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MASS AND CURVATURE

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Abstract. In this work we present a brief summary of the "geometry of mass". We show how the notion of mass of elementary particles is related to the geometrical concept of curvature. In particular, the bosonic mass matrices are related to the extrinsic curvature of specific sub-manifolds of the Higgs bundle and the underlying gauge bundle. In contrast, the mass matrix of the fermions is related to the intrinsic curvature of bundles that geometrically represent "free fermions" within the context of spontaneously broken Yang–Mills gauge theories.

1. Introduction

When seen from a geometrical viewpoint the electric charge of an electron can be considered as the coupling constant of a $U(1)$ Yang–Mills gauge theory. That means that the electrons charge parameterizes the most general **Killing form** on $\mathfrak{u}(1) \equiv \mathrm{Lie}(U(1))$. One of the still outstanding and deep secrets of nature is why all of the electric charges of free particles is given by an integer multiple of this coupling constant. In general, the electric charge of a particle may be defined as

$$\text{charge} = \int_{\mathcal{S}} *F_{\text{elm}} \tag{1}$$

where $F_{\text{elm}} \in \Omega^2(\mathcal{M})$ denotes the electromagnetic field generated by the charge and $\mathcal{S} \subset \mathcal{M}$ any closed two dimensional space like surface of an orientable spacetime $(\mathcal{M}, g_{\mathcal{M}})$.

In other words, electric charge is tied to the curvature of the total space of the underlying principal $U(1)$-bundle. However, by formula (1) electric charge is also tied to the metric on spacetime ($*$ denotes the Hodge map that is defined with respect to $g_{\mathcal{M}}$ and a choice of orientation of $\mathcal{M}$). No similar relation seems to hold true in the case of the "mass" of an elementary particle. Even more unsatisfying, there is no unique definition of mass at all in physics (see, e. g. the book by Jammer [3]). Nonetheless, in particle physics the mass of an elementary particle usually refers to the notion of "rest mass". Geometrically, this means that mass is defined by the four momentum $p \in \mathcal{C}^+ \subset T^*\mathcal{M}$ of the particle in question, i. e. by the well-known formula[(1)]

$$\mathrm{m}^2 := g_{\mathcal{M}}(p,p)\,. \tag{2}$$

This again, relates the notion of mass to that of the metric of spacetime. However, for this to make sense the particle must be considered as "free", i. e. as a closed system.[(2)] In relativistical mechanics (2) is being considered as evaluated along the particles world line γ. If the particle is free, then the corresponding value of the particle's four momentum squared is constant along γ. However, elementary particles are not considered as point like constituents of spacetime. Rather they are mathematically described by fields on spacetime. These fields obey specific equations which are in a semi-classical approximation of a full quantum theory determined by a differential operator of first or second order, depending on the spin of the corresponding particle. Then, the four momentum of the particle geometrically corresponds to the symbol of the differential operator in question. In particular, the four momentum of a free particle corresponds to the symbol of a differential operator that is determined by the metric only.

In 1928 P.A.M. Dirac introduced his famous equation

$$(\mathrm{i}\partial\!\!\!/ - \mathrm{m})\psi = 0 \tag{3}$$

that describes the dynamics of a free fermion of spin 1/2 (at that time identified with an electron). Since the non negative real parameter "m" in (3) is physically interpreted as mass the symbol of the Dirac operator $\partial\!\!\!/$ corresponds to the (positive) square root of the four momentum of the fermion, i. e.

$$\mathrm{Sym}(\mathrm{i}\partial\!\!\!/)(p) = \sqrt{g_{\mathcal{M}}(p,p)}\,. \tag{4}$$

Therefore, the Dirac operator has a clear interpretation both in geometry and in physics. However, besides of being a parameter in Dirac's equation what is the geometrical meaning of mass? Moreover, since the notion of mass is related to

(1) In what follows we consider $g_{\mathcal{M}}$ either as a metric on the tangent bundle $\xi_{T\mathcal{M}}$, or as a metric on the cotangent bundle $\xi_{T^*\mathcal{M}}$ of $\mathcal{M}$; $\mathcal{C}^+_x \subset T^*_x\mathcal{M}$ is the future oriented part of the light cone at $x \in \mathcal{M}$.

(2) Since quarks do not occur as free particles in nature the definition of mass in this case is different from (2) and will not be taken into account here.

the notion of "freeness" of a particle one also has to geometrically understand the meaning of a free particle within the context of gauge theories. This is far from being obvious indeed. In a gauge covariant description of elementary particle dynamics the symbol of the corresponding differential operator cannot only be determined by the metric. Hence, the notion of mass becomes gauge dependent. Of course, this is physically unacceptable. At a first glance, the concept of mass of an elementary particle seems to be in conflict with gauge symmetry and thus also with the notion of charge. In what follows we will briefly summarize some ideas that show that in fact charge and mass can live in "peaceful harmony" when neither of these classical concepts is being considered as fundamental, actually.

2. Orbit Bundles and Vacuum Pairs

Today, the notion of mass is not considered to be a fundamental one like in mechanics. Rather all particles are assumed to be massless in some sense. What we call "mass" is considered to be a manifestation of specific interactions of the (massless) particles with the so-called Higgs boson. Within the frame of the **Standard Model** of particle physics one has to distinguish three different kinds of gauge invariant interactions that give rise to the notion of mass. One of which is given by the gauge coupling. It yields the "masses of (some of) the gauge bosons". Another type of gauge invariant interaction is given by what is called the **Yukawa coupling** between the fermions and the Higgs boson and which give rise to the "masses of the fermions". The third kind of interaction to be considered is the interaction of the Higgs boson by itself and which leads to the "mass of the (physical) Higgs boson". Note that the latter two kinds of interaction are usually regarded as non-geometrical in contrast to the gauge coupling. Nonetheless, the self interaction of the Higgs boson is known to be of fundamental significance since it yields a reduction of the gauge symmetry which underlies all of the various couplings.

In what follows we want to present some ideas how such a reduction can be used to geometrically describe the notion of mass in terms of curvature.

3. Bosonic Mass and Extrinsic Curvature

In this section we briefly summarize the general notion of vacuum pairs without going into the technical details that can be found, e. g. in [6]. We will use the notion of vacuum pairs in order to describe a relation between the bosonic mass matrices and the extrinsic curvature of specific sub-manifolds that are defined by a vacuum section.

An **Yang–Mills–Higgs gauge theory** is specified by the following data:

$$(\mathcal{P}(\mathcal{M},G),\rho,V) \tag{5}$$

where $\mathcal{P}(\mathcal{M},G)$ denotes a principal G-bundle $P \xrightarrow{\pi} \mathcal{M}$ over a connected, orientable (pseudo)Riemannian manifold $(\mathcal{M},g_{\mathcal{M}})$. $G \xrightarrow{\rho} \mathrm{Aut}(\mathbb{C}^N)$ is a Hermitian representation of the semi-simple Lie group G with Lie algebra $\mathfrak{g}$. The smooth function $\mathbb{C}^N \xrightarrow{V} \mathbb{R}$ is supposed to be G-invariant and positive definite transversally to the orbit of minimum. We call V a **general Higgs potential**. Associated to these data are the **Higgs bundle** ξ_{H}:

$$\begin{aligned}\pi_{\mathrm{H}} \colon E := P \times_\rho \mathbb{C}^N &\to \mathcal{M} \\ [(p,\mathbf{z})] &\mapsto \pi(p)\end{aligned} \tag{6}$$

and the **orbit bundle** $\xi_{\mathrm{orb}(\mathbf{z}_0)} = (\mathcal{O}(\mathbf{z}_0),\pi_{\mathrm{orb}},\mathcal{M})$ with respect to a minimum $\mathbf{z}_0 \in \mathbb{C}^N$ of the Higgs potential:

$$\pi_{\mathrm{orb}} \colon \mathcal{O}(\mathbf{z}_0) := P \times_{\rho_{\mathrm{orb}}} \mathrm{orb}(\mathbf{z}_0) \to \mathcal{M} \tag{7}$$

where $\mathrm{orb}(\mathbf{z}_0) \subset \mathbb{C}^N$ denotes the orbit of $\mathbf{z}_0$ and ρ_{orb} is the restriction of ρ to the orbit.

The data (5) also give rise to the well-known **Yang–Mills–Higgs functional**. For this let us denote by $\mathcal{A}(\xi_{\mathrm{H}})$ the affine set of associated connections on the Higgs bundle and by $\Gamma(\xi_{\mathrm{H}})$ the module of sections of the Higgs bundle. Notice that the Higgs bundle serves as the geometrical object that represents the Higgs boson. Correspondingly, a section of the Higgs bundle may be interpreted as a state of the Higgs boson. The Yang–Mills–Higgs action reads

$$\begin{aligned}\mathcal{I}_{\mathrm{YMH}} \colon \mathcal{A}(\xi_{\mathrm{H}}) \times \Gamma(\xi_{\mathrm{H}}) &\to \mathbb{R} \\ (A,\Psi) &\mapsto s\langle F_A,F_A\rangle + \langle \partial_A\Psi,\partial_A\Psi\rangle + s\langle \Psi^*V_H,1\rangle \\ &= \mathcal{I}_{\mathrm{YM}} + \mathcal{I}_{\mathrm{H}}\,.\end{aligned} \tag{8}$$

Here, $s = \pm$ depends on the signature of the (pseudo) metric $g_{\mathcal{M}}$. The pairing $\langle\cdot,\cdot\rangle$ on $\Omega(\mathcal{M},E)$ and on $\Omega(\mathcal{M},\mathrm{End}(E))$ is defined with respect to the (pseudo) metric together with the choice of an orientation on $\mathcal{M}$ and the Hermitian structure on ξ_{H}. The covariant derivative and the curvature of the connection A are denoted by ∂_A and F_A, respectively. The function Ψ^*V is defined by the mapping (also denoted by V)

$$\begin{aligned}V \colon \Gamma(\xi_{\mathrm{H}}) &\to \mathcal{C}^\infty(\mathcal{M}) \\ \Psi &\mapsto \Psi^*V\end{aligned} \tag{9}$$

where $\Psi^*V(x) := V(\psi(p))|_{p\in\pi^{-1}(x)}$. Here, $\psi \in \mathcal{C}^\infty_{\mathrm{eq}}(P,\mathbb{C}^N)$ is the equivariant mapping that corresponds to the section Ψ, i. e. $\Psi(x) = [(p,\psi(p))]|_{p\in\pi^{-1}(x)}$.

Since $\xi_{\mathrm{orb}(\mathbf{z}_0)} \subset \xi_{\mathrm{H}}$ any section $\mathcal{V}$ of the orbit bundle can also be considered as a section of the Higgs bundle. It can be shown that there is a one-to-one correspondence between the set of sections $\Gamma(\xi_{\mathrm{orb}(\mathbf{z}_0)})$ and the H-reductions of $\mathcal{P}(\mathcal{M}, G)$, with $H \subset G$ being isomorphic to the isotropy group of the minimum $\mathbf{z}_0$. We denote such a reduction by $(\mathcal{Q}, \iota)$, where $\mathcal{Q}(\mathcal{M}, H)$ is the principal H-bundle (with projection π_Q) that reduces $\mathcal{P}(\mathcal{M}, G)$ and $Q \xrightarrow{\iota} P$ is the corresponding embedding (of principal bundles). Note that $\mathcal{V}(x) = [(\iota(q), \mathbf{z}_0)]|_{q \in \pi_Q^{-1}(x)}$. A connection A on $\mathcal{P}(\mathcal{M}, G)$ is called reducible if $\iota^* A$ is a connection on $\mathcal{Q}(\mathcal{M}, H)$. We call a Yang–Mills–Higgs pair $(A, \Phi) \in \mathcal{A}(\xi_{\mathrm{H}}) \times \Gamma(\xi_{\mathrm{H}})$ a **vacuum pair** iff $A \equiv \Theta$ is associated to a flat reducible connection on $\mathcal{P}(\mathcal{M}, G)$ and $\Phi = \mathcal{V}$ is a section of the orbit bundle. The covariant derivative associated to the connection Θ is denoted by ∂.

Clearly, every vacuum pair $(\Theta, \mathcal{V})$ defines a minimum of the energy functional that corresponds to (8). In particular, any **vacuum section** $\mathcal{V} \in \Gamma(\xi_{\mathrm{orb}(\mathbf{z}_0)})$ minimizes the functional associated to the mapping (9). It therefore corresponds to a possible ground state of the Higgs boson. We call a reduction $(\mathcal{Q}, \iota)$ a **vacuum** with respect to the minimum $\mathbf{z}_0$. Notice that in the case where two minima $\mathbf{z}_0$ and $\mathbf{z}'_0$ are on the same orbit the corresponding reductions are equivalent. However, the orbit bundle with respect to some minimum of a general Higgs potential may have gauge inequivalent vacuum sections. Thus, the Higgs boson may give rise to gauge inequivalent vacua even if the Higgs potential in question has only one orbit of minima. We call the gauge group of a reduced principal bundle $\mathcal{Q}(\mathcal{M}, H)$ an **invariance group** of the vacuum. The gauge symmetry defined by $\mathcal{P}(\mathcal{M}, G)$ is called **spontaneously broken** by a vacuum $(\mathcal{Q}, \iota)$ if the invariance group of the vacuum is a proper subgroup of the original gauge group. The gauge symmetry is called **completely broken** by the vacuum if the invariance group of the vacuum is trivial. Of course, a necessary condition for a gauge symmetry to be completely broken is that $\mathcal{P}(\mathcal{M}, G)$ is trivial. We call a vacuum $(\mathcal{Q}, \iota)$ trivial iff $\mathcal{Q}(\mathcal{M}, H)$ is trivial. Notice that the H-reductions of a trivial principal G-bundle are nontrivial in general. Thus, the possible vacua of a spontaneously broken gauge theory might be nontrivial even if $\mathcal{P}(\mathcal{M}, G)$ is trivial.

Let $(\mathcal{Q}, \iota)$ be a vacuum that spontaneously breaks the gauge symmetry of a Yang–Mills–Higgs gauge theory. Let ξ_E be an arbitrary associated fiber bundle with respect to $\mathcal{P}(\mathcal{M}, G)$ with total space E, typical fiber F and $G \xrightarrow{\rho} \mathrm{Diff}(F)$. We call the fiber bundle $\xi_{E,\mathrm{red}}$

$$\begin{aligned} \pi_{\mathrm{H,\,red}} : E_{\mathrm{red}} := Q \times_{\rho_{\mathrm{red}}} F &\to \mathcal{M} \\ [(q, y)] &\mapsto \pi_Q(q) \end{aligned} \tag{10}$$

the **reduced fiber bundle** with respect to the vacuum $(\mathcal{Q}, \iota)$. Here, $\rho_{\mathrm{red}} := \rho|_H$.

Notice that the vacuum section $\mathcal{V}$ that corresponds to $(\mathcal{Q}, \iota)$ is (covariantly) constant when considered as a section of the reduced Higgs bundle. Moreover, regarded as a real (orthogonal) vector bundle the reduced Higgs bundle is $\mathbb{Z}_2$-graded (cf. [6])

$$\xi_{E,\text{red}} = \xi_{\text{G}} \oplus \xi_{\text{H, phys}} \,. \tag{11}$$

The real vector bundle ξ_{G} is called the **Goldstone bundle**. It has rank equal to $\dim G - \dim H$. The real vector bundle $\xi_{\text{H, phys}}$ is called the **physical Higgs bundle**. It has rank equal to $2N + \dim H - \dim G$ (if this is a nonnegative number). Both the Goldstone and the physical Higgs bundle refer to a vacuum. However, the rank is independent of the vacuum chosen. In particular, it can be shown that the rank of the Goldstone bundle equals the number of "massive gauge bosons". The rank of the physical Higgs boson bundle equals the number of "massive Higgs bosons" (cf. l.c.).

Definition 1. *Let $(\mathcal{Q}, \iota)$ be a vacuum with respect to a minimum $\mathbf{z}_0 \in \mathbb{R}^{2N}$ of a general Higgs potential V. The global mass matrix of the Higgs boson is the section $\mathcal{V}^*\mathrm{M}^2_{\text{H}} \in \Gamma(\xi_{\mathrm{End}(E_{\text{H}})})$ defined by the equivariant mapping*

$$\begin{aligned} \nu^*\mathrm{M}^2_{\text{H}} \colon P &\to \mathrm{End}(\mathbb{R}^{2N}) \\ p = \iota(q)g &\mapsto \rho(g^{-1})\mathrm{M}^2_{\text{H}}(\mathbf{z}_0)\rho(g) \,. \end{aligned} \tag{12}$$

Here, $\mathrm{M}^2_{\text{H}}(\mathbf{z}_0) \in \mathrm{End}(\mathbb{R}^{2N})$ is given by $\mathrm{M}^2_{\text{H}}(\mathbf{z}_0)\mathbf{z}\cdot\mathbf{z}' := \mathrm{Hess}\, V(\mathbf{z}_0)(\mathbf{z}, \mathbf{z}')$ for all $\mathbf{z}, \mathbf{z}' \in \mathbb{R}^{2N}$. The equivariant mapping $\nu \in \mathcal{C}^\infty_{eq}(P, orb(\mathbf{z}_0))$ corresponds to the vacuum section of $(\mathcal{Q}, \iota)$, i. e. $\nu(p) = \rho(g^{-1})\mathbf{z}_0$ for all $p = \iota(q)g \in P$.

With respect to a vacuum pair $(\Theta, \mathcal{V})$ the set of (principal) connections on $\mathcal{P}(\mathcal{M}, G)$ can be identified with the module of sections of the **Yang–Mills bundle** $\tau^*_{\mathcal{M}} \otimes \xi_{\text{YM}}$ where $\tau^*_{\mathcal{M}}$ is the cotangent bundle of $\mathcal{M}$ and ξ_{YM} is defined by

$$E_{\text{YM}} := Q \times_H \mathfrak{g} \to \mathcal{M} \,. \tag{13}$$

Notice that the Yang–Mills bundle is also $\mathbb{Z}_2$-graded since

$$\xi_{\text{YM}} = \xi_{\mathrm{ad}\,Q} \oplus \xi_{\text{G}}, \tag{14}$$

where $\xi_{\mathrm{ad}\,Q}$ is the "adjoint bundle" with respect to $\mathcal{Q}(\mathcal{M}, H)$.
The Yang–Mills bundle serves as the geometrical quantity that corresponds to a "real gauge boson". Accordingly, $\Gamma(\tau^*_{\mathcal{M}} \otimes \xi_{\text{YM}})$ represents the possible states of the real gauge boson.

Definition 2. *The global mass matrix of the real gauge boson is the section* $\mathcal{V}^*\mathbf{M}^2_{\mathrm{YM}} \in \Gamma(\xi_{\mathrm{End}(\mathrm{ad}\,P)})$ *defined by the equivariant mapping*

$$\begin{aligned} \nu^*\mathbf{M}^2_{\mathrm{YM}} \colon P &\to \mathrm{End}(\mathfrak{g}) \\ p = \iota(q)g &\mapsto \mathrm{ad}_{g^{-1}} \circ \mathbf{M}^2_{\mathrm{YM}}(\mathbf{z}_0) \circ \mathrm{ad}_g \,. \end{aligned} \tag{15}$$

Here, $\mathbf{M}^2_{\mathrm{YM}}(\mathbf{z}_0) \in \mathrm{End}(\mathfrak{g})$ *is defined by* $\beta(\mathbf{M}^2_{\mathrm{YM}}(\mathbf{z}_0)\eta, \eta') := 2\rho'(\eta)\mathbf{z}_0 \cdot \rho'(\eta')\mathbf{z}_0$ *for all* $\eta, \eta' \in \mathfrak{g}$. *The ad-invariant bilinear form* β *denotes the most general Killing form on* $\mathfrak{g}$ *parameterized by the "Yang–Mills coupling constants". By* $\rho' := \mathrm{d}\rho(e)$ *we mean the "derived" representation of* $\mathfrak{g}$.

It can be shown that the spectrum of the global mass matrix of the gauge and the Higgs boson is constant and independent of the vacuum. Moreover, both endomorphisms are in the commutant of the invariance group of the vacuum, cf. [6]. In the same reference it is also shown how the spectrum of the global mass matrices can be interpreted as the masses of the gauge and Higgs boson.

In order to describe the geometrical meaning of the (nontrivial part of the) global mass matrix of the Higgs boson we consider the reduced Higgs bundle as a real vector bundle over $\mathrm{Im}\,\mathcal{V} \subset \mathcal{O}(\mathbf{z}_0)$, i. e.

$$\pi^*_{\mathrm{orb}} E_G \oplus \pi^*_{\mathrm{orb}} E_{\mathrm{H,\,phys}} \to \mathrm{Im}(\mathcal{V}) \subset \mathcal{O}(\mathbf{z}_0)\,. \tag{16}$$

Notice that $\pi^*_{\mathrm{orb}} E_G = V\mathcal{O}(\mathbf{z}_0)|_{\mathrm{Im}(\mathcal{V})}$ and that the tangent bundle of $\mathcal{O}(\mathbf{z}_0)$ along $\mathrm{Im}(\mathcal{V})$ splits into

$$\mathrm{T}\mathcal{O}(\mathbf{z}_0)|_{\mathrm{Im}(\mathcal{V})} = \mathrm{Im}(\mathrm{d}\mathcal{V}) \oplus \pi^*_{\mathrm{orb}} E_G\,. \tag{17}$$

Consequently, $\pi^*_{\mathrm{orb}} \xi_{\mathrm{H,\,phys}}$ can be regarded as the normal bundle along the vacuum $\mathrm{Im}(\mathcal{V})$ which in turn is a sub-manifold of the total space of the reduced Higgs bundle. Note that the vacuum gives rise to a specific embedding of spacetime into the total space of the Higgs bundle.

A vacuum pair admits to define a (pseudo) metric on the total space of the reduced Higgs bundle and which is denoted by g_E. For this let $\xi \in E_{\mathrm{red}}$, then

$$\begin{aligned} g_{E,\xi} \colon \mathrm{T}_\xi E_{\mathrm{red}} \times \mathrm{T}_\xi E_{\mathrm{red}} &\to \mathbb{R} \\ (\mathbf{w}_1, \mathbf{w}_2) &\mapsto h_{E,\xi}(\mathrm{ver}(\mathbf{w}_1), \mathrm{ver}(\mathbf{w}_2)) + (\pi^*_{\mathrm{H,\,red}} g_{\mathcal{M}})_\xi(\mathbf{w}_1, \mathbf{w}_2)\,. \end{aligned} \tag{18}$$

Here, $\mathbf{w} = \mathrm{ver}(\mathbf{w}) + \mathrm{hor}(\mathbf{w})$ is the decomposition of $\mathbf{w} \in \mathrm{T}_\xi E_{\mathrm{red}}$ into its vertical and horizontal part with respect to the connection Θ. The vertical metric (real form of the Hermitian product) is denoted by h_E. Consequently, the tangent bundle of $E \equiv E_{\mathrm{red}}$ decomposes as

$$\tau_E = \tau_{\mathrm{orb}(\mathbf{z}_0)} \oplus \nu_{\mathrm{orb}(\mathbf{z}_0)}\,. \tag{19}$$

Here, $\nu_{\mathrm{orb}(\mathbf{z}_0)}$ denotes the **normal bundle** defined by the metric g_E on the total space of the reduced Higgs bundle and $\tau_{\mathrm{orb}(\mathbf{z}_0)}$ denotes the tangent bundle of the total space of the reduced orbit bundle. Again, when restricted to $\mathrm{Im}(\mathcal{V})$ the normal bundle can be identified with $\pi^*_{\mathrm{orb}}\xi_{\mathrm{H,\,phys}}$.
The crucial point is that the reduced Higgs bundle decomposes into the Whitney sum of eigenbundles of the global mass matrix of the Higgs boson. In particular, the physical Higgs bundle decomposes into eigenbundles that correspond to nonzero eigenvalues of $\mathcal{V}^*\mathrm{M}^2_{\mathrm{H}}$

$$\xi_{\mathrm{H,\,phys}} = \bigoplus_{\mathrm{m}^2\in\mathrm{spec}(\mathcal{V}^*\mathrm{M}^2_{\mathrm{H}})\setminus\{0\}} \xi_{\mathrm{H,m}^2}\,. \tag{20}$$

Consequently, along the vacuum $\mathrm{Im}(\mathcal{V})\subset E_{\mathrm{red}}$ there exists a distinguished set of normal sections. Notice that the eigenbundles of $\mathcal{V}^*\mathrm{M}^2_{\mathrm{H}}$ are line bundles iff all of the (mass) eigenvalues are different. Also notice that the normal sections are nowhere vanishing iff the appropriate eigenbundles are trivial. We also mention that the restriction to $\mathrm{Im}(\mathcal{V})$ of any reducible connection A on $\mathcal{P}(\mathcal{M},G)$ coincides with the horizontal distribution that is defined by the vacuum section $\mathcal{V}$ (cf. [6]). Of course, this holds true especially in the case of $\pi^*_{\mathrm{orb}}\Theta$. Thus, when restricted to $\mathrm{Im}(\mathcal{V})$ the (pseudo) metric g_E reads

$$g_{E|_{\mathrm{Im}(\mathcal{V})}} = h_E \oplus \pi^*_{\mathrm{orb}} g_{\mathcal{M}} \tag{21}$$

i. e. $g_{E|_{[(q,\mathbf{z}_0)]}}([(\mathbf{u}_1,\mathbf{z}_1)],[(\mathbf{u}_2,\mathbf{z}_2)]) = \mathbf{z}_1\cdot\mathbf{z}_2 + g_{\mathcal{M},\pi_Q(q)}(\mathrm{d}\pi_Q(q)\mathbf{u}_1,\mathrm{d}\pi_Q(q)\mathbf{u}_2)$ for all $\mathbf{u}_1,\mathbf{u}_2\in\mathrm{T}_qQ$ and $\mathbf{z}_1,\mathbf{z}_2\in\mathrm{T}_{\mathbf{z}_0}\mathbb{R}^{2N}$.
Let, respectively, "nor" and "tang" be the normal and the tangential projection with respect to g_E and let ∇^E be the covariant derivative with respect to the connection on $E_{\mathrm{H,\,red}}$ that is defined by g_E. Also, let $\mathcal{Y}\in\Gamma(\nu_{\mathrm{orb}(\mathbf{z}_0)})$ be a normal vector field and $\mathcal{X},\mathcal{X}'\in\Gamma(\tau_{\mathrm{orb}(\mathbf{z}_0)})$ be tangential vector fields. Then, the metric connection on τ_E splits according to (see, e. g. [4])

$$\nabla^E_{\mathcal{X}}\mathcal{X}' = \nabla^{\mathrm{orb}}_{\mathcal{X}}\mathcal{X}' + \sigma_H(\mathcal{X},\mathcal{X}') \tag{22}$$

$$\nabla^E_{\mathcal{X}}\mathcal{Y} = \nabla^{\mathrm{nor}}_{\mathcal{X}}\mathcal{Y} - \alpha_H(\mathcal{X},\mathcal{Y})\,. \tag{23}$$

Here, ∇^{orb} is the covariant derivative that is defined by $\mathrm{tang}(\nabla^E_{\mathcal{X}}\mathcal{X}')$. It coincides with the metric connection that is defined by j^*g_E, where $j\colon \mathrm{orb}(\mathbf{z}_0)\hookrightarrow E$ is the inclusion mapping. Correspondingly, the covariant derivative ∇^{nor} is given by $\mathrm{nor}(\nabla^E_{\mathcal{X}}\mathcal{Y})$.
The two bilinear mappings σ_H and α_H are defined by

$$\begin{aligned}\sigma_H : \Gamma(\tau_{\mathrm{orb}(\mathbf{z}_0)})\times\Gamma(\tau_{\mathrm{orb}(\mathbf{z}_0)}) &\to \Gamma(\nu_{\mathrm{orb}(\mathbf{z}_0)})\\ (\mathcal{X},\mathcal{X}') &\mapsto \mathrm{nor}(\nabla^E_{\mathcal{X}}\mathcal{X}')\end{aligned} \tag{24}$$

$$
\begin{aligned}
\alpha_H : \Gamma(\tau_{\mathrm{orb}(\mathbf{z}_0)}) \times \Gamma(\nu_{\mathrm{orb}(\mathbf{z}_0)}) &\to \Gamma(\tau_{\mathrm{orb}(\mathbf{z}_0)}) \\
(\mathcal{X}, \mathcal{Y}) &\mapsto -\operatorname{tang}(\nabla^E_{\mathcal{X}} \mathcal{Y}) \,.
\end{aligned} \tag{25}
$$

They are related to each other by the relation

$$
g_E(\alpha_H(\mathcal{X}, \hat{\mathcal{Y}}_a), \mathcal{X}') = g_E(\sigma_H(\mathcal{X}, \mathcal{X}'), \hat{\mathcal{Y}}_a), \quad a = 1, \ldots, L \tag{26}
$$

where $(\hat{\mathcal{Y}}_1, \ldots, \hat{\mathcal{Y}}_L) \subset \Gamma(\nu_{\mathrm{orb}(\mathbf{z}_0)})$ is a set of (locally defined) orthonormal sections of the normal bundle of $\mathcal{O}(\mathbf{z}_0) \subset E_{\mathrm{H,\,red}}$.

For each $\hat{\mathcal{Y}}_a$ $(a = 1, \ldots, L)$ the mapping

$$
\begin{aligned}
\alpha^a_H : \Gamma(\tau_{\mathrm{orb}(\mathbf{z}_0)}) &\to \Gamma(\tau_{\mathrm{orb}(\mathbf{z}_0)}) \\
\mathcal{X} &\mapsto \alpha_H(\mathcal{X}, \hat{\mathcal{Y}}_a)
\end{aligned} \tag{27}
$$

is called a **second fundamental form** of the sub-manifold $\mathcal{O}(\mathbf{z}_0)$. It generalizes the usual second fundamental form in the case of a hyper-surface. Of course, in general (27) strongly depends on the choice of normal sections $(\hat{\mathcal{Y}}_1, \ldots, \hat{\mathcal{Y}}_L)$. However, along the vacuum $\mathrm{Im}(\mathcal{V}) \subset E_{\mathrm{H,\,red}}$ the global mass matrix of the Higgs boson gives rise to a distinguished set of normal sections because of the splitting (20) and the identification $\nu_{\mathrm{orb}(\mathbf{z}_0)}|_{\mathrm{Im}(\mathcal{V})} = \pi^*_{\mathrm{orb}} \xi_{\mathrm{H,\,phys}}$.

We call the set of bilinear forms $\Xi_H \equiv (\Xi^a_H)$ $(a = 1, \ldots, L = \mathrm{rank}(\xi_{\mathrm{H,\,phys}}))$ a *second fundamental form of the vacuum*, whereby $(\mathcal{V}^* M^2_{\mathrm{H}})\hat{\mathcal{Y}}_a = \mathrm{m}^2_a \hat{\mathcal{Y}}_a$ and

$$
\begin{aligned}
\Xi^a_H : \Gamma(\tau_{\mathrm{orb}(\mathbf{z}_0)|_{\mathrm{Im}(\mathcal{V})}}) \times \Gamma(\tau_{\mathrm{orb}(\mathbf{z}_0)|_{\mathrm{Im}(\mathcal{V})}}) &\to \mathcal{C}^\infty(\mathrm{Im}(\mathcal{V})) \\
(\mathcal{X}, \mathcal{X}') &\mapsto g_E(\alpha^a_H(\mathcal{X}), \mathcal{X}') \,.
\end{aligned} \tag{28}
$$

Notice that $\sigma_H(\mathcal{X}, \mathcal{X}') = \sum_{a=1}^L \Xi^a_H(\mathcal{X}, \mathcal{X}')\hat{\mathcal{Y}}_a$.

Clearly, the second fundamental forms (28) determine the extrinsic geometry of the ground state of the Higgs boson in terms of the physical Higgs bosons. In this sense the global mass matrix of the Higgs boson determines the extrinsic geometry of the vacuum.

In quite an analogous manner the global mass matrix $\mathcal{V}^* \mathrm{M}^2_{\mathrm{YM}}$ of the Yang–Mills gauge boson determines a set of second fundamental forms $\Xi_{\mathrm{YM}} \equiv (\Xi^a_{\mathrm{YM}})$ of the sub-manifold $\iota(Q) \subset P$, where $a = 1, \ldots, L = \mathrm{rank}(\xi_{\mathrm{G}})$, cf. [6]. The global mass matrix of the gauge boson thus determines the extrinsic geometry of the vacuum in terms of the Goldstone bosons like the global mass matrix of the Higgs boson determines the extrinsic geometry of the vacuum in terms of the physical Higgs bosons. This might be considered as a geometrical variant of the well-known **Higgs–Kibble mechanism** of a spontaneously broken gauge symmetry, cf. [2]. Notice that $\mathrm{rank}(\xi_{\mathrm{G}}) = \dim(\mathrm{Ker}(\mathcal{V}^* \mathrm{M}^2_{\mathrm{H}})) = \dim(\mathrm{Im}(\mathcal{V}^* \mathrm{M}^2_{\mathrm{YM}}))$.

4. Fermionic Mass and Intrinsic Curvature

In the preceding section we have discussed how the global bosonic mass matrices are related to the extrinsic geometry of the vacuum. In this section we briefly discuss how the global mass matrix of the fermions is related to the intrinsic curvature of the "reduced fermion bundle". For the technical details we refer to [7].(1)

Let $(\mathcal{M}, g_{\mathcal{M}})$ be a connected (pseudo)Riemannian spin-manifold of dimension $2n$. We denote by ξ_S the spinor bundle with respect to a chosen spin structure. Also, let ζ_F be an associated Hermitian vector bundle with respect to the geometrical data $(\mathcal{P}(\mathcal{M}, G), \rho_H, V_H)$ of a Yang–Mills–Higgs gauge theory. That is

$$\pi_F\colon E_F := P \times_{\rho_F} \mathbb{C}^{N_F} \to \mathcal{M} \tag{29}$$

where $\rho_F\colon G \to \mathrm{Aut}(\mathbb{C}^{N_F})$ is a unitary representation of the structure group G. In addition we will assume for physical reasons that the vector bundle ζ_F is $\mathbb{Z}_2$-graded.

We call the twisted spinor bundle

$$\xi_F := \xi_S \otimes \zeta_F \tag{30}$$

the **Fermionic bundle**. It geometrically represents a fermionic particle of spin 1/2. Notice that the fermion bundle is $\mathbb{Z}_2$-graded and admits a natural action of the Clifford bundle ξ_{Cl} associated to $(\mathcal{M}, g_{\mathcal{M}})$. We denote this Clifford action by γ.

Let $\mathcal{A}(\xi_F)$ be the affine set of all associated connections on the fermion bundle ξ_F. Moreover, let $\mathcal{D}(\xi_F)$ be the affine set of all Dirac type operators which are compatible with the Clifford action, i. e. $D \in \mathcal{D}(\xi_F)$ fulfills $[D, f] = \gamma(\mathrm{d}f)$ for all $f \in \mathcal{C}^\infty(\mathcal{M})$. The set $\mathcal{A}(\xi_F)$ of all connections on the fermion bundle has a distinguished affine subset $\mathcal{A}_{\mathrm{Cl}}(\xi_F)$. It consists of those connections that are compatible with the Clifford action. For this reason these connections are referred to as Clifford connections. We denote the covariant derivative of a Clifford connection by ∂_A. Then, any $D \in \mathcal{D}(\xi_F)$ reads

$$D = \not{\partial}_A + \Phi \tag{31}$$

where $\not{\partial}_A := \gamma \circ \partial_A$ is the twisted spin-Dirac operator with respect to the chosen Clifford connection and $\Phi \in \Omega^0(\mathcal{M}, \mathrm{End}(\mathcal{E}))$ (see, e. g. [1] and [5]). Here, $\mathcal{E} := S \otimes E_F$ is the total space of the fermion bundle (30).

(1) An appropriate English version is in preparation and will be published elsewhere.

A Dirac type operator (31) is called of *simple type* iff the zero order part of D reads $\Phi = \gamma_5 \otimes \phi$. Here, $\phi \in \Omega^0(\mathcal{M}, \mathrm{End}^-(E_F))$ and γ_5 is the canonical grading operator on the spinor bundle ξ_S.

Notice that in general $\mathcal{D}(\xi_F) \simeq \mathcal{A}(\xi_F)/\mathrm{Ker}(\gamma)$. However, the connection class representing a Dirac operator of simple type has a natural representative called a **Dirac connection.**[(1)] The appropriate covariant derivative is denoted by

$$\partial_{A,\phi} := \partial_A + \xi \wedge (\gamma_5 \otimes \phi)\,. \tag{32}$$

Here, $\xi \in \Omega^1(\mathcal{M}, \mathrm{End}(\mathcal{E}))$ is the *canonical one-form* which is defined on every Clifford module bundle (over an even dimensional base manifold). The two basic properties of this one form are: (a) it is covariantly constant with respect to every Clifford connection on ξ_F; (b) it is a right inverse of the Clifford action γ (cf. [5]).

Let ξ_{H} be the Higgs bundle with respect to the data $(\mathcal{P}(\mathcal{M}, G), \rho_H, V_H)$. A linear mapping

$$\begin{aligned} G_{\mathrm{Y}} \colon \Gamma(\xi_{\mathrm{H}}) &\rightarrow \Gamma(\xi_{\mathrm{End}^-(E_F)}) \\ \varphi &\mapsto \phi_{\mathrm{Y}} := G_{\mathrm{Y}}(\varphi) \end{aligned} \tag{33}$$

is called a **Yukawa mapping**. A Dirac type operator is called a **Dirac–Yukawa operator** D_{Y} iff its zero order part reads $\Phi \equiv \Phi_{\mathrm{Y}} := \gamma_5 \otimes \phi_{\mathrm{Y}}$. The corresponding Dirac connection on the fermion bundle is called a **Dirac–Yukawa connection** and the covariant derivative is denoted by ∂_{Y}.

Let $(\Theta, \mathcal{V})$ be a vacuum pair that spontaneously breaks the gauge symmetry that is defined by $\mathcal{P}(\mathcal{M}, G)$. Then, on the corresponding reduced fermionic bundle $\xi_{F,\mathrm{red}}$ we have a natural non-flat connection

$$\partial_{\mathcal{D}} := \partial + \xi \wedge (\gamma_5 \otimes \mathcal{D}) \tag{34}$$

with $\mathcal{D} := G_{\mathrm{Y}}(\mathcal{V})$. The (total) curvature on the reduced fermion bundle is given by

$$F_{\mathcal{D}} = R + \mathrm{m}_F^2 \xi \wedge \xi \tag{35}$$

where R denotes the lifted (pseudo)Riemannian curvature tensor of $g_{\mathcal{M}}$ and the section $\mathrm{m}_F := -\mathrm{i}\gamma_5 \otimes \mathcal{D}$ is the *global fermionic mass matrix.*

Like in the case of the global bosonic mass matrices the spectrum of the Hermitian operator m_F^2 is constant. Since this operator is in the commutant of

(1) In fact, it can be shown that this property fully characterizes Dirac operators of simple type.

the invariance group of the vacuum one may decompose the reduced fermion bundle into the Whitney sum of eigenbundles of m_F^2:

$$\xi_{F,\mathrm{red}} = \bigoplus_{\mathrm{m}^2 \in \mathrm{Spec}(\mathrm{m}_F^2)} \xi_{F,\mathrm{m}^2} \,. \tag{36}$$

The vector bundles ξ_{F,m^2} geometrically represent "free fermions of mass m". Of course, the curvatures on these bundles are given by $F_{\mathcal{D}} \equiv F_{\mathrm{m}} = R + \mathrm{m}^2 \xi \wedge \xi$. In other words, the mass of the fermion determines the relative curvature on the fermion bundle which represents a free fermion of the mass in question.

Instead of a summary we close our considerations by three remarks:

1. Dirac's original first order differential operator $\mathrm{i}\partial\!\!\!/ - \mathrm{m}$ is not an operator of Dirac type. However, the operator

$$\mathrm{i}\partial\!\!\!/_{\mathcal{D}} = \mathrm{i}\partial\!\!\!/ - \mathrm{m}_F \tag{37}$$

is, of course, a well-defined Dirac operator of simple type on the reduced fermion bundle. Notice that the odd zero order operator m_F (i. e. the global fermionic mass matrix) is defined with respect to a vacuum.

2. The **Dirac potential** $V_{\mathrm{D}} := D^2 - \triangle_{\mathrm{D}} \in \Omega^0(\mathcal{M}, \mathrm{End}(\mathcal{E}))$ defines a **universal Lagrangian** that is naturally associated to a Dirac type operator (cf. [7]):

$$\begin{aligned} \mathcal{L} \colon \mathcal{D}(\xi_F) &\to \Omega^{2n}(\mathcal{M}) \\ D &\mapsto * \,\mathrm{trace}\, V_{\mathrm{D}} \,. \end{aligned} \tag{38}$$

The second order differential operator $\triangle_{\mathrm{D}}$ is the Bochner–Laplacian. It is uniquely defined by the Dirac operator D and can be explicitly calculated by using, for instance, the generalized Lichnerowicz decomposition formula (see, e. g. [5]). In fact, it can be shown that the Dirac potential generalizes the Higgs potential (see [7]). In the case of the Dirac–Yukawa operator $\partial\!\!\!/_{\mathcal{D}}$ the Dirac potential reads

$$V_{\mathrm{D}} = \left(\frac{r_{\mathcal{M}}}{4} + \mathrm{m}_F^2 \right) \tag{39}$$

with $r_{\mathcal{M}}$ the scalar curvature on $\mathcal{M}$ with respect to $g_{\mathcal{M}}$. Therefore, $\mathcal{L}(\partial\!\!\!/_{\mathcal{D}})$ naturally gives rise to gravity with a cosmological constant in terms of the fermionic masses.

3. It can be shown that for a given minimum of a general Higgs potential there exist gauge inequivalent vacuum pairs only if spacetime is not simply connected. If spacetime is simply connected, then there exists a vacuum pair iff $\mathcal{P}(\mathcal{M}, G)$ is trivial. In this case all vacuum pairs are

gauge equivalent to the canonical one that is defined by the canonical vacuum section $\mathcal{M} \xrightarrow{\mathbf{z}_0} \mathcal{M} \times \mathrm{orb}(\mathbf{z}_0), x \mapsto (x, \mathbf{z}_0)$ and the trivial connection on $\mathcal{M} \times G \xrightarrow{\mathrm{pr}_1} \mathcal{M}$. Notice that in the case of $\pi_1(\mathcal{M}) = 0$ all vacua are also trivial.

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