres conjecture, this formal analogy has, in fact, a much deeper meaning.

4. Twistor Interpretation of Instantons

4.1. Basic Twistor Bundle over $\mathbb{S}^4$

We shall identify the four-sphere $\mathbb{S}^4$ with the quaternion projective line in the same way as the two-sphere $\mathbb{S}^2$ is identified with the complex projective line $\mathbb{CP}^1$.

Recall that the space of quaternions $\mathbb{H}$ consists of elements

$$q = x_1 + \mathrm{i}x_2 + \mathrm{j}x_3 + \mathrm{k}x_4$$

where $x_1, x_2, x_3, x_4 \in \mathbb{R}$, $\mathrm{i}^2 = \mathrm{j}^2 = \mathrm{k}^2 = -1$ and the multiplication law is defined by the relation

$$\mathrm{ij} = -\mathrm{ji} = \mathrm{k}.$$

The space $\mathbb{H}$ is a non-commutative field isomorphic, as a vector space, to $\mathbb{R}^4$. As a complex vector space $\mathbb{H}$ can be identified with $\mathbb{C}^2$ by writing quaternions in the form

$$q = z_1 + z_2\mathrm{j}$$

where $z_1 = x_1 + \mathrm{i}x_2, z_2 = x_3 + \mathrm{i}x_4 \in \mathbb{C}$.

Quaternion projective line $\mathbb{HP}^1$ consists of pairs $[q, q']$ of quaternions (not equal to zero simultaneously) which are defined up to multiplication (from the right) by a nonzero quaternion. We identify the Euclidean sphere $\mathbb{S}^4 = \mathbb{R}^4 \cup \{\infty\}$ with the quaternion projective line $\mathbb{HP}^1$ and define the **basic twistor bundle** over $\mathbb{S}^4$

$$\pi : \mathbb{CP}^3 \xrightarrow{\mathbb{CP}^1} \mathbb{HP}^1$$

by the tautological formula

$$[z_1, z_2, z_3, z_4] \longmapsto [z_1 + z_2\mathrm{j}, z_3 + z_4\mathrm{j}]$$

where the four-tuple $[z_1, z_2, z_3, z_4] \in \mathbb{CP}^3$ is defined up to multiplication by a nonzero complex number while the pair $[z_1 + z_2 j, z_3 + z_4 j] \in \mathbb{HP}^1$ is defined up to multiplication (from the right) by a nonzero quaternion. The fibre of π coincides with the complex projective line $\mathbb{CP}^1$, invariant under multiplication from the right by j, i.e., under the map

$$\mathrm{j} : [z_1, z_2, z_3, z_4] \longmapsto [-z_2, z_1, -z_4, z_3].$$

The constructed bundle $\pi : \mathbb{CP}^3 \to \mathbb{S}^4$ has a nice interpretation in terms of complex structures on $\mathbb{R}^4$ due to Atiyah. To describe it, consider the restriction of π to the Euclidean space $\mathbb{R}^4 \cong \mathbb{H}$

$$\pi : \mathbb{CP}^3 \backslash \mathbb{CP}^1_\infty \longrightarrow \mathbb{R}^4$$

where the omitted complex projective line $\mathbb{CP}^1_\infty$ is identified with the fibre $\pi^{-1}(\infty)$ of the twistor bundle at $\infty \in \mathbb{S}^4$.

The space $\mathbb{CP}^3 \backslash \mathbb{CP}^1_\infty$ is foliated by parallel complex projective planes $\mathbb{CP}^2$. These planes intersect in $\mathbb{CP}^3$ on the projective line $\mathbb{CP}^1_\infty$ so that each point p of $\mathbb{CP}^1_\infty$ defines one family of parallel planes. The tangent map π_* provides the tangent space $T_q\mathbb{R}^4$ at a point $q \in \mathbb{R}^4$ with the complex structure, induced from these parallel planes. Different families, determined by points $p \in \mathbb{CP}^1_\infty$, define different complex structures on $T_q\mathbb{R}^4$ so that the space of all complex structures on $T_q\mathbb{R}^4$, compatible with metric, can be identified with $\mathbb{CP}^1_\infty$. Summing up, we can consider the twistor bundle

$$\pi : \mathbb{CP}^3 \backslash \mathbb{CP}^1_\infty \longrightarrow \mathbb{R}^4$$

as a bundle of complex structures on $\mathbb{R}^4$, compatible with metric. The fibre of this bundle at a point $q \in \mathbb{R}^4$ consists of complex structures on the tangent space $T_q\mathbb{R}^4$, compatible with metric, and can be identified, as above, with $\mathbb{CP}^1_\infty$.

4.2. Atiyah–Hitchin–Singer Construction and Penrose Twistor Program

We shall use an interpretation of basic twistor bundle as a bundle of complex structures, given in the last Subsection, to extend the twistor bundle construction to general Riemannian manifolds.

Let N be an even-dimensional oriented Riemannian manifold of dimension $2n$. Consider the bundle $\pi : \mathcal{J}(N) \to N$ of complex structures on N, compatible with

Riemannian metric. The fibre of this bundle at a point $q \in N$ coincides with the space $\mathcal{J}(T_qN)$ of complex structures J_q on the tangent space T_qN, compatible with metric. The bundle $\pi : \mathcal{J}(N) \to N$ is associated with the principal bundle $\mathrm{O}(N) \to N$ of orthonormal frames on N and its fibre $\pi^{-1}(q)$ can be identified with the complex homogeneous space $\mathrm{O}(2n)/\mathrm{U}(n)$.

The bundle $\pi : \mathcal{J}(N) \to N$ can be always provided with a natural almost complex structure, introduced by Atiyah–Hitchin–Singer. Namely, the Levi-Civita connection ${}^N\nabla$ on N generates a natural connection on $\mathrm{O}(N)$, hence on $\mathcal{J}(N)$. This connection determines the corresponding vertical-horizontal decomposition

$$T\mathcal{J}(N) = V \oplus H.$$

Using this decomposition, introduce an almost complex structure $\mathcal{J}^1$ on $\mathcal{J}(N)$ by setting

$$\mathcal{J}^1 = \mathcal{J}^v \oplus \mathcal{J}^h$$

where the value $\mathcal{J}^v_z$ of $\mathcal{J}^v$ at $z \in \mathcal{J}(N)$ coincides with the canonical complex structure on the vertical space V_z, identified with $\mathrm{O}(2n)/\mathrm{U}(n)$. The value of the horizontal component $\mathcal{J}^h_z$ at z coincides with the complex structure $J(z)$ on the horizontal space H_z, given by the point z of the twistor bundle, where H_z is identified with the tangent space $T_{\pi(z)}N$ by π_*. We recall that the fibre $\pi^{-1}(q)$ of the bundle $\pi : \mathcal{J}(N) \to N$ at $q = \pi(z) \in N$ consists of complex structures on T_qN and we denote by $J(z)$ the complex structure on T_qN, corresponding to the point $z \in \pi^{-1}(q)$. This construction provides $(\mathcal{J}(N), \mathcal{J}^1)$ with the structure of an almost complex manifold.

We formulate now an heuristic *Penrose twistor program*:

> Construct for a given Riemannian manifold N a twistor bundle $\pi : Z \to N$, where the twistor space Z is an almost complex manifold, with the following characteristic property: there should be a one-to-one correspondence between

$$\left\{\begin{matrix}\text{objects of Riemannian} \\ \text{geometry on } N\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{objects of holomorphic} \\ \text{geometry on } Z\end{matrix}\right\}.$$

Such a correspondence, being established, would give a method of studying the *real* geometry of the Riemannian manifold N via the *complex* geometry of its twistor space Z.

The above Atiyah–Hitchin–Singer construction yields an example of such a twistor bundle $\mathcal{J}(N) \to N$ where the twistor space $Z = \mathcal{J}(N)$ is provided with the almost complex structure $\mathcal{J}^1$.

4.3. Atiyah–Ward and Donaldson Theorems

From now on we shall deal only with the *complex* projective spaces $\mathbb{CP}^1$ and $\mathbb{CP}^3$. By this reason, we shall shorten their notation to $\mathbb{P}^1$ and $\mathbb{P}^3$.

We return to the problem of description of

$$\left\{\begin{matrix}\text{moduli space of}\\ G\text{-instantons on } \mathbb{R}^4\end{matrix}\right\} = \frac{\{G\text{-instantons on } \mathbb{R}^4\}}{\{\text{gauge transforms}\}} .$$

Using the basic twistor bundle $\pi : \mathbb{P}^3 \setminus \mathbb{P}^1 \to \mathbb{R}^4$, Atiyah and Ward have reduced this problem to a problem of description of certain holomorphic bundles over the three-dimensional complex projective space $\mathbb{P}^3$. Namely, according to them, there is a one-to-one correspondence between

$$\left\{\begin{matrix}\text{moduli space of}\\ G\text{-instantons on } \mathbb{R}^4\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{based equivalence classes of holomorphic}\\ G^{\mathbb{C}}\text{-bundles over } \mathbb{P}^3\text{, trivial on } \pi\text{-fibers}\end{matrix}\right\} .$$

Here, $G^{\mathbb{C}}$ is the complexification of the group G and the term "based" means that the equivalence of $G^{\mathbb{C}}$-bundles over $\mathbb{P}^3$ is defined "modulo" $\mathbb{P}^1_\infty$, i.e., all mappings, defining the equivalence of the bundles, should be equal to identity on $\mathbb{P}^1_\infty$.

This result has the following two-dimensional reduction to the space $\mathbb{P}^1 \times \mathbb{P}^1$, given by the Donaldson theorem

$$\left\{\begin{matrix}\text{moduli space of}\\ G\text{-instantons on } \mathbb{R}^4\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{based equivalence classes of holomorphic}\\ G^{\mathbb{C}}\text{-bundles over } \mathbb{P}^1 \times \mathbb{P}^1\text{, trivial on } \mathbb{P}^1_\infty \cup \mathbb{P}^1_\infty\end{matrix}\right\}$$

where $\mathbb{P}^1_\infty \cup \mathbb{P}^1_\infty$ denotes the union of two complex projective lines "at infinity" of $\mathbb{P}^1 \times \mathbb{P}^1$.

5. Twistor Interpretation of Harmonic Spheres

5.1. Eells–Salamon Theorem

Guided by the Penrose twistor program, mentioned in Subsection 4.2, we may suppose that our original problem of construction of harmonic spheres $\varphi : \mathbb{P}^1 \to N$ in a given Riemannian manifold N should reformulate as a problem of construction of holomorphic spheres $\psi : \mathbb{P}^1 \to Z$ in its twistor space $(Z = \mathcal{J}(N), \mathcal{J}^1)$ such that $\varphi = \pi \circ \psi$

$$\begin{array}{ccc} & & Z = \mathcal{J}(N) \\ & \overset{\psi}{\nearrow} & \downarrow \pi \\ \mathbb{P}^1 & \xrightarrow{\ \varphi\ } & N \end{array}$$

And it is almost true. In fact, projections of holomorphic spheres $\psi : \mathbb{P}^1 \to Z$ to N do satisfy some partial differential equations of second order on N. However, these equations are not harmonic but ultrahyperbolic, i.e., "harmonic with a wrong signature" (n, n) instead of the required signature $(2n, 0)$.

By this reason, we have to change the definition of the almost complex structure on Z if we want to construct harmonic spheres in N as projections of holomorphic spheres in Z. Namely, we shall provide Z with a new almost complex structure $\mathcal{J}^2$ which is given in terms of the vertical-horizontal decomposition

$$T\mathcal{J}(N) = V \oplus H$$

by

$$\mathcal{J}^2 = (-\mathcal{J}^v) \oplus \mathcal{J}^h.$$

It is precisely this almost complex structure, introduced by Eells and Salamon, which is used for the twistor description of harmonic spheres.

Before we formulate the main result of this Subsection, let us give a formal definition of the twistor bundle.

Definition 12. *A smooth bundle $\pi : Z \to N$ of an almost complex manifold $(Z, \mathcal{J})$ over a Riemannian manifold N will be called the twistor bundle of N if the projection $\varphi := \pi \circ \psi$ of any holomorphic sphere $\psi : \mathbb{P}^1 \to Z$ to N is a harmonic sphere $\varphi : \mathbb{P}^1 \to N$.*

Theorem 13 (Eells–Salamon theorem)**.** *The twistor bundle*

$$\pi : Z = \mathcal{J}(N) \longrightarrow N$$

provided with the almost complex structure $\mathcal{J}^2$, is the twistor bundle, i.e., projection $\varphi := \pi \circ \psi$ of any holomorphic sphere $\psi : \mathbb{P}^1 \to Z$ to N is a harmonic sphere $\varphi : \mathbb{P}^1 \to N$.

Using this theorem, we can construct harmonic spheres in the manifold N from holomorphic spheres in its twistor space Z.

However, we note that the almost complex structure $\mathcal{J}^1$ on $\mathcal{J}(N)$ is *integrable* $\Leftrightarrow$ *N is conformally flat* while the almost complex structure $\mathcal{J}^2$ is *never integrable.*

Remark 14. *Taking this into account, the Eells–Salamon theorem may look not helpful as a method of construction of harmonic spheres in N. Indeed, it reduces the problem of construction of harmonic spheres in the Riemannian manifold N to the problem of construction of holomorphic spheres in the almost complex manifold $(Z, \mathcal{J}^2)$. But the almost complex structure $\mathcal{J}^2$, being non-integrable, might be quite bizarre. For example, such a structure may have no non-constant holomorphic functions even locally. But our advantage is that we are dealing not with holomorphic functions, i.e., holomorphic maps $f : Z \to \mathbb{C}$, but with a dual object – holomorphic maps $\psi : \mathbb{C} \to Z$. Such a map is holomorphic with respect to the* *almost complex structure $\mathcal{J}^2$ on $Z \iff$ it satisfies the Cauchy–Riemann equation $\bar\partial_J \psi = 0$ with respect to the pulled-back almost complex structure $J := \psi^*(\mathcal{J}^2)$ on $\mathbb{C}$. This structure J is integrable (as any almost complex structure in complex dimension one). In particular, the above Cauchy–Riemann equation has many local solutions.*

5.2. Complex Grassmann Manifolds and Flag Bundles

We apply the twistor approach to the description of harmonic spheres in the complex **Grassmann manifold** $G_r(\mathbb{C}^d)$. In this case it is natural to choose for its twistor spaces the bundles of complex structures over $G_r(\mathbb{C}^d)$, invariant under the action of the unitary group $\mathrm{U}(d)$. Such bundles coincide with the flag bundles defined below.

Definition 15. *The **flag manifold** $F_{\mathbf{r}}(\mathbb{C}^d)$ in $\mathbb{C}^d$ of type $\mathbf{r} = (r_1, \ldots, r_n)$ with $d = r_1 + \ldots + r_n$ consists of flags $\mathcal{W} = (W_1, \ldots, W_n)$, i.e., nested sequences of complex subspaces*

$$W_1 \subset \ldots \subset W_n = \mathbb{C}^d$$

such that the dimension of the subspace $V_1 := W_1$ is equal to r_1 and dimensions of the subspaces $V_i := W_i \ominus W_{i-1}$ are equal to r_i for $1 < i \le n$.

The flag manifold $F_{\mathbf{r}}(\mathbb{C}^d)$ admits the following description as a homogeneous space of the unitary group $\mathrm{U}(d)$

$$F_{\mathbf{r}}(\mathbb{C}^d) = \mathrm{U}(d)/\,\mathrm{U}(r_1) \times \cdots \times \mathrm{U}(r_n).$$

It is a compact Kähler manifold which has an $\mathrm{U}(d)$-invariant complex structure, denoted by $\mathcal{J}^1$.

Definition 16. *For the construction of a flag bundle over the Grassmann manifold $G_r(\mathbb{C}^d)$ we fix an ordered subset $\sigma \subset \{1, \ldots, n\}$, such that $\sum_{i\in\sigma} r_i = r$, and define the flag bundle*

$$\pi_\sigma : F_{\mathbf{r}}(\mathbb{C}^d) \longrightarrow G_r(\mathbb{C}^d)$$

by

$$\pi_\sigma : \mathcal{W} = (W_1, \ldots, W_n) \longmapsto W := \bigoplus_{i\in\sigma} V_i.$$

5.3. Harmonic Spheres in Grassmann Manifolds: Burstall–Salamon Theorem

The flag bundle π_σ, introduced in the previous Subsection, can be provided, as before, with an almost complex structure $\mathcal{J}^2_\sigma$ so that the following analogue of Eells–Salamon Theorem 13 will hold.

Theorem 17 (Burstall-Salamon). *The flag bundle*

$$\pi_\sigma : (F_{\mathbf{r}}(\mathbb{C}^d), \mathcal{J}^2_\sigma) \longrightarrow G_r(\mathbb{C}^d)$$

provided with an almost complex structure $\mathcal{J}^2_\sigma$, is a twistor bundle, i.e., the projection $\varphi = \pi_\sigma \circ \psi$ of any holomorphic sphere $\psi : \mathbb{P}^1 \to F_{\mathbf{r}}(\mathbb{C}^d)$ to $G_r(\mathbb{C}^d)$ is a harmonic sphere $\varphi : \mathbb{P}^1 \to G_r(\mathbb{C}^d)$ in $G_r(\mathbb{C}^d)$. Moreover, the converse is also true: any harmonic sphere $\varphi : \mathbb{P}^1 \to G_r(\mathbb{C}^d)$ in $G_r(\mathbb{C}^d)$ may be obtained in this way from some flag bundle $\pi_\sigma : F_{\mathbf{r}}(\mathbb{C}^d) \to G_r(\mathbb{C}^d)$.

Using the above twistor interpretation of harmonic spheres in $G_r(\mathbb{C}^d)$, we can reduce their description to the description of holomorphic spheres in flag manifolds $F_{\mathbf{r}}(\mathbb{C}^d)$. The latter problem was solved by Wood. The idea of his construction can be roughly described as follows. A map $\psi : \mathbb{P}^1 \to F_{\mathbf{r}}(\mathbb{C}^d)$ may be considered as a decomposition of the trivial bundle $\mathbb{P}^1 \times \mathbb{C}^d$ into the direct sum of subbundles

$$\mathbb{P}^1 \times \mathbb{C}^d = \psi_1 \oplus \ldots \oplus \psi_n$$

where $\psi_i := \psi^* T_i$ with T_i being the ith tautological bundle over $F_{\mathbf{r}}(\mathbb{C}^d)$. A map $\psi : \mathbb{P}^1 \to F_{\mathbf{r}}(\mathbb{C}^d)$ is $\mathcal{J}^1$-holomorphic $\Longleftrightarrow$ all subbundles $\psi_1, \ldots, \psi_n$ are holomorphic. Wood has proposed a procedure how to rebuild this decomposition into a decomposition

$$\mathbb{P}^1 \times \mathbb{C}^d = \tilde{\psi}_1 \oplus \ldots \oplus \tilde{\psi}_m$$

corresponding to a $\mathcal{J}^2$-holomorphic sphere, where subbundles $\tilde{\psi}_i$ are either holomorphic or anti-holomorphic.

6. Atiyah Theorem and Harmonic Spheres Conjecture

6.1. Loop Spaces of Compact Lie Groups

We switch now to the infinite-dimensional target manifolds N, namely we take for N the loop space ΩG of a compact Lie group G.

Definition 18. *Let G be a compact Lie group. Then its loop space is*

$$\Omega G = LG/G$$

where $LG = C^\infty(\mathbb{S}^1, G)$ is the loop group of G, i.e., the space of C^∞-smooth maps $\mathbb{S}^1 \to G$ and G in the denominator is identified with the subgroup of constant maps $S^1 \to g_0 \in G$. Otherwise, ΩG can be thought of as the space of based loops, i.e., the maps $\mathbb{S}^1 \to G$, sending $1 \in \mathbb{S}^1 \mapsto e \in G$.

The space ΩG is an infinite-dimensional Kähler manifold. A complex structure on ΩG is induced from its representation as a homogeneous space of a complex Lie group

$$\Omega G = LG^{\mathbb{C}} \Big/ L_{+}G^{\mathbb{C}}$$

where $G^{\mathbb{C}}$ is the complexification of G, $LG^{\mathbb{C}} = C^{\infty}(\mathbb{S}^1, G^{\mathbb{C}})$ is the complexified loop group of G, and $L_{+}G^{\mathbb{C}} = \mathrm{Hol}(\Delta, G^{\mathbb{C}})$ is a subgroup of $LG^{\mathbb{C}}$, consisting of the maps which may be holomorphically extended to the unit disc Δ.

6.2. Holomorphic Spheres in Loop Spaces: Theorem of Atiyah

Recall that, according to Donaldson theorem

$$\left\{\begin{array}{l}\text{moduli space of}\\ G\text{-instantons on } \mathbb{R}^4\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{based equivalence classes of holomorphic}\\ G^{\mathbb{C}}\text{-bundles over } \mathbb{P}^1 \times \mathbb{P}^1\text{, trivial on the union}\\ \mathbb{P}^1_{\infty} \cup \mathbb{P}^1_{\infty}\end{array}\right\}.$$

Atiyah theorem asserts that the right hand side of this correspondence can be identified with the space of based holomorphic spheres in ΩG. In other words, there is a one-to-one correspondence

$$\left\{\begin{array}{l}\text{based equivalence classes of holo-}\\ \text{morphic } G^{\mathbb{C}}\text{-bundles over } \mathbb{P}^1 \times \mathbb{P}^1,\\ \text{trivial on the union } \mathbb{P}^1_{\infty} \cup \mathbb{P}^1_{\infty}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{based holomorphic spheres}\\ f : \mathbb{P}^1 \to \Omega G\text{, sending } \infty \text{ to}\\ \text{the origin of } \Omega G\end{array}\right\}.$$

The proof of Atiyah theorem is based on the following construction.

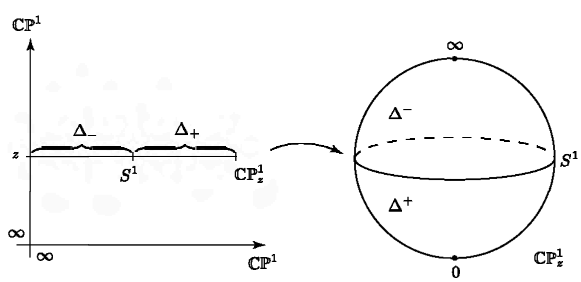

Figure 2. Holomorphic $G^{\mathbb{C}}$-bundle over $\mathbb{CP}^1 \times \mathbb{CP}^1$.

Restrict a given holomorphic $G^{\mathbb{C}}$-bundle over $\mathbb{P}^1 \times \mathbb{P}^1$ to the projective line $\mathbb{P}^1_z$, passing through a point $\mathbb{P}^1 \times \{z\}$ parallel to $\mathbb{P}^1_{\infty}$. This restricted bundle is determined by a transition function

$$\tilde{f}_z : \mathbb{S}^1 \longrightarrow G^{\mathbb{C}}$$

which is holomorphic in a neighborhood of the equator $\mathbb{S}^1$ in $\mathbb{P}^1_z$. Hence, $\tilde{f}_z \in LG^{\mathbb{C}}$ and we have a map

$$f : \mathbb{P}^1 \ni z \longmapsto \tilde{f}_z \in LG^{\mathbb{C}} \longmapsto f(z) \in \Omega G = LG^{\mathbb{C}}/L_+G^{\mathbb{C}}.$$

This map is holomorphic and based $\Longleftrightarrow$ the original $G^{\mathbb{C}}$-bundle over $\mathbb{P}^1 \times \mathbb{P}^1$ is holomorphic and trivial on $\mathbb{P}^1_\infty \cup \mathbb{P}^1_\infty$.

6.3. Harmonic Spheres Conjecture

Atiyah and Donaldson theorems imply that there is a one-to-one correspondence between

$$\left\{\begin{matrix}\text{moduli space of} \\ G\text{-instantons on } \mathbb{R}^4\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{based holomorphic spheres} \\ f : \mathbb{P}^1 \to \Omega G\end{matrix}\right\}.$$

So we have a correspondence between local minima of two functionals, namely

$$\left\{\begin{matrix}\text{Yang–Mills action on} \\ \text{gauge } G\text{-fields on } \mathbb{R}^4\end{matrix}\right\} \quad \text{and} \quad \left\{\begin{matrix}\text{energy of smooth} \\ \text{spheres in } \Omega G\end{matrix}\right\}$$

with local minima given respectively by

$$\left\{\begin{matrix}\text{instantons and anti-} \\ \text{instantons}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{holomorphic and anti-} \\ \text{holomorphic spheres}\end{matrix}\right\}.$$

If we replace here the local minima by the critical points of the corresponding functionals, we shall arrive at the formulation of the **harmonic spheres conjecture**, namely it should exist a one-to-one correspondence between

$$\left\{\begin{matrix}\text{moduli space of Yang–} \\ \text{Mills } G\text{-fields on } \mathbb{R}^4\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{based harmonic spheres} \\ f : \mathbb{P}^1 \to \Omega G\end{matrix}\right\}.$$

Remark 19. *We can consider the described transition from the local minima to the critical points of our functionals as a "realification" procedure. Indeed, if we replace smooth spheres in the right hand side of the above diagram by smooth functions $f : \mathbb{C} \to \mathbb{C}$ then the described transition will reduce to the replacement of holomorphic and anti-holomorphic functions by arbitrary harmonic functions (which are the sums of holomorphic and anti-holomorphic functions). In the case of smooth spheres in ΩG this transition from holomorphic and anti-holomorphic spheres to harmonic ones becomes non-trivial due to the non-linear character of Euler–Lagrange equations for the energy.*

Unfortunately, a direct extension of Atiyah–Donaldson proof to the harmonic case is not possible since the proof of Donaldson theorem, based on the monad method of construction of holomorphic vector bundles on complex projective spaces, is purely holomorphic. However, one can attempt to reduce the proof of the harmonic spheres conjecture to the holomorphic case by "pulling-up" both sides of the correspondence in this conjecture to their twistor spaces.

7. Twistor Bundle over the Loop Space

7.1. Hilbert–Schmidt Grassmannian

In order to construct a twistor bundle over the loop space ΩG we shall first embed ΩG into an infinite-dimensional Grassmannian, and then construct its twistor bundle by analogy with the finite-dimensional case.

The role of an infinite-dimensional Grassmannian will be played by the Hilbert–Schmidt Grassmannian of a complex Hilbert space H, provided with a *polarization*. That is a complex Hilbert space H together with a decomposition

$$H = H_+ \oplus H_-$$

into the direct orthogonal sum of closed infinite-dimensional subspaces $H_\pm$. In the case of the space $H = L_0^2(\mathbb{S}^1, \mathbb{C})$ of square integrable functions on $\mathbb{S}^1$ with zero average one can take for such subspaces

$$H_\pm = \{\gamma \in H\ ;\ \gamma = \sum_{\pm k>0} \gamma_k \mathrm{e}^{\mathrm{i}k\theta}\}.$$

Definition 20. *The Hilbert–Schmidt Grassmannian $Gr_{HS}(H)$ consists of closed subspaces $W \subset H$ such that the orthogonal projection $\pi_+ : W \to H_+$ is a Fredholm operator and orthogonal projection $\pi_- : W \to H_-$ is a Hilbert–Schmidt operator.*

For a given subspace $W \in \mathrm{Gr_{HS}}(H)$ the Fredholm index of the projection $\pi_+ : W \to H_+$ is called the **virtual dimension** of W.

Similar to the finite-dimensional case, the Hilbert–Schmidt Grassmannian $\mathrm{Gr_{HS}}(H)$ admits the following homogeneous representation

$$\mathrm{Gr_{HS}}(H) = \frac{\mathrm{U_{HS}}(H)}{\mathrm{U}(H_+) \times \mathrm{U}(H_-)}$$

where the **unitary Hilbert–Schmidt group** $\mathrm{U_{HS}}(H)$ is defined by

$$\mathrm{U_{HS}}(H) = \{A \in \mathrm{U}(H)\ ;\ \pi_- \circ A \circ \pi_+ \text{ is Hilbert–Schmidt}\}.$$

The Grassmannian $\mathrm{Gr_{HS}}(H)$ is a Hilbert Kähler manifold, consisting of a countable number of connected components of a fixed virtual dimension:

$$\mathrm{Gr_{HS}}(H) = \bigcup_d G_d(H)$$

where

$$G_d(H) = \{W \in \mathrm{Gr_{HS}}(H)\ ;\ \text{virtual dim}\, W = d\}.$$

7.2. Virtual Flag Bundles and Harmonic Spheres in the Hilbert–Schmidt Grassmannian

The virtual flag manifold and flag bundles are defined by analogy with the finite-dimensional case.

Definition 21. *The virtual flag manifold $F^d_{\mathbf{r}}(H)$ in H of type $\mathbf{r} = (r_1, \dots, r_n)$ with $d = r_1 + \dots + r_n$ consists of flags $\mathcal{W} = (W_1, \dots, W_n)$, i.e., nested sequences of complex subspaces*

$$W_1 \subset \dots \subset W_n \subset H$$

such that the virtual dimension of the subspace $V_1 := W_1$ is equal to r_1, and dimensions of subspaces $V_i := W_i \ominus W_{i-1}$ are equal to r_i for $1 < i \leq n$.

Definition 22. *For the construction of a flag bundle over the Grassmann manifold $G_r(H)$ we fix an ordered subset $\sigma \subset \{1, \dots, n\}$, so that $\sum_{i\in\sigma} r_i = r$, and define the virtual flag bundle*

$$\pi_\sigma : F^d_{\mathbf{r}}(H) \longrightarrow G_r(H)$$

by

$$\pi_\sigma : \mathcal{W} = (W_1, \dots, W_n) \longmapsto W := \bigoplus_{i\in\sigma} V_i.$$

As in the finite-dimensional case, we can provide the virtual flag bundle π_σ with an almost complex structure $\mathcal{J}^2_\sigma$ so that the following analogue of Burstall–Salamon Theorem 17 holds.

Theorem 23. *The virtual flag bundle*

$$\pi_\sigma : (F^d_{\mathbf{r}}(H), \mathcal{J}^2_\sigma) \longrightarrow G_r(H)$$

provided with the almost complex structure $\mathcal{J}^2_\sigma$, is a twistor bundle, i.e., the projection $\varphi = \pi_\sigma \circ \psi$ of any almost holomorphic sphere $\psi : \mathbb{P}^1 \to F^d_{\mathbf{r}}(H)$ to $G_r(H)$ is a harmonic sphere $\varphi : \mathbb{P}^1 \to G_r(H)$ in $G_r(H)$.

We think that the converse of this Theorem is also true, as in the finite-dimensional case.

7.3. Embedding of Loop Spaces into the Hilbert–Schmidt Grassmannian

Suppose that our compact Lie group G is realized as a subgroup of the unitary group $\mathrm{U}(N)$ and construct an embedding of ΩG into the Grassmannian $\mathrm{Gr}_{\mathrm{HS}}(H)$ where $H = L^2_0(\mathbb{S}^1, \mathbb{C}^N)$.

Construct first an embedding of the loop group LG into the unitary Hilbert–Schmidt group $\mathrm{U}_{\mathrm{HS}}(H)$. For that we associate with a loop γ, belonging to the space

$LG = C^\infty(\mathbb{S}^1, G) \subset C^\infty(\mathbb{S}^1, \mathrm{U}(N))$, the multiplication operator M_γ in the Hilbert space $H = L_0^2(\mathbb{S}^1, \mathbb{C}^N)$, acting by the formula

$$h \in H = L_0^2(\mathbb{S}^1, \mathbb{C}^N) \longmapsto M_\gamma h(z) := \gamma(z)h(z), \qquad z \in \mathbb{S}^1.$$

In other words, $M_\gamma h$ is a vector function from $H = L_0^2(\mathbb{S}^1, \mathbb{C}^N)$, obtained by the pointwise application of the matrix function $\gamma \in C^\infty(\mathbb{S}^1, \mathrm{U}(N))$ to the vector function $h \in H = L_0^2(\mathbb{S}^1, \mathbb{C}^N)$. The operator M_γ belongs to the unitary group $\mathrm{U}_{\mathrm{HS}}(H)$ if $\gamma \in C^\infty(\mathbb{S}^1, \mathrm{U}(N))$.

The constructed embedding $LG \hookrightarrow \mathrm{U}_{\mathrm{HS}}(H)$ generates an isometric embedding

$$\Omega G \longrightarrow \mathrm{Gr}_{\mathrm{HS}}(H).$$

8. Idea of the Proof of Harmonic Spheres Conjecture

8.1. Harmonic Analogue of Atiyah Theorem

Using the constructed isometric embedding $\Omega G \hookrightarrow \mathrm{Gr}_{\mathrm{HS}}(H)$, we can consider an arbitrary harmonic map $\varphi : \mathbb{P}^1 \to \Omega G$ as taking its values in the Grassmannian $\mathrm{Gr}_{\mathrm{HS}}(H)$, hence, in one of its connected components $G_r(H)$ and use the twistor method.

We start from a harmonic version of Atiyah theorem, relating based harmonic spheres $\varphi : \mathbb{P}^1 \to \Omega G$ to harmonic $G^{\mathbb{C}}$-bundles over $\mathbb{P}^1 \times \mathbb{P}^1$. For a fixed $z \in \mathbb{P}^1$ we pull back the value $\varphi(z) \in \Omega G$ to $\tilde{\varphi}(z) \in LG^{\mathbb{C}}$ and consider $\tilde{\varphi}(z)$ as a transition function of a bundle over projective line $\mathbb{P}^1_z$. By changing $z \in \mathbb{P}^1$, we obtain a $G^{\mathbb{C}}$-bundle E over $\mathbb{P}^1 \times \mathbb{P}^1$ which is harmonic and trivial over $\mathbb{P}^1 \cup \mathbb{P}^1$ if and only if the original map φ is based and harmonic.

We note that if we consider the map $\varphi : \mathbb{P}^1 \to \Omega G$ as taking values in $\mathrm{Gr}_{\mathrm{HS}}(H)$ then the value $\varphi(z)$ for a fixed $z \in \mathbb{P}^1$ is interpreted in terms of $\mathrm{Gr}_{\mathrm{HS}}(H)$ as a subspace $W_z = M_{\tilde{\varphi}(z)} H_+$.

8.2. Twistor Interpretation of the Moduli Space of Yang–Mills Fields

The twistor interpretation of the above construction has the following form. A harmonic sphere $\varphi : \mathbb{P}^1 \to \Omega G$ may be considered as a harmonic sphere in a submanifold $G_r(H) \subset \mathrm{Gr}_{\mathrm{HS}}(H)$, consisting of subspaces $W \subset H$ of some fixed virtual dimension r. Assuming that the converse of Theorem 23 is true, the harmonic sphere $\varphi : \mathbb{P}^1 \to G_r(H)$ in terms of the twistor flag bundle should coincide with the projection of some $\mathcal{J}^2_\sigma$-holomorphic sphere $\psi : \mathbb{P}^1 \to F^d_{\mathbf{r}}(H)$ so that there is a commutative diagram

$$\begin{array}{ccc} & & F^d_{\mathbf{r}}(H) \\ & \overset{\psi}{\nearrow} & \Big\downarrow \pi_\sigma \\ \mathbb{P}^1 & \xrightarrow[\varphi]{} & G_r(H)\,. \end{array}$$

Then for a fixed $z \in \mathbb{P}^1$ we pull back the value $\psi(z) = (\psi_1(z), \ldots, \psi_n(z))$ to $\tilde{\psi}(z) = (\tilde{\psi}_1(z), \ldots, \tilde{\psi}_n(z))$ with $\tilde{\psi}_i(z) \in LG^{\mathbb{C}}$.

In terms of $F^d_{\mathbf{r}}(H)$ the value $\psi(z) = (\psi_1(z), \ldots, \psi_n(z))$ is given by the virtual flag $\mathcal{W}(z) = (W_1(z), \ldots, W_n(z)$ where $W_i(z) = M_{\tilde{\psi}_i(z)} H_+$.

The functions $\tilde{\psi}_i(z) \in LG^{\mathbb{C}}$, being considered as transition functions, determine some bundles over $\mathbb{P}^1_z$. By changing $z \in \mathbb{P}^1$, we obtain for $i = 1, \ldots, n$ the $G^{\mathbb{C}}$-bundles E_i over $\mathbb{P}^1 \times \mathbb{P}^1$, trivial over $\mathbb{P}^1_\infty \cup \mathbb{P}^1_\infty$. It follows from the definition of the almost complex structure $\mathcal{J}^2_\sigma$ that these bundles E_i should be either holomorphic or anti-holomorphic. So by Atiyah theorem they should correspond either to instantons or anti-instantons on $\mathbb{R}^4$.

In this way we can associate with any Yang–Mills field on $\mathbb{R}^4$ a finite collection of instantons and anti-instantons on $\mathbb{R}^4$. This construction may be considered as a twistor description of the moduli space of Yang–Mills fields on $\mathbb{R}^4$.

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