

NEWS ON IMMERSIONS OF THE LOBACHEVSKY SPACE INTO EUCLIDEAN SPACE

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Abstract. An exposition of the new results concerning the nonexistence of local isometric immersions of 3-dimensional Lobachevsky space L^3 into 5-dimensional Euclidean space E^5 with constant curvature of the Grassmannian image metric, on connections between curvatures of asymptotic lines on a domain of $L^3 \subset E^5$, on regularity theorems for surfaces obtained by Backlund transformation of a domain of $L^2 \subset S^3$ and $L^2 \subset E^3$.

Isometric immersions of the domains of the n -dimensional Lobachevsky space L^n into the $(2n - 1)$ -dimensional Euclidean space E^{2n-1} for $n > 2$ were considered in works by Moore, Tenenblat, Terng, the present author and others. It is well-known, that L^n cannot be locally immersed into E^{2n-2} . So the dimension $(2n - 1)$ is the least possible one. In this case there exist relations between the extrinsic and intrinsic properties of the submanifolds. It is possible to prove that on an immersed domain of L^n there exist coordinates of curvature u^1, \dots, u^n such that the metric of L^n is expressed in the form

$$ds^2 = \sum_{i=1}^n \sin^2 \sigma_i (du^i)^2 \quad (1)$$

with the condition

$$\sum_{i=1}^n \sin^2 \sigma_i = 1. \quad (2)$$

The functions σ_i satisfy some system of nonlinear differential equations which expresses that the curvature of ds^2 is equal to -1 and the condition (2) is fulfilled.

For convenience we shall use the following notation: $H_i = \sin \sigma_i$, $\beta_{ij} = \frac{1}{H_i} \frac{\partial H_j}{\partial u_i}$, $i \neq j$.

Then the following system of differential equations describes the isometric immersions of Lobachevsky space L^n into E^{2n-1} (see [1, 2])

$$\begin{aligned} \text{a) } \frac{\partial H_j}{\partial u_i} &= \beta_{ij} H_i, & \text{d) } \frac{\partial \beta_{ij}}{\partial u_j} + \frac{\partial \beta_{ji}}{\partial u_i} + \sum_q \beta_{iq} \beta_{jq} &= 0, \\ \text{b) } \frac{\partial H_i}{\partial u_i} &= - \sum_q \beta_{iq} H_q, & \text{e) } \frac{\partial \beta_{ij}}{\partial u_i} + \frac{\partial \beta_{ji}}{\partial u_j} + \sum_q \beta_{qi} \beta_{qj} &= H_i H_j. \quad (3) \\ \text{c) } \frac{\partial \beta_{ij}}{\partial u_k} &= \beta_{ik} \beta_{kj}, \end{aligned}$$

where $i \neq j \neq k$. It is natural to call this system the *Lobachevsky-Euclid system*, or, briefly, the *LE system*.

This system is a generalization of the well-known equation

$$\omega_{uu} - \omega_{vv} = \sin \omega, \quad (4)$$

called “sin-Gordon equation”, but in fact Gordon has no relation either to this equation or to the *LE system*. In [8] the authors indicated that the first scientist who wrote this equation was Enneper. So the correct title of this equation must be the **Enneper equation**.

Various classes of solutions for the *LE system* and the corresponding immersions of domains of L^n into E^{2n-1} were considered.

- 1) Immersions with one family of curvature curves consists of geodesic curves.
- 2) Immersions with one family of curvature surfaces consists of totally-geodesic surfaces.
- 3) Functionally degenerate immersions, when the functions H_i depend on less than $n - 1$ variables.
- 4) Immersions of L^3 into E^5 with a hyperflat Grassmannian image.

Now I would like to consider the immersions with restrictions on the Grassmannian image.

Let Γ^n be the Grassmannian image of an immersed domain $D \subset E^{2n-1}$. The Grassmannian manifold $G_{p,q}$ can be considered as a submanifold in the Euclidean space. The metric of this Euclidean space induces the metric of $G_{p,q}$ and consequently the metric of the Grassmannian image. The Grassmannian

mapping ϕ transfers coordinates u^1, \dots, u^n from L^n onto Γ^n . The standard metric of the Grassmannian manifold $G_{n-1, 2n-1}$ induces the metric dl^2 of Γ^n

$$dl^2 = \sum_{i=1}^n \cos^2 \sigma_i (du^i)^2 \tag{5}$$

with the condition

$$\sum_{i=1}^n \cos^2 \sigma_i = n - 1.$$

The following question arises in the natural way: do there exist local immersions of L^n into E^{2n-1} with constant curvature of the metric of the Grassmannian image?

I have obtained the answer to this question only for $n = 3$, [3]

Theorem 1. *There is no local C^3 isometric immersion of L^3 into E^5 with constant curvature of the metric of the Grassmannian image.*

In other words, forgetting about immersion and Grassmannian image, we can reformulate this statment only in terms of metrics: If the metric (1) with condition (2) has a constant curvature equal to -1 , then the metric (5) does not have a constant curvature.

This time there exists a local immersion L^3 into E^5 such that the Grassmannian image has a constant curvature along a line.

The proof of this theorem is based on three lemmas.

Let Λ_{ijkh} be the components of the curvature tensor of the metric (4). These components have expressions in terms of the functions H_i and β_{ij} .

Let Λ be the matrix of curvature of (5)

$$\Lambda = \begin{vmatrix} \Lambda_{2323} & \Lambda_{3132} & \Lambda_{2123} \\ * & \Lambda_{1313} & \Lambda_{1213} \\ * & * & \Lambda_{1212} \end{vmatrix}.$$

Lemma 1. *If the matrix Λ is diagonal, then there exist functions $\alpha_i = \alpha_i(u^i)$ and a function $\Phi(\alpha_1, \alpha_2, \alpha_3)$ such that the coefficients of the metric ds^2 are*

$$H_i^2 = \frac{\partial \Phi}{\partial \alpha_i}, \quad i = 1, 2, 3.$$

Lemma 2. *The functions α_i satisfy the following equations*

$$\left(\frac{\partial \alpha_i}{\partial u_i} \right)^2 = c\alpha_i + \rho_i,$$

where c and ρ_i are constants.

Lemma 3. *If the metric of the Grassmannian image has a constant curvature, then under the Grassmannian map the ratio of the volume elements of the image and the pre-image is constant.*

This conclusion can be written in the form

$$\frac{\cos \sigma_1 \cos \sigma_2 \cos \sigma_3}{\sin \sigma_1 \sin \sigma_2 \sin \sigma_3} = \text{const} .$$

We thus passed from the condition on the second derivatives of the functions H_i to the condition on the functions H_i themselves. So the immersion is some special subclass of functionally degenerate immersions. Later we can easily obtain a contradiction.

Asymptotic Lines

It is well-known that on the surface of negative curvature $F^2 \subset E^3$ the torsion of asymptotic curve is

$$\kappa_1 = \sqrt{-K}$$

for one family and

$$\kappa_2 = -\sqrt{-K} ,$$

for the second family, where K is the Gaussian curvature of F^2 .

These relations are very useful for theorems in the “large”.

Let us consider the generalization of these relations for n -dimensional submanifolds F^n in the Riemannian space M^{n+p} . The following theorem is proved in [5].

Theorem 2. *Let γ be an asymptotic curve on the submanifold $F^n \subset M^{n+p}$. Let ξ_1, \dots, ξ_{n+p} be the natural frame along γ . Let ϕ be the angle between ξ_3 and the normal space to F^n . Then for the second curvature k_2 of γ we have the expression*

$$k_2 = \frac{\sqrt{\bar{K}(\xi_1, \xi_2) - K_i(\xi_1, \xi_2)}}{\cos \phi}$$

where $\bar{K}(\xi_1, \xi_2)$ is the curvature of M^{n+p} , $K_i(\xi_1, \xi_2)$ is the curvature of F^n .

If ξ_3 belongs to the tangent space of F^n (for example, if the nominator and denominator are equal to zero) then this formula does not work. But in this case it is possible to obtain an expression for the next curvature, see [5].

Let us consider now the asymptotic line γ on a domain of L^3 immersed into E^5 . Let k_1, k_2, k_3 and k_4 be the curvatures of this curve $\gamma \subset E^5$, then [4]:

$$k_2 = \frac{1}{\cos \phi}$$

Hence, for the regular immersion, ξ_3 does not belong to tangent space $\cos \phi \neq 0$. Moreover, there exists a relation between k_2, k_3 and k_4

$$k_4 = k_2 \left(\pm 1 \pm k_3 \frac{\frac{d^2 \sin \phi}{dt^2} + \sin \phi}{\sqrt{1 - \left(\frac{d\phi}{dt}\right)^2}} \right),$$

where $\frac{d}{dt} = \frac{1}{k_3} \frac{d}{ds}$. The different signs in this formula probably correspond to four different asymptotic curves through one point.

So, an arbitrary curve in E^5 could not be an asymptotic curve on immersed domain of L^3 into E^5 .

On Regularity of Backlund Transform

Now let us consider immersions of L^3 into 3-dimensional sphere S^3 with radius R and its Backlund transformation.

We are interested in the question of regularity of the surface obtained by this transformation. The system describing of the Backlund transformation has been written for the first time by Bianchi. Let F^2 be represented by the position vector $r = r(u, v)$. In terms of the curvature coordinates u, v , the metric of F^2 can be written as follows

$$ds^2 = \cos^2 \omega du^2 + \sin^2 \omega dv^2.$$

The Backlund transformation can be constructed in the following way: let an arc of a great circle of the sphere S^3 be going from a point $x \in F^2$ along a tangent direction τ which makes an angle ϕ with the first principal direction. Let the length of this arc be a constant number. The end point of this arc is a point of a new surface \bar{F}^2

Define $\tau = \cos \phi \tau_1 + \sin \phi \tau_2$, where τ_1 and τ_2 are the unit vectors along the principal directions. Let ϕ satisfy the following system of equations

$$\begin{aligned} \frac{\partial \phi}{\partial u} + \frac{\partial \omega}{\partial v} &= \alpha \cos \omega \sin \phi + \beta \sin \omega \cos \phi, \\ \frac{\partial \phi}{\partial v} + \frac{\partial \omega}{\partial u} &= -\alpha \sin \omega \cos \phi - \beta \cos \omega \sin \phi, \end{aligned}$$

where α and β are such constants that $\alpha^2 - \beta^2 = 1$.

Then the transformation $\psi: r \rightarrow \bar{r}$, where

$$\bar{r} = r \sin \sigma + R\tau \cos \sigma, \quad \sigma = \arctan R\alpha$$

transforms F^2 into a surface \bar{F}^2 with $\bar{K} = -1$.

In our joint article with Cieslinski [6] we have considered the question of regularity of the surface \bar{F}^2 .

Let $T(t)$ be a geodesic disc on F^2 with a radius t and ϕ_0 be an initial value of ϕ at the center O of $T(t)$. We assume $0 < \phi_0 < \pi/2$. Then the surface \bar{F}^2 is regular at the point corresponding to O .

What is the lower bound for the radius t_r such that the Backlund transformation of $T(t_r)$ gives a regular surface?

Theorem 3. *Let a pseudospherical immersion F^2 with principal curvatures k_1, k_2 belongs to the class C^k ($k \geq 2$), and let there exist positive constants p and N such that*

$$\sqrt{1 + k_i^2} \leq p, \quad \frac{\partial}{\partial s} \ln \sqrt{1 + k_i^2} \leq N, \quad i = 1, 2,$$

where s is the arc length of a curve on F^2 . Let $M_1(\phi_0) = \min(\phi_0, \pi/2 - \phi_0)$.

If
$$t_r \leq \frac{M_1(\phi_0)}{\alpha + \beta p + 2N},$$

then the image of the geodesic disc $T(t_r)$ under the Backlund transformation is a regular surface of the class C^{k-1} .

I remark that similar estimate exists for the Backlund transformation of $L^2 \subset E^3$.

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