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SEPARABLE NON-PARALLEL AND UNSTEADY FLOW STABILITY PROBLEMS

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Abstract. The governing equations of the hydrodynamic stability theory are separable only with the parallel steady-state flow assumption, when they can be reduced to an ordinary differential equation, the Orr-Sommerfeld equation. For nonparallel flows, a basic flow and the equations for disturbance flow are dependent on the downstream coordinate so that the corresponding operator does not separate unless certain terms are ignored. If the basic flow is non-steady, this brings about great difficulties in theoretical studies of the instability since the normal modes containing an exponential time factor $\exp t$ are not applicable here. The objective of this work was to obtain new results in the problem of linear stability of non-parallel and unsteady flows by applying the recently developed symmetry-based approach to the separation of variables in PDEs with variable coefficients.

1. Introduction

Problems of hydrodynamic stability are of great theoretical and practical interest, as evidenced by the number of publications devoted to this subject. The prediction of the location of transition from a laminar flow to a turbulent one plays a fundamental role in the analysis of the flow field around most configurations of engineering interest. The flow stability problem has also many applications in meteorology and oceanography, and in astrophysics and geophysics.

The underlying notion is that transition from one type of flow to another results from spontaneous amplification of disturbances present in the original flow. Because of the mathematical simplifications associated with linearization, the stability theory has largely been developed with the restriction of infinitesimal disturbances. Infinitesimal disturbances are always present, even in the most carefully controlled experiments, and if these tend to be amplified, the flow will necessarily break down. Thus, linear stability theory should indicate conditions in which the flow cannot remain in its postulated form and must undergo to another type of motion. Whether a small disturbance that is superimposed upon a known primary flow will be amplified or damped depends on the properties of the fluid, the pattern of the primary flow, and the nature of the disturbance.

The linear stability theory (see, e.g., [8]) for a particular flow starts with a solution (or approximate solution) of the equations of motion representing the flow. One then considers this solution with a small perturbation superimposed. Substituting the perturbed solution into the equations of motion and neglecting all terms that involve the square of the perturbation amplitude yield the linear stability equations which govern the behavior of the perturbation. If the perturbation dies away the original flow is said to be stable, and if the perturbation grows and results in what is permanently different from zero, the flow is said to be unstable. The linearization provides a means of allowing for the many different forms that the disturbance can take. In the method of normal modes, small disturbances are resolved into modes, which may be treated separately because each satisfies the linear equations and there are no interactions between different modes. For a steady-state flow, normal modes depend on time exponentially with a complex exponent, and the sign of its real part indicates whether the disturbance grows or decays in time. In practice, however, it is not only one normal mode but some superposition of normal modes that determines the nature of the initial disturbance. So, in general, one should distinguish between the normal modes and the development of a localized disturbance in time and space.

The mathematical problem of the determination of the stability of a given flow involves deriving a set of perturbation equations obtained from the Navier-Stokes equations by linearization and finding a set of possible solutions (normal modes) which would permit splitting any general perturbation into normal modes. The perturbation equations represent linear partial differential equations with variable coefficients depending on the form of the basic flow. For a steady-state basic flow, when the equation coefficients do not depend on time, the normal modes containing an exponential time factor $\exp(\lambda t)$ can be taken. The boundary conditions on the disturbance equations require the vanishing at the boundaries of quantities like the disturbance velocity components, so that we have an eigenvalue problem for the determination of the quantity λ. Thus, in general, stability analysis of a given flow requires the evaluation of the eigenvalues of a set of partial differential equations. In certain cases, the basic flow field is defined as a function of only one spatial variable and the set of partial differential equations defining the perturbation is separable, so that the problem reduces to that of solving a set of ordinary differential equations. A steady-state parallel basic flow is such a case. With the parallel flow assumption, separation of variables in the governing stability equations leads to an ordinary differential equation, the Orr-Sommerfeld equation. When this equation is solved with proper boundary conditions, the problem of linear stability of parallel flows is reduced to a 2-point boundary (eigen)value problem.

Exact parallel flows are idealized models. In general, all flows occurring in nature and engineering are nonparallel. In contrast to the situation for parallel flows, relatively little is known on the linear stability of non-parallel flows. For nonparallel flows, a basic flow and the equations for disturbance flow are dependent not only on the normal coordinate (in a two-dimensional case) but also on the downstream coordinate so that the corresponding operator does not separate unless certain terms are ignored. For example, even though the mean flow cannot be exactly treated as a parallel, but the dependence on the downstream coordinate is weak, the approximation is commonly made that the stability characteristics are related in some sense to those of the equivalent parallel flow. Boundary layers, jets and far wakes are examples of this type (see review in [16]). Those parallel theories and even weakly nonparallel theories (see reviews in [16], [12], [10]) did not fully agree with the experimental measurements. The attempts to investigate nonparallel stability effects on the basis of the complete Navier-Stokes equations by using a numerical procedure (e.g., [9], [17]) were made. Such an approach in a sense represents a direct numerical simulation of the laboratory experiments and it does not allow one to draw general conclusions concerning the flow stability properties. In addition, it is constrained by artificial boundary conditions, which need to be introduced in order to limit the computational domain.

There exist a number of the boundary layer type flows, interesting from both theoretical and engineering point of views, that cannot be treated as weakly nonparallel. Among them, the flows described by exact solutions of the Navier-Stokes equations have evident advantages from the point of view of performing the stability analysis. The nonparallel Taylor flow (a channel flow driven by injection through the two porous walls that models some flows in engines) provides such an example (see discussion in [11]). Exact solutions of the Navier-Stokes equations found in [4–6], which, like the Taylor flow, describe the flows in the presence of suction or injection through the boundaries, also represent essentially nonparallel flows.

There are many flow stability problems, which are not of the boundary layer type, so that the linear stability analysis may only be performed in the fully nonparallel linear analysis framework. Several numerical methods have been designed for direct solution of the two-dimensional stability problem. Global stability analysis of nonparallel flows, based on eigensolutions and physical modes (see, e.g., [14]) was often pointed out as the most suitable numerical method of analyzing the onset of flow stability (the ideas of this method, that are commonly referred to [18], were first introduced and applied to a convective stability problem in [7]).

If the basic flow is non-steady (many flows interesting from both theoretical and engineering points of view are essentially unsteady), this brings about great difficulties in theoretical studies of the instability since the normal modes containing an exponential time factor $\exp(\lambda t)$ are not applicable here. If an unsteady flow is non-parallel, it should further complicate matters since even a separation of the space and time variables becomes impossible so that the method of normal modes may not be applicable at all. There exist only isolated examples of the unsteady flows for which the linear stability problem permit separable solutions. As a rule, the separation becomes possible only after making some coordinate transformation (see, for example, stability analysis of certain flows of viscous and conducting gas in [1–3]).

All the above said shows that the method of separation of variables, is of a fundamental importance for the hydrodynamic stability problems. However, till now, the method of separation of variables was used for stability analysis in an intuitive way as a normal mode approach.

There exist at least two possible approaches to variable separation in linear PDEs.

The first one is based on symmetry properties of equation under study: it makes use of the fact that each solution with separated variables is a common eigenfunction of three mutually commuting linear second order symmetry operators of the equation, and separation constants are eigenvalues of these operators. This method starts with a set of commuting symmetry operators of the equation and ends by constructing the coordinate systems (see, e.g., [15]).

Another approach is the so-called direct approach to separation of variables which is closer to the original concept of separation of variables in PDEs. It has been developed in a series of papers by Zhdanov, Fushich and Zhalij (see, e.g., [21]). By a proper formalizing the features of the notion of separation of variables, they have formulated a constructive algorithm for separation of variables in linear PDEs (for more details see the next section of our paper). In this approach, a form of the Ansätz for a solution with separated variables as well as a form of reduced ODEs, that should be obtained as a result of the variable separation, are postulated from the beginning. Upon obtaining the solutions, they can be related to a set of mutually commuting symmetry operators of the equations under consideration.

In the present study, we applied the second approach, since a computation of symmetry operators is an extra step which is not, in fact, necessary for obtaining solution with separated variables. As distinct from the classical approach to variable separation, which gives no general routine, this method is completely algorithmic. The method has been successfully applied to several equations of mathematical physics in [19–22]. The application of this method to the problems of stability of nonparallel and/or unsteady flows offers considerable promise for the exact stability analysis of certain flows.

In the next Section we give a description of the application of the method to the problem of linear stability of an unsteady nonparallel basic flow.

2. Separation of Variables in the Linear Stability Equations

We will describe the application of the method using, as an example, the problem of linear stability of a two-dimensional basic flow to two-dimensional perturbations. It should be noted that, although such a problem is considered in the literature, its solution cannot be used for drawing final conclusions concerning stability properties of the flow under consideration in the case when the flow is unsteady and/or nonparallel. The point is that, in such a case, Squire's theorem, reducing the problem of stability of a parallel flow with respect to general three-dimensional perturbations to an equivalent problem of stability with respect to two-dimensional perturbations, is not valid. Nevertheless, we will describe the technique for the two-dimensional case since all the important details of the method are present here but the relations with the common normal mode approach can be more easily traced.

The problem of linear stability of a two-dimensional basic flow to two-dimensional perturbations can be reduced to a single equation for a streamfunction of perturbation u of the form

$$\frac{\partial \Delta u}{\partial t} + U\frac{\partial \Delta u}{\partial x} + V\frac{\partial \Delta u}{\partial y} = \Delta V\frac{\partial u}{\partial y} + \Delta U\frac{\partial u}{\partial x} + \nu\Delta^2 u \tag{1}$$

where Δ is Laplace operator and $U(x, y)$ and $V(x, y)$ are arbitrary smooth functions of indicated variables.

With all the variety of approaches to a separation of variables in PDEs, one can notice the three generic principles respected by all of them, namely, the following

1. Representation of a solution to be found in a separated (factorized) form via several functions of one variable.
2. Requirement that the above mentioned functions of one variable should satisfy some ordinary differential equations.
3. Dependence of the so found solution on several arbitrary (continuous or discrete) parameters, called spectral parameters or separation constants.

By a proper formalizing the above features we have formulated in [19], [21] an algorithm for variable separation in linear PDEs. Below we apply this algorithm in order to classify separable equations of the form (1).

To have a right to speak about description of *all* basic flows and *all* coordinate systems enabling us to separate the equation (1), one needs to provide a rigorous definition of separation of variables. The definition we intend to use is based on ideas from [13].

We say that equation (1) is separable in a coordinate system t, $\omega_1 = \omega_1(t,x,y)$, $\omega_2 = \omega_2(t,x,y)$ if the separation Ansätz,

$$u(t,x,y) = Q(t,x,y)\varphi_0(t)\varphi_1\left(\omega_1(t,x,y),\lambda_1,\lambda_2\right)\varphi_2\left(\omega_2(t,x,y),\lambda_1,\lambda_2\right) \quad (2)$$

reduces PDE (1) to three ordinary differential equations for the functions φ_0, φ_1 and φ_2,

$$\begin{aligned} \varphi_0' &= U_0(t,\varphi_0;\,\lambda_1,\lambda_2) \\ \varphi_1' &= U_1(\omega_1,\varphi_1;\,\lambda_1,\lambda_2) \quad (3) \\ \varphi_2^{(4)} &= U_2(\omega_2,\varphi_2,\varphi_2',\varphi_2'',\varphi_2''';\,\lambda_1,\lambda_2). \quad (4) \end{aligned}$$

Here U_0, U_1, U_2 are some smooth functions of the indicated variables, λ_1, λ_2 are separation constants and, what is more,

$$\operatorname{rank}\left\|\begin{matrix} \frac{\partial U_0}{\partial\lambda_1} & \frac{\partial U_0}{\partial\lambda_2} \\ \frac{\partial U_1}{\partial\lambda_1} & \frac{\partial U_1}{\partial\lambda_2} \\ \frac{\partial U_2}{\partial\lambda_1} & \frac{\partial U_2}{\partial\lambda_2} \end{matrix}\right\| = 2. \quad (5)$$

The above condition secures the essential dependence of a solution with separated variables on the separation constants $\vec{\lambda}$.

Formulas (2)–(5) form the input data of the method. The principal steps of the procedure of variable separation in equation (1) are as follows.

1. We insert the Ansätz (2) into equation (1) and express the derivatives φ_0', φ_1', $\varphi_2^{(4)}$, in terms of functions φ_0, φ_1, φ_2, φ_2', φ_2'', φ_2''', using equations (3).
2. We regard φ_0, φ_1, φ_2, φ_2', φ_2'', φ_2''', λ_1, λ_2 as the new independent variables. As the functions $Q, \omega_1, \omega_2, U, V$ are independent on these variables, we can demand that the obtained equality is transformed into identity under arbitrary φ_0, φ_1, φ_2, φ_2', φ_2'', φ_2''', λ_1, λ_2. In other words, we should split the equality with respect to these variables. After splitting we get an overdetermined system of nonlinear partial differential equations for unknown functions $Q, \omega_1, \omega_2, U, V$.
3. After solving the above system we get an exhaustive description of coordinate systems providing separability of equation (1).

Thus, the problem of variable separation in equation (1) reduces to integrating the overdetermined system of PDEs for five functions Q, ω_1, ω_2, U, V. This can be done with the aid of *Mathematica* package.

A possibility of variable separation in equation (1) is intimately connected to its symmetry properties [15]. Namely, solutions with separated variables are common eigenfunctions of three mutually commuting symmetry operators of equation (1).

For all cases of variable separation in equation (1) these operators will be constructed in explicit form. They are expressed in terms of the coefficients of the separation equations (3).

Note, that formulas (2)–(5) form the input data of the method. We can change these conditions and thereby modify the definition of separation of variables. For instance, we can change the order of the reduced equations (3) or the number of essential parameters. So, our claim of obtaining the *complete description* of basic flows and coordinate systems providing separation of variables in (1) makes sense only within the framework defined by the formulas (2)–(5). If one uses a more general definition, it might be possible to construct new coordinate systems and basic flows providing separability of equation (1).

3. Some results

Here we will concentrate on the results for stability problem formulated for the linear stability equations written in cylindrical coordinates. It should be clarified why the problem in cylindrical coordinates should be treated separately of that in Cartesian coordinates. Of course, if our method enabled us to give a complete description of all the coordinate systems providing a separability of the linear stability equations, then the results obtained with the use of Cartesian coordinates should include those obtained with the use of cylindrical coordinates. However, our results are restricted by the form of the separation Ansätz and therefore they are not obligatory include all the coordinate systems – in particular, those obtained by a deformation of the cylindrical system induced by the separation Ansätz for the equations in cylindrical coordinates. It is worth also noting that, even though the results coincided, separate treating equations in cylindrical coordinates would be preferable since the results for the flows possessing some symmetries (for example, for axially symmetrical flows) are then represented in a more natural form.

Equations for perturbations v_r, v_φ, v_z in the cylindrical coordinates are:

$$\begin{aligned}\frac{\partial v_r}{\partial t} + V_r\frac{\partial v_r}{\partial r} + v_r\frac{\partial V_r}{\partial r} + \frac{V_\varphi}{r}\frac{\partial v_r}{\partial \varphi} + \frac{v_\varphi}{r}\frac{\partial V_r}{\partial \varphi} + V_z\frac{\partial v_r}{\partial z} + v_z\frac{\partial V_r}{\partial z} - 2\frac{V_\varphi v_\varphi}{r} \\ = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \varphi^2} + \frac{\partial^2 v_r}{\partial z^2} + \frac{1}{r}\frac{\partial v_r}{\partial r} - \frac{2}{r^2}\frac{\partial v_\varphi}{\partial \varphi} - \frac{v_r}{r^2}\right)\end{aligned} \tag{6}$$

$$\begin{aligned}\frac{\partial v_\varphi}{\partial t} + V_r\frac{\partial v_\varphi}{\partial r} + v_r\frac{\partial V_\varphi}{\partial r} + \frac{V_\varphi}{r}\frac{\partial v_\varphi}{\partial \varphi} + \frac{v_\varphi}{r}\frac{\partial V_\varphi}{\partial \varphi} + V_z\frac{\partial v_\varphi}{\partial z} + v_z\frac{\partial V_\varphi}{\partial z} + \frac{V_r v_\varphi}{r} + \frac{v_r V_\varphi}{r} \\ = -\frac{1}{\rho r}\frac{\partial p}{\partial \varphi} + \nu\left(\frac{\partial^2 v_\varphi}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 v_\varphi}{\partial \varphi^2} + \frac{\partial^2 v_\varphi}{\partial z^2} + \frac{1}{r}\frac{\partial v_\varphi}{\partial r} + \frac{2}{r^2}\frac{\partial v_r}{\partial \varphi} - \frac{v_\varphi}{r^2}\right)\end{aligned} \tag{7}$$

$$\frac{\partial v_z}{\partial t} + V_r\frac{\partial v_z}{\partial r} + v_r\frac{\partial V_z}{\partial r} + \frac{V_\varphi}{r}\frac{\partial v_z}{\partial \varphi} + \frac{v_\varphi}{r}\frac{\partial V_z}{\partial \varphi} + V_z\frac{\partial v_z}{\partial z} + v_z\frac{\partial V_z}{\partial z}$$
$$= -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 v_z}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \varphi^2} + \frac{\partial^2 v_z}{\partial z^2} + \frac{1}{r}\frac{\partial v_z}{\partial r}\right) \tag{8}$$

$$\frac{\partial v_r}{\partial r} + \frac{1}{r}\frac{\partial v_\varphi}{\partial \varphi} + \frac{\partial v_z}{\partial z} + \frac{v_r}{r} = 0 \tag{9}$$

where V_r, V_φ, V_z are the basic flows.

Applying the above described method (modified for a system of equations) to the system (6–9) yields the following results.

The most general forms for the basic flows, that permit separation of variables in the stability equations, are:

$$V_z = A(\eta)T(t) - \frac{c'(t) + zT'(t)}{T(t)}$$
$$V_r = B(\eta)T(t) - r\frac{T'(t)}{T(t)}$$
$$V_\varphi = C(\eta)T(t)$$

where

$$\eta = T(t)r, \qquad \gamma = T(t)z + c(t).$$

The corresponding Ansätzes for perturbations v_r, v_φ, v_z and pressure p are:

$$v_r = T(t)\exp\left(a\gamma + m\varphi + s\int T(t)^2\,\mathrm{d}t\right)f(\eta)$$
$$v_\varphi = T(t)\exp\left(a\gamma + m\varphi + s\int T(t)^2\,\mathrm{d}t\right)g(\eta)$$
$$v_z = T(t)\exp\left(a\gamma + m\varphi + s\int T(t)^2\,\mathrm{d}t\right)h(\eta)$$
$$p = \rho T(t)^2\exp\left(a\gamma + m\varphi + s\int T(t)^2\,\mathrm{d}t\right)k(\eta).$$

The ordinary differential equations defining the solutions with separated variables are

$$f(\eta)(\eta^2 s + \nu - m^2\nu - a^2\eta^2\nu + a\eta^2 A(\eta) + m\eta C(\eta) + \eta^2 B'(\eta))$$
$$+ 2(m\nu - \eta C(\eta))g(\eta) + \eta((-\nu + \eta B(\eta))f'(\eta) + \eta(k'(\eta) - \nu f''(\eta))) = 0$$
$$(\eta^2 s + \nu - m^2\nu - a^2\eta^2\nu + a\eta^2 A(\eta) + \eta B(\eta) + m\eta C(\eta))g(\eta)$$
$$+f(\eta)(-2m\nu + \eta C(\eta) + \eta^2 C'(\eta)) + \eta(mk(\eta) + (-\nu + \eta B(\eta))g'(\eta) - \eta\nu g''(\eta)) = 0$$

$$(\eta^2 s - m^2\nu - a^2\eta^2\nu + a\eta^2 A(\eta) + m\eta C(\eta))h(\eta) + \eta(a\eta k(\eta) + \eta f(\eta)A'(\eta)$$
$$- \nu h'(\eta) + \eta B(\eta)h'(\eta) - \eta\nu h''(\eta)) = 0$$
$$f(\eta) + mg(\eta) + \eta(ah(\eta) + f'(\eta)) = 0.$$

In general, the basic flows, defined above from the point of view of separability of the corresponding stability problem, should represent the solutions of the Navier-Stokes equations. Note, however, that in many hydrodynamic stability studies, because of the great difficulties in solving exact stability problems, the basic flow is taken either as an approximate solution of the Navier-Stoke equation (for example, boundary layer solution) or as an arbitrary flow approximating some features of the flow of interest, which, of course, is not completely justified from theoretical point of view. Since our study is aimed in obtaining *exact results* in the stability problem of nonparallel and unsteady flows, we will consider below only the basic flows that represent *exact solutions* of the Navier-Stokes equations.

The restrictions on the forms of the basic flows following from the requirement that they should satisfy Navier-Stokes equations lead to the two following cases:

Case 1:
$$T(t) = \frac{1}{\sqrt{t}}, \qquad B(\eta) = -\frac{3\eta}{4} + \frac{k}{\eta}$$
and functions $A(\eta)$ and $C(\eta)$ satisfy the equations
$$(4k + 3\eta^2 - 4\nu)A'(\eta) + \eta(-4k + 3\eta^2 + 4\nu)A''(\eta) + 4\eta^2\nu A'''(\eta) = 0$$
$$8(k + \nu)C(\eta) + \eta(2(3\eta^2 - 4\nu)C'(\eta) + \eta((-4k + 3\eta^2 + 4\nu)C''(\eta)$$
$$+ 4\eta\nu C'''(\eta) = 0.$$

Case 2:
$$T(t) = 1, \qquad B(\eta) = \frac{k}{\eta}$$
and functions $A(\eta)$ and $C(\eta)$ satisfy the equations
$$(k - \nu)A'(\eta) + \eta(-k + \nu)A''(\eta) + \eta^2\nu A'''(\eta) = 0$$
$$2(k + \nu)C(\eta) - 2\nu\eta C'(\eta) + \eta^2(-k + \nu)C''(\eta) + \eta^3\nu C'''(\eta) = 0.$$

These ODEs can be explicitly solved in terms of the hypergeometric functions.

It should be noted that the invariance properties of the equations may be used to enrich the solutions. In particular, using the invariance with respect to translation in time we may replace t by $t + t_0$ where t_0 is a constant.

Below we will discuss the fluid dynamics interpretation of some basic flows defined above as the *exact solutions of the Navier-Stokes equations* having *exactly* *solvable stability problems*. We will consider a particular case of the class of the exact solutions of the Navier-Stokes equations identified above as Case 1. It represents an axially symmetric flow ($V_\varphi = 0$) outside an expanding and stretching cylinder - unsteady flows near the stretching and expanding surfaces are of particular importance for polymer industry. The radius of the cylinder changes with time as

$$R = R_0\sqrt{1+at}$$

and the surface is stretched according to the law

$$V = \frac{bz}{1+at}$$

where a, b and R_0 are constants. Restricting ourselves to the case of impermeable cylinder, we formulate the boundary conditions, as follows

$$V_z = V, \quad V_r = \frac{\mathrm{d}R}{\mathrm{d}t} \qquad \text{at } r = R. \tag{10}$$

The flow at infinity represents a combination of the stagnation line and uniform flows.

We will introduce the dimensionless variables with the following scales x^*, t^* and V^*, for coordinates, time and velocities, respectively:

$$x^* = R_0, \quad t^* = \frac{1}{a}, \quad V^* = aR_0.$$

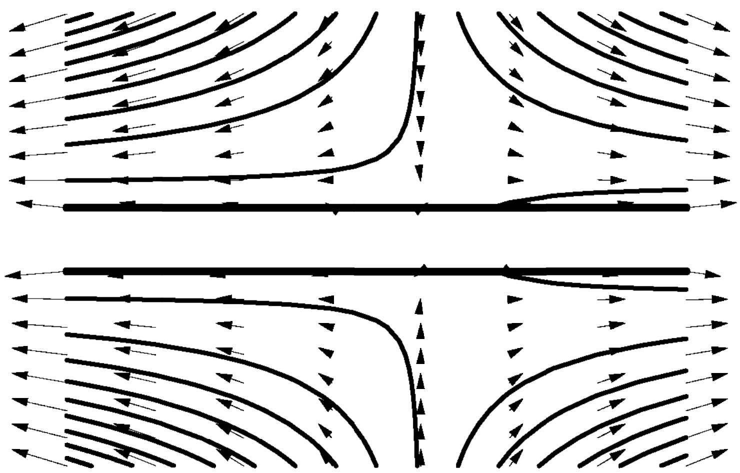

Figure 1. Flow outside an expanding impermeable cylinder; $U = -0.5$, $Re = 100$, $t = 1$.

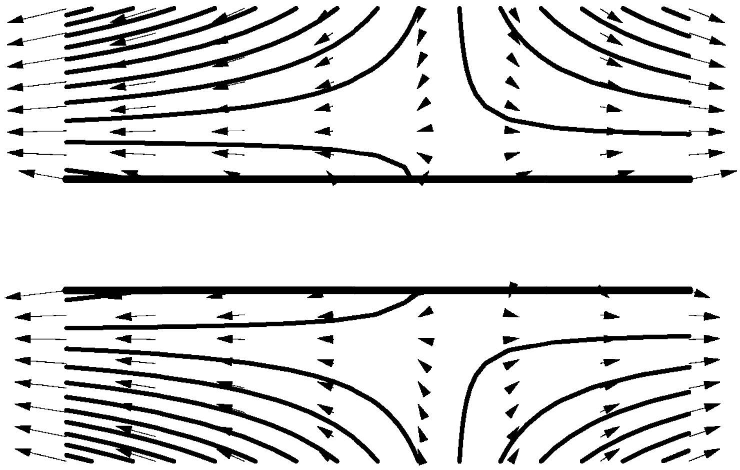

Figure 2. Flow outside an expanding impermeable cylinder; $U = -0.5$, $Re = 100$, $t = 5$.

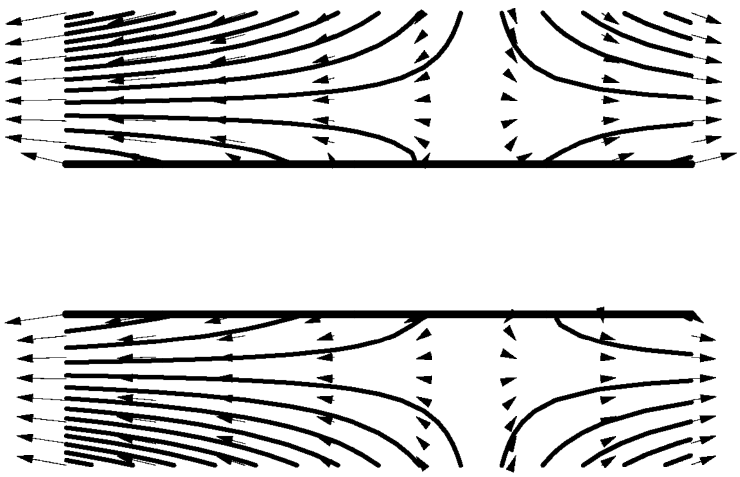

Figure 3. Flow outside an expanding impermeable cylinder; $U = -0.5$, $Re = 100$, $t = 10$.

Then the solution takes the form (we retain the same notation for the dimensionless variables):

$$V_z = \frac{x}{2(1+t)} + \frac{U}{\sqrt{1+t}}\left[1 - \frac{\Gamma\left(\frac{3Re}{8}, \frac{3Re\eta^2}{8}\right)}{\Gamma\left(\frac{3Re}{8}, \frac{3Re}{8}\right)}\right]$$

$$V_r = -\frac{r}{4(1+t)} + \frac{3}{4r}, \quad V_\varphi = 0, \quad \eta = \frac{r}{\sqrt{1+t}}$$

where $\Gamma(a, z)$ is the incomplete Gamma function, U is the velocity of the uniform flow at infinity and Re is the Reynolds number defined by

$$Re = \frac{V^* x^*}{\nu} = \frac{a R_0^2}{\nu}.$$

It is easily checked that the solution satisfies the boundary conditions (10) rewritten in the dimensionless form.

The structure of the flow described by the above solution is illustrated in Figures 1–3 where the streamlines and flow fields superimposed are shown for one set of parameters and different time moments.

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