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$\mathfrak{g}$-SYMPLECTIC ORBITS AND A SOLUTION OF THE BRST CONSISTENCY CONDITION

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Abstract. For any Lie algebra $\mathfrak{g}$ we introduce the notion of $\mathfrak{g}$-symplectic structures and show that every orbit of a principal G-bundle carries a natural $\mathfrak{g}$-symplectic form and an associated momentum map induced by the Maurer–Cartan form on G. We apply this to the BRST bicomplex and show that the associated momentum map is a solution of the Wess–Zumino consistency condition for the anomaly.

1. Introduction

We first introduce the notion of Lie algebra $\mathfrak{g}$-valued symplectic structures and we show that every orbit of a principal G-bundle carries a natural $\mathfrak{g}$-symplectic form, which is exact and induced from the Maurer–Cartan form on the Lie group G. The G-action has a natural momentum map which is an invariant for any fundamental vector field. In order to give a solution to the BRST (Wess–Zumino) consistency condition, we generalize these results to infinite dimensional group $\mathcal{G}$ of gauge transformations which acts on $\mathfrak{g}$-valued differential forms. On these orbit spaces we have the natural $\mathfrak{g}$-valued 1-form Θ, induced by the Maurer–Cartan form on the Lie group $\mathcal{G}$, and the corresponding momentum map. We summarize the classical BRST transformations described as coboundary operator of the Chevalley–Eilenberg complex of the infinite dimensional Lie algebra $\mathfrak{g}$ of infinitesimal gauge transformations, [10–12]. Next we describe the chiral anomaly as element of the first cohomology of the local BRST complex [11, 12] using an induced representation of $\mathfrak{g}$ on local forms. We consider the Wess–Zumino consistency condition as a problem in this BRST cohomology. To find a solution we combine the BRST bicomplex with the idea of $\mathfrak{g}$-valued symplectic geometry and momentum maps. We show that this momentum map is a solution of the Wess–Zumino consistency condition for the 1-form Θ.

2. The $\mathfrak{g}$-symplectic Structure on Orbits

For a principal G-bundle (P, π, M) we denote by $\Omega^k(P, \mathfrak{g})$ the space of $\mathfrak{g}$ valued k-forms on P, called $\mathfrak{g}$-forms for short. For $\mathfrak{g}$-forms the usual Cartan calculus holds like for classical real valued forms. For instance, if f is a $\mathfrak{g}$-function on P (0-form), $f\colon P \to \mathfrak{g}$, then $\mathrm{d}f$ is a $\mathfrak{g}$ one-form on P, i. e. $\mathrm{d}f(p) := T_p f\colon T_pP \to T_{f(p)}\mathfrak{g} \simeq \mathfrak{g}$. If $\varphi\colon P \to P$ is a smooth map and $\alpha \in \Omega^k(P, \mathfrak{g})$, then the pull back $\varphi^*\alpha \in \Omega^k(P, \mathfrak{g})$. The Lie derivative L_X with respect to a vector field X, the inner product operator i_X and exterior derivative d are defined analogous to the classical real valued case and we have the Cartan formula $L_X\alpha = \mathrm{d}i_X\alpha + i_X\,\mathrm{d}\alpha$ for any $\alpha \in \Omega^k(P, \mathfrak{g})$.

We define $\mathfrak{g}$-valued symplectic forms as follows:

Definition 1. *A $\mathfrak{g}$-**symplectic structures** on P is a $\mathfrak{g}$-form $\Omega \in \Omega^2(P, \mathfrak{g})$ which is closed and nondegenerate, i. e. $\mathrm{d}\Omega = 0$ and for each $p \in P$ the map $\Omega(p)\colon T_pP \times T_pP \to \mathfrak{g}$ is bilinear and nondegenerate, meaning that if $\Omega(p)(u, v) = 0$ for all $u \in T_pP$ then $v \equiv 0$.*

A $\mathfrak{g}$-symplectic form Ω on P induces a linear injective map

$$\Omega(p)^{\#}\colon T_pP \to L(T_pP, \mathfrak{g}), \qquad \Omega(p)^{\#}(v)\cdot w = \Omega(p)(v, w).$$

A vector field X on P is called **$\mathfrak{g}$-Hamiltonian** if there exists a $\mathfrak{g}$-function $f\colon P \to \mathfrak{g}$ such that $\mathrm{d}f = i_X\Omega$, or equivalently $\Omega(p)^{\#}X(p) = \mathrm{d}f(p)$.

A $\mathfrak{g}$-vector field X is locally $\mathfrak{g}$-Hamiltonian if and only if its flow φ_t is $\mathfrak{g}$-symplectic, i. e. $\varphi_t^*\Omega = \Omega$, indeed $\varphi_t^*\Omega = \Omega$ iff $0 = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \varphi_t^*\Omega = L_X\Omega = \mathrm{d}i_X\Omega + i_X\,\mathrm{d}\Omega = \mathrm{d}i_X\Omega$ and by the Poincare lemma for $\mathfrak{g}$-forms, there exists locally a $\mathfrak{g}$-function f such that $\mathrm{d}f = i_X\Omega$. We will later use the explicit formula for f. There is a general local formula. For any closed $\mathfrak{g}$-form $\alpha \in \Omega^p(\mathbb{R}^n, \mathfrak{g})$, $\mathrm{d}\alpha = 0$, there exists locally a $\beta \in \Omega^{p-1}(\mathbb{R}^n, \mathfrak{g})$ such that $\alpha = \mathrm{d}\beta$ and β is given by

$$\beta(x) = \int_0^1 i_x\alpha(tx)\,\mathrm{d}t \tag{1}$$

Let θ be the *right* invariant Maurer–Cartan form on G, that is $\theta(g) = T_g r_{g^{-1}}\colon T_gG \to \mathfrak{g}$, where $r_g\colon G \to G$ is right multiplication on G, $r_g(h) = hg$. Note that with respect to the left multiplication l_g on G the right Maurer–Cartan form θ satisfies $l_g^*\theta = \mathrm{Ad}_g \circ\theta$. Furthermore we have $\mathrm{d}\theta(v, w) =$

$v \cdot \theta(w) - w \cdot \theta(v) - \theta[v, w]$ and if v, w are vertical tangent vectors we get $\mathrm{d}\theta(v, w) = -\theta[v, w] = -[\theta(v), \theta(w)]$.

Let $R\colon P \times G \to P$ denotes the right action of G on P and for each $p \in P$ let $\mathcal{O}_p$ be the G-orbit through p, i. e. $\mathcal{O}_p = R(p, G)$ with the diffeomorphism $R_p\colon G \to \mathcal{O}_p; R_p(g) = R(p, g)$ and the G-action $R_g\colon \mathcal{O}_p \to \mathcal{O}_p$, $g \in G$.

Theorem 1. *If G is semi-simple, then every G-orbit $\mathcal{O}_p$ is a $\mathfrak{g}$-symplectic manifold.*

Proof: Let $p \in P$ and define the $\mathfrak{g}$ one-form Θ_p on the orbit $\mathcal{O}_p$ by $\Theta_p := R_{p*}\theta$. The $\mathfrak{g}$ 2-form Ω_p on the orbit $\mathcal{O}_p$ defined by $\Omega_p := \mathrm{d}\Theta_p$ is clearly closed, $\mathrm{d}\Omega_p = 0$ and Ω_p is nondegenerate; indeed let $\Omega_p(q)(v_q, w_q) = 0$ for all $v_q \in T_q\mathcal{O}_p$, $q \in \mathcal{O}_p$. Then

$$\begin{aligned} 0 = \Omega_p(q)(v_q, w_q) &= \mathrm{d}\Theta_p(q)(v_q, w_q) = \mathrm{d}(R_{p*}\theta)(q)(v_q, w_q) \\ &= R_{p*}(\mathrm{d}\theta)(q)(v_q, w_q) = -R_{p*}[\theta(q)v_q, \theta(q)w_q]\,. \end{aligned}$$

This implies that $[\theta(q)v_q, \theta(q)w_q] = 0$ for all v_q and since $\mathfrak{g}$ is semi-simple, this implies that $\theta(q)w_q = 0$ for all q and w_q, but w_q is vertical, hence $w_q = 0$. Therefore Ω_p is a $\mathfrak{g}$-symplectic form. □

Remark 1. *We don't need to assume that G is semi-simple, only that $\mathfrak{g}$ has no center.*

Proposition 1. *The canonical $\mathfrak{g}$ one-form Θ_p and the $\mathfrak{g}$-symplectic form Ω_p are G-invariant. For each $R_g\colon \mathcal{O}_p \to \mathcal{O}_p$, $g \in G$*

$$R_g^*\Theta_p = \Theta_p R_g^*\Omega_p = \Omega_p\,.$$

The $\mathfrak{g}$-Poisson bracket for any two $\mathfrak{g}$-functions $f, g\colon \mathcal{O}_p \to \mathfrak{g}$ such that $\mathrm{d}f(q), \mathrm{d}g(q) \in \Omega^\#(p)(T_qP)$ (i. e. X_f, X_g exist) is defined by

$$\{f, g\}(p) = \Omega(p)(X_f(q), X_g(q)) \in \mathfrak{g}\,.$$

This bracket makes $C^\infty(\mathcal{O}_p, \mathfrak{g})$ into a Lie algebra.

2.1. The Canonical Momentum Map on $\mathcal{O}_p$

Proposition 2. *For every $\xi \in \mathfrak{g}$ the fundamental vector field ξ_P on $\mathcal{O}_p$ defined by*

$$\xi_P(q) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} R_{\exp t\xi}(q) \tag{2}$$

is locally $\mathfrak{g}$-Hamiltonian.

Proof: The flow of ξ_P is $\varphi_t(q) = R_{\exp t\xi}(q)$, but $R^*_{\exp t\xi}\Theta_p = \Theta_p$, hence $R^*_{\exp t\xi}\Omega_p = \Omega_p$. So the flow is $\mathfrak{g}$-symplectic, hence ξ_P is locally $\mathfrak{g}$-Hamiltonian. □

Therefore, for every $\xi \in \mathfrak{g}$, there exists a $\mathfrak{g}$-function $H\colon \mathcal{O}_p \to \mathfrak{g}$ such that $\xi_P = X_H$ i. e. $\mathrm{d}H = i_{\xi_P}\Omega_p$. We explicitly compute the function H. Locally from (1) we get for $\alpha = i_{\xi_P}\Omega_p$, $X_H = \xi_P = T_eR_p(\xi)$, $\xi \in \mathfrak{g}$, $\Omega = \Omega_p = \mathrm{d}\Theta_p$, $\Theta_p = R_{p*}\theta$. Then locally H is given by

$$
\begin{aligned}
H(x) &= \int_0^1 i_x i_{\xi_P}\Omega_p(tx)\,\mathrm{d}t = \int_0^1 \Omega_p(tx)(x,\xi_P(tx))\,\mathrm{d}t \\
&= \int_0^1 \mathrm{d}\Theta_p(tx)(x,\xi_P(tx))\,\mathrm{d}t = \int_0^1 \mathrm{d}R_{p*}\theta(tx)(x,\xi_P(tx))\,\mathrm{d}t \\
&= \int_0^1 (R_{p*}\,\mathrm{d}\theta)(tx)(x,\xi_P(tx))\,\mathrm{d}t \\
&= \int_0^1 \mathrm{d}\theta(R_p^{-1}(tx))(TR_p^{-1}(x), TR_p^{-1}TR_{tx}(\xi))\,\mathrm{d}t \\
&= \int_0^1 -[TR_p^{-1}(x), T(R_p^{-1}R_{tx})(\xi)]\,\mathrm{d}t \\
&= -\int_0^1 t[TR_p^{-1}(x), T(R_p^{-1}R_x)(\xi)]\,\mathrm{d}t \\
&= -\frac{1}{2}[TR_p^{-1}(x), T(R_p^{-1}R_x)(\xi)] \in \mathfrak{g}\,, \qquad x \in \mathcal{O}_p\,.
\end{aligned}
$$

For simplicity we write $H(x) = -\frac{1}{2}[x, x\cdot\xi]$.

The $\mathfrak{g}$-momentum map of the right action of G on $\mathcal{O}_p$ is the map $J\colon \mathcal{O}_p \to \mathcal{L}(\mathfrak{g},\mathfrak{g})$ defined by $\langle J(q),\xi\rangle = H(q), q \in \mathcal{O}_p$, $\xi \in \mathfrak{g}$. Notice that $\mathcal{L}(\mathfrak{g},\mathfrak{g})$ replaces the dual $\mathfrak{g}^*$ from the real case.

Proposition 3. *The momentum map of the action of G on $\mathcal{O}_p$ is given by*

$$J(q) = \mathrm{ad}_\eta \circ TR_q \tag{3}$$

where $\eta = R_p^* X_t(g)$, $q = p\cdot g$.

Proof: We have $R_p^*\xi_P(g) = R_p^* Z_\xi(g) = Z_\xi(R_p g) = Z_\xi(pg) = TR_{pg}(\xi) \in T_{pg}\mathcal{O}_p$, hence $\langle J(q), \xi\rangle = -[R_p^*\xi_P(g), R_p^* X_t(g)] = \mathrm{ad}_{R_p^* X_t(g)}(R_p^*\xi_P(g)) = \mathrm{ad}_\eta(TR_{pg}(\xi))$, where $\eta = R_p^* X_t(g)$. □

3. Local Differential Forms

A Lagrangian (or variational principle [16, 20]) on a fiber bundle $\pi\colon P \to M^n$ is an operator L which assigns to each local section s of π an n-form $L(s)$ on the domain of s such that $L(s)(x)$ depends smoothly on the value of $s(x)$ and on only a finite number of derivatives $D^j s(x)$, $0 \le j \le k < \infty$. This leads to the notion of local differential forms, [16, 20, 19], and to the associated variational bicomplex and its cohomology, [1, 11, 12], which we will recall.

Let $\pi\colon P \to M$ be a smooth fiber bundle and let $\Gamma^\infty(\pi)$ denote the manifold of smooth sections of π. The spaces of k-jets $J^k(\pi)$, $0 \le k \le \infty$, of local sections of π are smooth manifolds and we have the canonical projections, for $0 \le l \le k$, $\pi_k^l\colon J^k(\pi) \to J^l(\pi)$, and $\pi_k\colon J^k(\pi) \to M$, as well as the k-jet extension maps $j^k\colon M \times \Gamma^\infty(\pi) \to J^k(\pi); j^k(x,s) = [x,s]_k$ the k-jet equivalence class of (x,s). Note that $J^0(\pi) = M$ and $\pi_0 = \pi$.

There is a natural splitting of the tangent space $T_s J^\infty(\pi) = H_s \oplus V_s$ at each $s \in J^\infty(\pi)$ and hence of the space of vector fields on $J^\infty(\pi)$ — $\mathrm{Vec}(J^\infty(\pi)) = \mathbf{H} \oplus \mathbf{V}$ as follows: $\mathbf{H}$ is the space of horizontal vector fields, i. e. lifts of vector fields $\bar{X}$ on M; $\bar{X} \in \mathrm{Vec}(M) \mapsto X \in \mathrm{Vec}(J^\infty(\pi))$ defined by $(X(f))(s) = \bar{X}(f \circ S)(\pi_\infty(s))$ where $f \in C^\infty(J^\infty(\pi)), s \in J^\infty(\pi)$ and S is a local section at of π such that $j^\infty(x,S) = [x,s]_\infty$, i. e. the ∞-jet of S in $\pi_\infty(s)$ equals s. The subspace $\mathbf{V}$ is the space of vertical vector fields on $J^\infty(\pi)$; i. e. $Y \in \mathbf{V}$ if and only if $Y(f \circ \pi_\infty) = 0$ for all $f \in C^\infty(M)$. It should be remarked that such a canonical splitting of $\mathrm{Vec}(J^\infty(\pi)) = \mathbf{H} \oplus \mathbf{V}$ cannot be constructed for $J^k(\pi)$ if $k < \infty$, [9, 16].

We denote by $\Omega_p^q(\pi)$ the vector space of those $(q+p)$-forms ω on $J^\infty(\pi)$ with $\omega(X_1, \dots, X_{q+p}) = 0$ if more than q of the vector fields X_i, $1 \le i \le q+p$, are vertical or more than p of them are horizontal. Elements of $\Omega_p^q(\pi)$ are called **local forms** on $J^\infty(\pi)$. If $\omega \in \Omega_p^q(\pi)$ then $\mathrm{d}\omega \in \Omega_p^{q+1}(\pi) \oplus \Omega_{p+1}^q(\pi)$, which imp Lies that the (total) exterior derivative $\mathrm{d}\colon \Omega_p^q(\pi) \to \Omega_p^{q+1}(\pi) \oplus \Omega_{p+1}^q(\pi)$ splits into the **vertical exterior derivative** $\mathrm{d}_V\colon \Omega_p^q(\pi) \to \Omega_p^{q+1}(\pi)$ and the **horizontal exterior derivative** $\mathrm{d}_H\colon \Omega_p^q(\pi) \to \Omega_{p+1}^q$ defined by $\mathrm{d} = \mathrm{d}_H + \mathrm{d}_V$. Then $\mathrm{d}^2 = \mathrm{d}_H^2 = \mathrm{d}_V^2 = \mathrm{d}_H\,\mathrm{d}_V + \mathrm{d}_V\,\mathrm{d}_H = 0$. This bicomplex of local forms is often called the **variational bicomplex**, (see e. g. Anderson [1], Saunders [9]). Horizontal and vertical derivatives satisfy certain exactness theorems and

Poincaré lemmas [16], so that we have the variational bicomplex [1, 16]:

$$
\begin{array}{ccccccccccccc}
 & & & & \uparrow{\scriptstyle d_V} & & & & & & & & \uparrow{\scriptstyle d_V} \\
 & & 0 & \longrightarrow & \Omega_0^3 & & & & \cdots\cdots & & & & \Omega_n^3 \\
 & & & & \uparrow{\scriptstyle d_V} & & & & & & & & \uparrow{\scriptstyle d_V} \\
 & & 0 & \longrightarrow & \Omega_0^2 & \xrightarrow{d_H} & \Omega_1^2 & \xrightarrow{d_H} & \cdots & \Omega_{n-1}^2 & \xrightarrow{d_H} & & \Omega_n^2 \\
 & & & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} & & & \uparrow{\scriptstyle d_V} & & & \uparrow{\scriptstyle d_V} \\
 & & 0 & \longrightarrow & \Omega_0^1 & \xrightarrow{d_H} & \Omega_1^1 & \xrightarrow{d_H} & \cdots & \Omega_{n-1}^1 & \xrightarrow{d_H} & & \Omega_n^1 \\
 & & & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} & & & \uparrow{\scriptstyle d_V} & & & \uparrow{\scriptstyle d_V} \\
0 & \longrightarrow & \mathbb{R} & \longrightarrow & \Omega_0^0 & \xrightarrow{d_H} & \Omega_1^0 & \xrightarrow{d_H} & \cdots & \Omega_{n-1}^0 & \xrightarrow{d_H} & & \Omega_n^0
\end{array}
$$

The **local cohomology** of π is defined as $H^p(\pi) = \operatorname{Ker} \mathrm{d} \,/\operatorname{Im} \mathrm{d}$. These are local cohomologies in the physical sense as the following shows.

There is another characterization of local forms, which justifies their names. Consider the de Rham complex $\Omega(M\times\Gamma^\infty(\pi))$ of smooth differential forms on $M\times\Gamma^\infty(\pi)$ with exterior derivative d. From the product structure of $M\times\Gamma^\infty(\pi)$ the space $\Omega(M\times\Gamma^\infty(\pi))$ inherits a bigradation and we can write

$$\Omega(M\times\Gamma^\infty(\pi)) = \coprod_{p,q} \Omega^{p,q}(M\times\Gamma^\infty(\pi))\,.$$

Corresponding to this bigradation the exterior derivative d on $M \times \Gamma^\infty(\pi)$ breaks into two operators; D of type (1, 0), $D\colon \Omega^{p,q}(M\times\Gamma^\infty(\pi)) \to \Omega^{p+1,q}(M\times\Gamma^\infty(\pi))$, and ∂ of type (0, 1), $\partial\colon \Omega^{p,q}(M\times\Gamma^\infty(\pi)) \to \Omega^{p,q+1}(M\times\Gamma^\infty(\pi))$. With these we have $\mathrm{d} = D + \partial$ and $\mathrm{d}^2 = D^2 = \partial^2 = D\partial + \partial D = 0$. If $A \in \Omega^{p,0}(M\times\Gamma^\infty(\pi))$ and $s \in \Gamma^\infty(\pi)$ we can define a p-form $A(s)$ on M by $A(s)(x) = A(x,s)$, $x \in M$. Then $DA \in \Omega^{p+1,0}(M\times\Gamma^\infty(\pi))$ and we have $(DA)(s) = \mathrm{d}_M(A(s))$ where d_M is the exterior derivative on M. More generally, if $A \in \Omega^{p,q}(M\times\Gamma^\infty(\pi))$, $s \in \Gamma^\infty(\pi)$ and $X_1,\ldots,X_q \in \mathrm{Vec}(J^\infty(\pi))$ we can define a p-form $A(s,X_1,\ldots,X_q)$ on M by $A(s,X_1,\ldots,X_q)(x) = (i_{X_1}\ldots i_{X_q}A)(x,s)$. Again $DA \in \Omega^{p+1,q}(M\times\Gamma^\infty(\pi))$ is given by $(DA)(s,X_1,\ldots,X_q) = \mathrm{d}_M(A(s,X_1,\ldots,X_q))$. $\Omega(M\times\Gamma^\infty(\pi))$ has a canonical sub-bicomplex $\Omega_{\text{loc}}(M\times\Gamma^\infty(\pi))$ defined as follows: The ∞-jet extension map $j_\infty\colon M\times\Gamma^\infty(\pi) \to J^\infty(\pi)$ induces a map of the de Rham complexes $j_\infty^*\colon \Omega(J^\infty(\pi)) \to \Omega(M\times\Gamma^\infty(\pi))$. The image $j_\infty^*\Omega(j^\infty(\pi))$ in $\Omega(M\times\Gamma^\infty(\pi))$ is stable under both D and ∂, and hence is a sub-bicomplex which we denote by $\Omega_{\text{loc}}(M\times\Gamma^\infty(\pi))$. We write

$$\Omega_{\text{loc}}(M\times\Gamma^\infty(\pi)) = \coprod_{p,q} \Omega_{\text{loc}}^{p,q}(M\times\Gamma^\infty(\pi))\,.$$

The map j^*_∞ induces an isomorphism of bicomplexes between local forms in $\Omega(J^\infty(\pi))$ as defined above, and $\Omega_{\text{loc}}(M\times\Gamma^\infty(\pi))$, i. e. $\Omega^q_p(\pi)\simeq\Omega^{p,q}_{\text{loc}}(M\times\Gamma^\infty(\pi))$ [20].

We call a form A on $M\times\Gamma^\infty(\pi)$ local if A lies in $\Omega^{p,q}_{\text{loc}}(M\times\Gamma^\infty(\pi)$. Thus if $A\in\Omega^{p,q}_{\text{loc}}(M\times\Gamma^\infty(\pi))$, then for $s\in\Gamma^\infty(\pi)$ and $X_1,\dots,X_q\in\text{Vec}(J^\infty(\pi))$ the p-form $A(s,X_1,\dots,X_q)$ on M depends on s, $X_1(s),\dots,X_q(s)$ in a *local fashion*, that means $A(s,X_1,\dots,X_q)(x)$ depends only on finite jets (i. e. finitely many derivatives) of $s,X_1(s),\dots,X_q(s)$ at x. In local coordinates of M, a local form A can be written as follows:

$$A=\sum_{i,j}A_{i_1\dots i_p,j_1\dots j_q}\,\mathrm{d}x_{i_1}\wedge\cdots\wedge\mathrm{d}x_{i_p}\wedge\partial u_{j_1}\wedge\cdots\wedge\partial u_{j_q}$$

where the coordinates $A_{i_1,\dots,i_p,j_1,\dots,j_q}$ are local (0, 0)-forms, the $\mathrm{d}x_i$'s are local (1, 0)-forms and the ∂u_j's are local (0, 1)-forms. This justifies the terminology of local forms.

We see from our discussion that a Lagrangian L on π is an element of $\Omega^{n,0}_{\text{loc}}(M\times\Gamma^\infty(\pi))$, $n=\dim M$. Indeed, any $L\in\Omega^{n,0}_{\text{loc}}(M\times\Gamma^\infty(\pi))$ defines an n-form $L(s)$ on M by $L(s)(x)=L(x,s)$ which is local in the physical sense above. Interpreting this n-form $L(s)$ as Lagrangian density (we fix a volume on M) the action $\mathcal{L}(s)$ in a domain $U\subset M$ is defined by

$$\mathcal{L}(s)=\int_U L(s)\,.$$

4. BRST Transformations

The BRST bicomplex described in [3, 11, 12] is related to the variational bicomplex as follows: Let $\pi\colon P\to M$ be a principal G-bundle, let $\mathfrak{g}$ be the Lie algebra of G and $\pi^p\colon\Omega^p(P,\mathfrak{g})\to M$ the bundle of Lie algebra valued p-forms. Let $\mathcal{G}$ denote the Lie group of gauge transformations and $\mathfrak{g}$ its Lie algebra. Set $\mathbf{C}^{q,p}_{\text{loc}}:=\mathbf{C}^q_{\text{loc}}(\mathfrak{g},\Omega^p(P,\mathfrak{g}))$ the space of local q-cochains with values in $\mathfrak{g}$-valued p-forms. $\phi\in\mathbf{C}^{q,p}_{\text{loc}}$ is local in the sense of differential operators $\phi\colon\bigotimes_q\mathfrak{g}\to\Omega^p(P,\mathfrak{g})$, i. e. decreasing the supports. Define $\delta\colon\mathbf{C}^{q,p}_{\text{loc}}\to\mathbf{C}^{q+1,p}_{\text{loc}}$ to be the Chevalley–Eilenberg coboundary operator with respect to a representation ρ of $\mathfrak{g}$:

$$\begin{aligned}(\delta\phi)(\xi_0,\dots,\xi_q)=&\sum_{i=0}^{q}(-1)^i\rho'(\xi_i)\phi(\xi_0,\dots,\hat{\xi}_i,\dots,\xi_q)\\&+\sum_{i<j}(-i)^{i+j}\phi([\xi_i,\xi_j],\dots,\hat{\xi}_i,\dots,\hat{\xi}_j,\dots,\xi_q)\end{aligned}\tag{4}$$

where ρ' is the induced derived representation of $\mathfrak{g}$ on $\Omega^p(P,\mathfrak{g})$. We have $\delta_{\rm loc}^2 = 0$. Then we define the **BRST operator** $\mathrm{s}\colon \mathbf{C}_{\rm loc}^{q,p} \to \mathbf{C}_{\rm loc}^{q+1,p}$ as

$$\mathrm{s} \equiv \frac{(-1)^{p+1}}{q+1}\delta_{\rm loc}\,. \tag{5}$$

It is clear that s is nilpotent, $\mathrm{s}^2 = 0$. We call $\{\mathbf{C}_{\rm loc}^{q,p}, \mathrm{s}\}$ the BRST bicomplex. In [11, 12] we derived the classical BRST transformations using the Chevalley–Eilenberg differential for $\rho = \mathrm{ad}_x$, $x \in \mathfrak{g}$, the **adjoint representation** of $\mathfrak{g}$:

Theorem 2. (Schmid [11]) *For a vector potential $A \in \mathbf{C}_{loc}^{0,1}$ and the ghost field $\eta \in \mathbf{C}_{loc}^{1,0}$ (being the Maurer–Cartan form on $\mathcal{G}$), the classical BRST transformations are:*

$$\mathrm{s}A = \mathrm{d}\eta + [A,\eta]\,, \qquad \mathrm{s}\eta = -\frac{1}{2}[\eta,\eta]\,, \qquad \mathrm{s}\bar{\eta} = b \quad \mathrm{s}b = 0\,. \tag{6}$$

In this case, the derived representation ρ' of the Lie algebra $\mathfrak{g}$ on $\Omega^0(P,\mathfrak{g}) \simeq \mathfrak{g}$ is identical with the adjoint representation of $\mathfrak{g}$, $\rho' = \mathrm{ad}_\xi$, $\xi \in \mathfrak{g}$.

5. Anomalies

Next we describe the cohomology which accommodates the Adler–Bardeen anomalies as elements of its first cohomology group [11, 12]. We combine the BRST bicomplex with local forms.

Consider $\Omega_{\rm loc}(M \times \Gamma^\infty(\pi))$ with $\Gamma^\infty(\pi) = \Omega^p(P,\mathfrak{g})$ i. e. $\pi = \pi^p$. Let C be a smooth p-chain on M and $\omega \in \Omega_{\rm loc}^{p,0}$. We consider functionals $\mathcal{L}$ on $\Omega^*(P,\mathfrak{g})$ given by

$$\mathcal{L}(A) = \int_C \omega(A)\,, \qquad A \in \Omega^p(P,\mathfrak{g})$$

and denote the space of all such functionals by $\Gamma_{\rm loc}^p$,

$$\Gamma_{\rm loc}^p = \{\mathcal{L}\colon \Omega^p(P,\mathfrak{g}) \to \mathbb{R};\, \mathcal{L}(A) = \int_C \omega(A)\}\,, \quad \omega \in \Omega_{\rm loc}^{p,0}\,.$$

We define the representation $\rho_{\rm loc}$ of the gauge group $\mathcal{G}$ on the space $\Gamma_{\rm loc}^p$ by

$$(\rho_{\rm loc}(\phi)\mathcal{L})(A) = \mathcal{L}(\rho(\phi^{-1})A)\,, \qquad \phi \in \mathcal{G}\,, \qquad A \in \Omega^p(P,\mathfrak{g})\,. \tag{7}$$

For short $\phi \cdot \mathcal{L}(A) = \mathcal{L}(\phi \cdot A) = \int_C \omega(\phi \cdot A)$. Then the derived representation $\rho'_{\rm loc}$ of the gauge algebra $\mathfrak{g}$ on $\Gamma_{\rm loc}^p$ is given by

$$(\rho'_{\rm loc}(\xi)\mathcal{L})(A) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathcal{L}(\rho_{\rm loc}(\mathrm{e}^{-t\xi})A) = \mathcal{L}(\rho(Z_\xi)A) \tag{8}$$

where $\xi \in \mathfrak{g}$, $A \in \Omega^p(P, \mathfrak{g})$, and Z_ξ denotes the fundamental vector field generated by ξ.

Now we consider the Chevalley–Eilenberg complex of $\mathfrak{g}$ with respect to the representation ρ'_{loc} on Γ^p_{loc}. That means that the coboundary operator $\delta_{\text{loc}}\colon C^q(\mathfrak{g}, \Gamma^p_{\text{loc}}) \to C^{q+1}(\mathfrak{g}, \Gamma^p_{\text{loc}})$ is given by

$$\begin{aligned}(\delta_{\text{loc}}\omega)(\xi_0, \dots, \xi_q) = \sum_{i=0}^{q}(-1)^i \rho'_{\text{loc}}(\xi_i)\omega(\xi_0, \dots, \hat{\xi}_i, \dots, \xi_q) \\ + \sum_{i<j}(-i)^{i+j}\omega([\xi_i, \xi_j], \dots, \hat{\xi}_i, \dots, \hat{\xi}_j, \dots, \xi_q)\,.\end{aligned} \tag{9}$$

Again $\delta^2_{\text{loc}} = 0$ and we have the double complex $\mathcal{C}^{q,p} = C^q(\mathfrak{g}, \Gamma^p_{\text{loc}})$ with $\delta_{\text{loc}}\colon \mathcal{C}^{q,p} \to \mathcal{C}^{q+1,p}$, and $\mathrm{d}\colon \mathcal{C}^{q,p} \to \mathcal{C}^{q,p+1}$ induced by the exterior derivative d_M on M as follows: define $\mathrm{d}\colon \Gamma^p_{\text{loc}} \to \Gamma^{p+1}_{\text{loc}}$ by

$$(\mathrm{d}\mathcal{L})(A) = \int_D (\mathrm{d}_M\omega)(A) \tag{10}$$

where $\mathcal{L}(A) = \int_C \omega(A)$, $\omega \in \Omega^{p,0}_{\text{loc}}$, $\mathrm{d}_M\omega \in \Omega^{p+1,0}_{\text{loc}}$ and D is a $p+1$ cocycle such that C is the boundary of D. The two operators δ_{loc} and d anticommute:

Proposition 4. $\delta_{loc}\,\mathrm{d} + \mathrm{d}\delta_{loc} = 0$.

We define the **total differential** $\Delta = \delta_{\text{loc}} + (-1)^p\,\mathrm{d}$. We have $\delta^2_{\text{loc}} = \mathrm{d}^2 = \delta_{\text{loc}}\,\mathrm{d} + \mathrm{d}\delta_{\text{loc}} = 0$, which implies $\Delta^2 = 0$. We denote the total cohomology with respect to Δ by $\mathbf{H}^*_{\text{loc}}(\mathfrak{g})$.

A homotopy formula ("Russian formula") on this bicomplex is derived in [11, 12] and with the introduction of Chern–Simons type forms $\omega^{i-1}_{2q-i} = a_i p(A, [A, A]^{i-1}, F^{q-1}_A)$ we obtain the associated descent equations $\delta_{\text{loc}}\omega^0_{2q-1} = -\,\mathrm{d}\omega^1_{2q-2}$, $\delta_{\text{loc}}\omega^1_{2q-2} = -\,\mathrm{d}\omega^2_{2q-3}$, $\dots$, $\delta_{\text{loc}}\omega^{2q-1}_0 = 0$. We identify the non-Abelian anomaly as a cohomology class in $H^1_{\text{loc}}(\mathfrak{g})$ represented by ω^1_{2q-2} in $n = 2q-2$ dimensions.

For example, for $q = 2, q = 3$ we get the 2- and 4-dimensional non-Abelian anomaly respectively, represented by $\omega^1_2 = \mathrm{Tr}(\eta\delta_{\text{loc}}\tilde{A})$ and $\omega^1_4 = \mathrm{Tr}(\eta\delta_{\text{loc}}(\tilde{A}\delta_{\text{loc}}\tilde{A} + \frac{2}{3}\tilde{A}^3))$ resp., where $\tilde{A} = A + \eta$. For $M = S^3$ the Chern–Simon form $\omega^0_5 = Tp(A)$, where p is an invariant polynomial and T the transgression operator, moreover $\mathrm{d}Tp(A) = p(F) = \mathrm{Tr}\,F^3$. We get the stair case equations [14]:

$$
\begin{array}{c|cccc}
q=3 & \mathrm{Tr}\,F^3 & & & \\
 & \uparrow \mathrm{d} & & & \\
q=2 & \omega_5^0 \quad \xrightarrow{\delta_{\mathrm{loc}}} & \bullet & & \\
 & & \uparrow \mathrm{d} & & \\
q=1 & & \omega_4^1 \quad \xrightarrow{\delta_{\mathrm{loc}}} & \bullet & \\
 & & & \uparrow \mathrm{d} & \\
q=0 & & & \omega_3^2 \quad \xrightarrow{\delta_{\mathrm{loc}}} & 0 \\
\hline
 & p=0 & p=1 & p=2 & p=3
\end{array}
$$

ω_3^2 represents the anomaly.

6. The Consistency Condition

We consider the Wess–Zumino consistency condition as a problem in *local cohomology* [21]. In our bicomplex $\mathcal{C}^*_{\mathrm{loc}} = \{\mathcal{C}^{q,p}_{\mathrm{loc}}, \Delta\}_{q,p\in\mathbf{N}}$ we have the differentials $\Delta = \delta_{\mathrm{loc}} + (-1)^p d$, where $\delta_{\mathrm{loc}}\colon \mathcal{C}^{q,p}_{\mathrm{loc}} \to \mathcal{C}^{q+1,p}_{\mathrm{loc}}$ and $\mathrm{d}\colon \mathcal{C}^{q,p}_{\mathrm{loc}} \to \mathcal{C}^{q,p+1}_{\mathrm{loc}}$ satisfying $\Delta^2 = \delta_{\mathrm{loc}}\,\mathrm{d} + \mathrm{d}\delta_{\mathrm{loc}} = \delta^2_{\mathrm{loc}} = \mathrm{d}^2 = 0$.

The Wess–Zumino *consistency condition* on $\omega \in \mathcal{C}^*_{\mathrm{loc}}$ means there exists a $\alpha \in \mathcal{C}^*_{\mathrm{loc}}$ such that

$$\delta_{\mathrm{loc}}\omega + \mathrm{d}\alpha = 0\,. \tag{11}$$

Any solution of (11) of the form $\omega = \delta_{\mathrm{loc}}\beta + \mathrm{d}\gamma$, β, $\gamma \in \mathcal{C}^*_{\mathrm{loc}}$ is considered to be trivial, since then $\delta_{\mathrm{loc}}\omega = \delta^2_{\mathrm{loc}}\beta + \delta_{\mathrm{loc}}\,\mathrm{d}\gamma$ so $\delta_{\mathrm{loc}}\omega - \mathrm{d}(\delta_{\mathrm{loc}}\gamma) = 0$. We restrict ourselves to the subalgebra $\Omega^p_{inv}(P, \mathfrak{g})$ of G-invariant $\mathfrak{g}$-forms. The consistency condition (11) produces the so called descent equations. If $\delta_{\mathrm{loc}}\omega + \mathrm{d}\alpha = 0$ then taking δ_{loc} of (11) we get $\delta^2_{\mathrm{loc}}\omega + \delta_{\mathrm{loc}}\,\mathrm{d}\alpha = 0$ hence $\delta_{\mathrm{loc}}\,\mathrm{d}\alpha = 0 = -\,\mathrm{d}\delta_{\mathrm{loc}}\alpha$. So by the Poincare lemma there exists a local form β such that $\delta_{\mathrm{loc}}\alpha = -\,\mathrm{d}\beta$, or $\delta_{\mathrm{loc}}\alpha + \mathrm{d}\beta = 0$. By definition $\delta_{\mathrm{loc}}[\omega] = [\alpha]$. If ω is trivial, i. e. $\omega = \delta_{\mathrm{loc}}\beta + \mathrm{d}\gamma$ then $\delta_{\mathrm{loc}}\,\mathrm{d}\gamma = -\,\mathrm{d}\alpha$, and $\mathrm{d}\alpha = -\,\mathrm{d}\delta_{\mathrm{loc}}\gamma$, hence α is of the form $\alpha = \delta_{\mathrm{loc}}\gamma + \mathrm{d}\lambda$, that is $[\alpha] = 0$. We get the descent equations

$$
\begin{aligned}
\delta_{\mathrm{loc}}\omega + \mathrm{d}\omega_1 &= 0\\
\delta_{\mathrm{loc}}\omega_1 + \mathrm{d}\omega_2 &= 0\\
&\vdots\\
\delta_{\mathrm{loc}}\omega_{k-1} + \mathrm{d}\omega_k &= 0
\end{aligned}
$$

where k is the smallest integer such that $[\omega] \in \mathbf{H}^k_{\text{loc}}(\mathfrak{g})$ with $\delta_{\text{loc}}\omega = 0$.

7. A Solution to the Consistency Condition

We now combine the ideas of the previous sections. The costruction of $\mathfrak{g}$-symplectic orbits and the momentum map is generalized from the finite dimensional case to the infinite dimesnional situation as follows (we use the same notation with script symbols): we consider the principal $\mathcal{G}$ bundle $(\mathcal{P}, \pi, \mathcal{M})$, where $\mathcal{P} = \Omega^*(P, \mathfrak{g})$ with the $\mathcal{G}$ action ρ_{loc} defined by (7), and $\mathcal{M} = \mathcal{P}/\mathcal{G}$ is the orbit space (under the usual assumptions on the action and topologies). Then for $A \in \Omega^*(P, \mathfrak{g})$ the canonical one-form Θ_A on the orbit $\mathcal{O}_A$ induced from the Maurer–Cartan form on $\mathcal{G}$ becomes a map

$$\Theta_A \colon \mathcal{O}_A \to \Omega^1(P, \mathfrak{g}) \simeq \mathcal{C}^{0,1}_{\text{loc}}$$

and the momentum map

$$J \colon \mathcal{O}_A \to \mathcal{L}(\mathfrak{g}, \mathfrak{g}) = \mathcal{C}^{1,0}_{\text{loc}}\,.$$

Theorem 3. *The momentum map J satisfies the consistency condition for the canonical one-form (Maurer–Cartan) Θ*

$$\delta_{loc}\Theta_A + \mathrm{d}J = 0\,. \tag{12}$$

Proof: We have $\delta_{\text{loc}}\Theta_A \in \mathcal{C}^{1,1}_{\text{loc}}$ and $\mathrm{d}J(A) \in \mathcal{C}^{1,1}_{\text{loc}}$ and from (6) we get for any $\xi \in \mathfrak{g}$

$$\delta_{\text{loc}}\Theta_A(\xi) = \mathrm{d}J(A)(\xi) + L_{Z_{J(\xi)}}\Theta_A$$

where $Z_{J(\xi)}$ is the induced fundamental vector field. From (2) we conclude that $L_{Z_{J(\xi)}}\Theta_A = 0$. □

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