GEOMETRICAL APPROACHES TO THE QUANTIZATION OF GAUGE THEORIES

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Abstract. In the Batalin-Vilkovisky field-antifield formalism a classical mechanical system is described by a solution of the classical master equation. The quantization of this general gauge theory in the Lagrangian approach can be accomplished in closed form [2]. The AKSZ-formalism is a geometrical construction of such a solution as a QP-manifold [1]. This can be extended and applied to topological quantum field theories.

1. Introduction

After a short review including the main topics of general gauge theory and the notion of fields and antifields in the Batalin-Vilkovisky formalism I will introduce the geometrical approach to these problems. This is the formalism of Alexandrov, Kontsevich, Schwarz and Zaboronsky (AKSZ) which constructs in a geometrical way the solutions of the master equation which are of physical interest by the use of QP-manifolds [1]. A special QP-manifold, $E = \Pi T^*X \times \Pi \mathfrak{g} \times \mathfrak{g}^*$, which leads to the Batalin-Vilkovisky action of an irreducible theory with gauge invariance will be constructed and discussed in detail.

2. General Gauge Theory

2.1. Canonical Formalism

The non-abelian Yang-Mills theory is the most familiar example of a gauge structure. In this case, when a choice of basis is made, the structure constants of the underlying Lie group determine the commutator algebra. The Jacobi identity, which expresses the associativity of the Lie group, must be satisfied.

Now I will recall the canonical formalism in a compact notation [11]. Consider a system whose dynamics is governed by a classical action $S_0[\phi]$, depending on n different fields ϕ^i with $i = 1, \ldots, n = n_+ + n_-$, where n_+ is the number

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of bosons and n_- is the number of fermions. In general i can label space-time indices of tensor fields, spinor indices of fermion fields or distinguish between different types of generic fields. Let $\epsilon(\phi^i) = \epsilon_i$ denote the statistical parity, i.e. the Grassmann parity of ϕ^i. Each ϕ^i is either a commuting bosonic field with parity $\epsilon_i = 0$ or an anticommuting fermionic field with $\epsilon_i = 1$, so one has $\phi^i(x)\phi^j(y) = (-1)^{\epsilon_i \epsilon_j} \phi^j(y)\phi^i(x)$ according to the Koszul sign rule. This has to be taken into consideration whenever two indices are interchanged. Here the new variables are introduced at the classical level.

Assume that the action $S_0[\phi]$ is invariant under a set of m, $m \leq n$, of non-trivial gauge transformations, which read in infinitesimal form

$$\delta\phi^i = T_\alpha^i \varepsilon^\alpha, \quad \text{where} \quad \alpha = 1, 2, \ldots, m.$$

This is the compact notation, where ε^α is an infinitesimal gauge parameter with parity $\epsilon_\alpha = 0, 1$ and the T_α^i are generators of gauge transformations with parity $\epsilon(T_\alpha^i) = \epsilon_i + \epsilon_\alpha \pmod 2$. Later on the gauge parameters will be turned into ghosts [9] and the generators into the generators of a underlying Lie algebra of a QP-manifold, which will be constructed in the AKSZ-formalism (see section 4.5). Let $S_{0,i}[\phi]$ denote the variation of the action with respect to ϕ^i

$$S_{0,i}[\phi] \equiv \left.\frac{\partial_r S}{\partial \phi^i}\right|_{\phi_0}$$

restricted to a stationary point ϕ_0. The index r denotes the right derivative, the distinction between left and right derivatives is necessary in the context of Grassmann algebras with fermionic and bosonic variables. The statement that the action is invariant under gauge transformations of the form $\delta\phi^i = T_\alpha^i \varepsilon^\alpha$ means that the Noether identities hold

$$S_{0,i} T_\alpha^i = 0.$$

Now consider the commutator of two gauge transformations of this form

$$[\delta_1, \delta_2]\phi^j = \delta_1(\delta_2\phi^j) - \delta_2(\delta_1\phi^j)$$
$$= \delta_1(T_\beta^j \varepsilon_2^\beta) - \delta_2(T_\alpha^j \varepsilon_1^\alpha)$$

with

$$\delta_1(T_\beta^j \varepsilon_2^\beta) = \frac{\partial}{\partial \phi^i}(T_\beta^j \varepsilon_2^\beta)\delta_1\phi^i = \frac{\partial T_\beta^j}{\partial \phi^i}\varepsilon_2^\beta \delta_1\phi^i + T_\beta^j \frac{\partial \varepsilon_2^\beta}{\partial \phi^i}\delta_1\phi^i = \frac{\partial T_\beta^j}{\partial \phi^i}\varepsilon_2^\beta T_\alpha^i \varepsilon_1^\alpha.$$

It follows that

$$[\delta_1, \delta_2]\phi^i = T_{\alpha,j}^i T_\beta^j \varepsilon_2^\beta \varepsilon_1^\alpha - T_{\beta,j}^i T_\alpha^j \varepsilon_1^\alpha \varepsilon_2^\beta$$
$$= (T_{\alpha,j}^i T_\beta^j - (-1)^{\epsilon_\alpha \epsilon_\beta} T_{\beta,j}^i T_\alpha^j)\varepsilon_1^\alpha \varepsilon_2^\beta.$$

The commutator of two gauge transformations is also a gauge symmetry of the action, so it has to satisfy the Noether identity

$$S_{0,i}(T^i_{\alpha,j}T^j_\beta - (-1)^{\epsilon_\alpha \epsilon_\beta}T^i_{\beta,j}T^j_\alpha) = 0.$$

Then the most general solution to the Noether identities implies the following important relation among the generators

$$T^i_{\alpha,j}T^j_\beta - (-1)^{\epsilon_\alpha \epsilon_\beta}T^i_{\beta,j}T^j_\alpha = 2T^i_\gamma f^\gamma_{\alpha\beta} + S_{0,j}E^{ij}_{\alpha\beta} \qquad (1)$$

for some gauge structure tensors $f^\gamma_{\alpha\beta}$ and $E^{ij}_{\alpha\beta}$. In the cases which will be considered here the $f^\gamma_{\alpha\beta}$ will be the structure constants of a Lie algebra, not depending on the fields, and the $E^{ij}_{\alpha\beta}$ vanish, so our algebra is closed. Then the bracket relation reads

$$\frac{1}{2}\left(\frac{\partial T^i_\alpha}{\partial \phi^j}T^j_\beta - (-1)^{\epsilon_\alpha \epsilon_\beta}\frac{\partial T^i_\beta}{\partial \phi^j}T^j_\alpha\right) = T^i_\gamma f^\gamma_{\alpha\beta}. \qquad (2)$$

2.2. BRST Formalism

Variational principles lead to the classical Poisson bracket on the phase space $\{q^i; p_i\}$. Gauge transformations G are local symmetry transformations $\delta q^i, \delta p_i$; they leave the Hamiltonian invariant

$$\delta H = \{H, G\} = 0.$$

The original phase space must be extended by one degree of freedom for each symmetry transformation. In the BRST formalism [5, 20] the solutions of the constraints will be identified with the cohomology classes of a nilpotent operator Ω. The construction of Ω and the extension of the phase space will be done with ghosts. Define the generalized Poisson bracket

$$\{F, G\} = \frac{\partial F}{\partial q^i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q^i} + (-1)^{\epsilon_F}\frac{\partial_l F}{\partial \theta^\alpha}\frac{\partial_l G}{\partial \pi_\alpha} - \frac{\partial_l F}{\partial \pi_\alpha}\frac{\partial_l G}{\partial \theta^\alpha}$$

on the extended phase space $\{q^i; p_i; \theta^\alpha; \pi_\alpha\}$, where q^i are the coordinates, p_i their conjugate momenta, and θ^α the ghosts with their conjugate momenta π_α. F, G are functions on this space. The BRST operator Ω generates ghost dependent symmetry transformations of the classical phase variables $\delta_\Omega q^i, \delta_\Omega p_i$. Now define BRST transformations for the ghosts and require nilpotence, $\delta_\Omega^2 = 0$.

Let F be a gauge invariant physical quantity. F has to be BRST invariant, i.e.

$$\delta_\Omega F = -\{\Omega, F\} = 0$$

and is said to be BRST-closed. The non-trivial solutions

$$F_0 = \delta_\Omega F_1 = -\{\Omega, F_1\}$$

are called BRST-exact. Hence F_0 depends on the ghosts, it is non-physical and must be divided out. This defines the BRST cohomology

$$\mathbb{H}(\delta_\Omega) = \frac{\ker \delta_\Omega}{\mathrm{im}\, \delta_\Omega}.$$

With the requirement of BRST invariance one can construct an effective Hamiltonian in the extended phase space

$$H_{\mathrm{eff}} = H_0 - \delta_\Omega \psi$$

which consists of the classical part modulo a BRST-exact term. The latter acts only in the non-physical sector, ψ is a function with ghost number $\mathrm{gh}[\psi] = -1$, called a gauge fermion. It ensures a zero ghost number of the whole Hamiltonian. Here a variable with negative ghost number enters for the first time, from now on it will be called an antighost. The construction of an equivalent BRST invariant Lagrangian is straightforward

$$L_{\mathrm{eff}} = L_0 - \delta_\Omega \psi.$$

It is quite natural to construct with these variables an effective action on a doubled configuration space for fields

$$S_{\mathrm{eff}}[\Phi^A; \Phi^*_A] = S_0 + \int \mathrm{d}t\; \delta_\Omega \Phi^A \Phi^*_A \Big|_{\Phi^*_A = \frac{\partial \psi}{\partial \Phi^A}}$$

where $\Phi^*_A = \frac{\partial \psi}{\partial \Phi^A}$ is the restriction to the physical hypersurface. The "antifields" are the sources of the BRST variations of the fields

$$\delta_\Omega \Phi^A = (-1)^{\epsilon_A} \frac{\delta S}{\delta \Phi^*_A}.$$

This leads to the Batalin-Vilkovisky field-antifield formalism [2, 3], which is presented in the next section. For more details see [12, 15].

3. The Batalin-Vilkovisky Formalism

3.1. Fields and Antifields

Introduce a system

$$z^a = \{\Phi^A; \Phi^*_A\} \quad \text{with} \quad A = 1, \ldots, N \quad \text{and} \quad a = 1, \ldots, 2N$$

of fields Φ^A with Grassmann parity

$$\epsilon_A = \begin{cases} 0 & \text{boson} \\ 1 & \text{fermion} \end{cases}$$

and antifields Φ^*_A with opposite statistics $\epsilon^*_A = \epsilon_A + 1 \pmod 2$, which carry ghost number

$$\mathrm{gh}[\Phi^A] \quad \text{and} \quad \mathrm{gh}[\Phi^*_A] = -\mathrm{gh}[\Phi^A] - 1.$$

The collection of fields and antifields for an irreducible theory is

$$\Phi^A = \{\phi^i; C^\alpha\} \quad \text{and} \quad \Phi^*_A = \{\phi^*_i; C^*_\alpha\}$$

with $\alpha = 1, \ldots, m$ for m gauge invariances. In the space of fields and antifields, one can define a bracket relation, the so-called antibracket or the Batalin-Vilkovisky bracket

$$(F, G) = \frac{\partial_r F}{\partial \Phi^A} \frac{\partial_l G}{\partial \Phi^*_A} - \frac{\partial_r F}{\partial \Phi^*_A} \frac{\partial_l G}{\partial \Phi^A} = \frac{\partial_r F}{\partial z^a} \xi^{ab} \frac{\partial_l G}{\partial z^b}$$

with

$$\xi^{ab} = \begin{pmatrix} 0 & \delta^A_B \\ -\delta^A_B & 0 \end{pmatrix}, \qquad \xi^{ab} = -\xi^{ba}, \qquad \epsilon\left(\xi^{ab}\frac{\partial}{\partial z^b}\right) = \epsilon_a + 1 \quad (\text{mod } 2).$$

It is analogous to the generalized Poisson bracket with the replacement

$$\epsilon_F \longrightarrow \epsilon_F + 1, \qquad \epsilon_G \longrightarrow \epsilon_G + 1.$$

The antibracket carries the ghost number 1

$$\text{gh}[(F, G)] = \text{gh}(F) + \text{gh}(G) + 1$$

and has odd statistics

$$\epsilon[(F, G)] = \epsilon_F + \epsilon_G + 1 \quad (\text{mod } 2).$$

It is graded antisymmetric

$$(F, G) = -(-1)^{(\epsilon_F+1)(\epsilon_G+1)}(G, F)$$

fulfils a graded Jacobi identity

$$((F, G), H) + (-1)^{(\epsilon_F+1)(\epsilon_G+\epsilon_H)}((G, H), F) + (-1)^{(\epsilon_H+1)(\epsilon_F+\epsilon_G)}((H, F), G) = 0$$

and a graded Leibniz rule

$$(F, GH) = (F, G)H + (-1)^{\epsilon_F \epsilon_G} G(F, H).$$

Therefore $(\cdot, \cdot)$ defines a so-called odd symplectic structure (see Section 4.2). Its properties are

$$(F, F) = 0 \quad \text{for any fermion}$$

$$\text{and} \quad (B, B) = 2\frac{\partial_r B}{\partial \Phi^A} \frac{\partial_l B}{\partial \Phi^*_A} \neq 0 \quad \text{for any boson,}$$

which is opposite to the expectation for the Poisson bracket. The functions on the space of fields and antifields form together with the Batalin-Vikovisky bracket a graded algebra, as do the functions on phase space in the case of the generalized Poisson bracket.

3.2. The Master Equation and its BRST Symmetry

Start with a functional $S[\Phi^A; \Phi^*_A]$, which has the dimension of an action, zero ghost number $\mathrm{gh}[S] = 0$ and even statistics $\epsilon_S = 0$. The classical master equation requires that the Batalin-Vilkovisky bracket of this bosonic functional vanishes,

$$(S, S) = 2\frac{\partial_r S}{\partial \Phi^A} \frac{\partial_l S}{\partial \Phi^*_A} = 0.$$

Not every solution of the master equation produces a dynamical system, only the proper solution is of interest. This kind of solution contains the original action and is obtained with the ghost number restriction, and boundary conditions which guarantee the postulates of a gauge theory.

The proper solution for an irreducible theory with a closed gauge algebra reads

$$S[\Phi, \Phi^*] = S_0[\phi] + \phi^*_i T^i_\alpha C^\alpha + C^*_\alpha f^\alpha_{\beta\gamma}(-1)^{\epsilon_\beta} C^\beta C^\gamma + \cdots \tag{3}$$

with the boundary conditions $S[\Phi, \Phi^*]|_{\Phi^*=0} = S_0[\phi]$, which guarantee the classical limit and

$$\frac{\partial_l \partial_r S}{\partial \phi^*_i \partial C^\alpha}\bigg|_{\Phi^*=0} = T^i_\alpha(\phi)$$

which reflect the Noether identities. For the commutator relation (1) a shift in the Grassmann parity due to the shift in the antibracket is taken into account. The reason for and consequences of this shift can be found in [6] in the chapter about graded algebras. The master equation $(S, S) = 0$ is a very compact notation, with certain requirements it determines all gauge structure equations. The proper solution S is unique up to canonical transformations and the addition of trivial pairs.

The Batalin-Vilkovisky bracket between a field and an antifield is

$$(\Phi^A, \Phi^*_B) = \delta^A_B.$$

The canonical transformations for fields and antifields are

$$\Phi^A \longrightarrow \bar{\Phi}^A = \Phi^A + \varepsilon(\Phi^A, F) + \mathcal{O}(\varepsilon^2)$$

$$\Phi^*_A \longrightarrow \bar{\Phi}^*_A = \Phi^*_A + \varepsilon(\Phi^*_A, F) + \mathcal{O}(\varepsilon^2)$$

which preserves the antibracket up to the order of ε^2

$$(\bar{\Phi}^A, \bar{\Phi}^*_B) = \delta^A_B + \mathcal{O}(\varepsilon^2).$$

The proper solution S has classical BRST-symmetry, which is a substitute for the gauge invariances

$$\delta_S F \equiv (F, S) \quad \text{with} \quad F = F[\Phi, \Phi^*]$$

where the generator δ_S for the symmetry is S, the proper solution itself. The transformation rules are

$$\delta_S \Phi^A = \frac{\partial_l S}{\partial \Phi_A^*} \quad \text{and} \quad \delta_S \Phi_A^* = -\frac{\partial_l S}{\partial \Phi^A} = (-1)^{(\epsilon_A + 1)} \frac{\partial_r S}{\partial \Phi^A}.$$

The symmetry of the action is guaranteed by the master equation

$$\delta_S S = 0 \quad \Longleftrightarrow \quad (S, S) = 0.$$

δ_S is a nilpotent graded derivation

$$\delta_S^2 F = 0$$
$$\delta_S(FG) = F(\delta_S G) + (-1)^{\epsilon_G}(\delta_S F)G$$

as follows from the properties of the antibracket, which imply

$$((F, S), S) = -((S, S), F) + (-1)^{\epsilon_F + 1}((S, F), S) \qquad \text{(Jacobi identity)}$$
$$= -((S, S), F) - ((F, S), S) \qquad\qquad\qquad \text{(antisymmetry)}.$$

This leads to

$$((F, S), S) = -\frac{1}{2}((S, S), F) = 0$$

and therefore

$$\delta_S^2 F = ((F, S), S) = 0.$$

4. The AKSZ Formalism

The Alexandrov-Kontsevich-Schwarz-Zabronsky formalism [1] reflects the geometry of the master equation. The solution S of the classical master equation $(S, S) = 0$, which specifies a classical mechanical system, can be geometrically considered as a QP-manifold. This is a supermanifold N, equipped with an odd self-commuting vector field Q, $[Q, Q] = 0$ and an odd symplectic structure ω, which is Q-invariant. $\mathcal{F}(N)$ will denote the $\mathbb{Z}_2$-graded algebra of functions on this supermanifold. First I will introduce the concept of supermanifolds in this context and use the definition of DeWitt [8].

4.1. Supermanifolds

An m-dimensional **manifold** M is a topological space provided with a collection of ordered pairs (U_i, φ_i). The U_i are a family of open sets which covers M

$$U_i \subset M, \quad \cup_i U_i = M$$

and φ_i is a homeomorphism onto a subspace of $\mathbb{R}^m$,

$$\varphi_i : U_i \subset M \longrightarrow U_i' \subset \mathbb{R}^m.$$

The coordinate functions are

$$\varphi_i : M \longrightarrow \mathbb{R}^m$$
$$p \longmapsto (x^1(p), \ldots, x^m(p))$$

where the x^i are the coordinates in M. For $U_i \cap U_j \neq \emptyset$, $\varphi_i \circ \varphi_j^{-1}$ is differentiable. A (m, n)-dimensional **supermanifold** M is a space provided with a collection of ordered pairs (U_A, ϕ_A). The U_A are subsets of M with the property

$$U_A \subset M, \quad \cup_A U_A = M$$

and ϕ_A is an one-to-one mapping onto an open set in $\mathbb{R}_c^m \times \mathbb{R}_a^n$, the commuting and anticommuting subspaces of a Grassmann algebra Λ_∞

$$\phi_A : U_A \subset M \longrightarrow U'_A \subset \mathbb{R}_c^m \times \mathbb{R}_a^n.$$

The coordinates on a supermanifold are defined by

$$\phi_A : M \longrightarrow \mathbb{R}_c^m \times \mathbb{R}_a^n$$
$$p \longmapsto (q^1, \ldots, q^m; \theta^1, \ldots, \theta^n).$$

For $U_A \cap U_B \neq \emptyset$, $\phi_A \circ \phi_B^{-1}$ is differentiable. The main difference to ordinary manifolds is that the underlying space M is not topological, e.g. on the supermanifold no distance exists.

A Grassmann algebra Λ_∞ is generated by anticommuting generators ξ

$$\xi^a \xi^b + \xi^b \xi^a = 0, \quad \text{with} \quad (\xi^a)^2 = 0.$$

An element of the Grassmann algebra can be represented as

$$z = c_0 + c_a \xi^a + \frac{1}{2!} c_{ab} \xi^a \xi^b + \frac{1}{3!} c_{abc} \xi^a \xi^b \xi^c + \cdots$$

a so-called supernumber. This can be decomposed into terms with an odd or even numbers of generators

$$z = q + \theta$$

with

$$q = c_0 + \frac{1}{2!} c_{ab} \xi^a \xi^b + \cdots, \quad \theta = c_a \xi^a + \frac{1}{3!} c_{abc} \xi^a \xi^b \xi^c \cdots,$$

i.e. in commuting c-numbers $\mathbb{R}_c$, $\mathbb{C}_c$ and anticommuting a-numbers $\mathbb{R}_a$, $\mathbb{C}_a$

$$\Lambda_N = \mathbb{R}_c \oplus \mathbb{R}_a$$

where only $\mathbb{R}_c$ forms a subalgebra.

4.2. The Geometry of the Master Equation

A symplectic structure on a supermanifold N is defined as a closed, non-degenerate 2-form

$$\omega = \frac{1}{2}\, \mathrm{d}z^a \omega_{ab}(z)\, \mathrm{d}z^b$$

with the local coordinates

$$\{z^1, \ldots, z^n\} \text{ in } N, \quad \text{with parity} \quad \epsilon_a = \epsilon(z^a).$$

The **symplectic structure** can be **even** or **odd** with respect to the $\mathbb{Z}_2$-grading of $\mathcal{F}(N)$.

In the even case the degree of the symplectic form is

$$\deg \omega_{ab} = \epsilon_a + \epsilon_b \quad (\mathrm{mod}\ 2)$$

$$\omega_{ab} = (-1)^{(\epsilon_a+1)(\epsilon_b+1)} \omega_{ba}$$

while an odd symplectic structure satisfies

$$\deg \omega_{ab} = \epsilon_a + \epsilon_b + 1 \quad (\mathrm{mod}\ 2)$$

$$\omega_{ab} = (-1)^{\epsilon_a \epsilon_b + 1} \omega_{ba}.$$

The change of sign in $\mathrm{d}z^a\, \mathrm{d}z^b = -(-1)^{\epsilon_a \epsilon_b}\, \mathrm{d}z^b\, \mathrm{d}z^a$ is according to the Koszul sign rule. Define a bracket for functions $f, g \in \mathcal{F}(N)$

$$(f, g) = \frac{\partial_r f}{\partial z^a} \omega^{ab} \frac{\partial_l g}{\partial z^b}.$$

In the even case it corresponds to the generalized Poisson bracket, because its degree is 0. In the odd case the bracket equips the algebra with a grading $\epsilon_{(\cdot,\cdot)} = 1$, like the BV-bracket. In general it will be called an **odd Poisson bracket**. The odd case is the one of main interest, because it produces the BV-bracket.

Associate to each $f \in \mathcal{F}(N)$ a vector field X_f, defined by

$$(f, g) = X_f(g) \quad \text{for all} \quad g \in \mathcal{F}(N).$$

For an even bracket X_f has the same parity as f, while in the odd case X_f has the opposite parity to f. The connection between the symplectic form, the vector field and the bracket is

$$\iota_{X_f} \omega = \mathrm{d}f$$

$$(f, g) =: X_f(g) = \iota_{X_f} \iota_{X_g} \omega.$$

The fundamental construction of a QP-manifold is given by the following definitions.

Definition 1. *The supermanifold N, which is considered in this context, can be constructed by associating to an ordinary manifold Σ its tangent bundle $T\Sigma$ and reversing the parity of the vector fields in the fiber. This is a simple turning of even (bosonic) vector fields into odd (fermionic) vector fields by a change of the corresponding variables, the result is denoted by $N = \Pi T\Sigma$. The construction is also possible with the cotangent bundle, which leads to $N = \Pi T^*\Sigma$.*

Definition 2. *A* **Q***-manifold is a supermanifold N with an odd self-commuting vector field Q*

$$\hat{Q} = Q^a \frac{\partial}{\partial z^a} \quad \text{in local coordinates, with} \quad \deg Q^a = \epsilon_a + 1 \quad (\text{mod } 2)$$

$$[Q, Q] = 0 \Longleftrightarrow \hat{Q}^2 = 0$$

where $\hat{Q}$ denotes the first order differential operator corresponding to the vector field Q. Then $[Q, Q] = 2Q^2 = 2(Q^b \partial_b Q^a)\partial_a$, so a Q-structure is the choice of a differential on $\mathcal{F}(N)$.

Definition 3. *A* **P***-manifold is a supermanifold N with an odd symplectic structure ω*

$$\omega = \frac{1}{2} dz^a \omega_{ab}(z)\, dz^b, \quad \deg \omega_{ab} = \epsilon_a + \epsilon_b + 1 \quad (\text{mod } 2).$$

Proposition 1. *There exists a Lie algebra homomorphism*

$$f \longrightarrow X_f$$
$$(f, g) \longrightarrow [X_f, X_g]$$

which maps an odd Poisson or BV-bracket of functions into a super commutator bracket of vector fields.

Definition 4. *A vector field X can be represented in the form X_f iff ω is X-invariant, i.e.*

$$\mathcal{L}_X \omega = 0.$$

Then X and ω are said to be **compatible***.*

Cartan's magic formula for the Lie derivative is

$$\mathcal{L}_X = d\iota_X + \iota_X d$$

so

$$\mathcal{L}_X \omega = d\iota_X \omega + \iota_X d\omega$$

and

$$\mathcal{L}_{X_f} \omega = d(\iota_{X_f} \omega) = d^2 f = 0$$

since $d\omega = 0$ because ω is a symplectic form, and $\iota_{X_f}\omega = df$ for a Hamiltonian f and a Hamiltonian vector field X_f.

Definition 5. *A* **QP**-*manifold is a supermanifold with a compatible Q- and P-structure, i.e. the supermanifold is equipped with an odd self-commuting vector field Q and an odd symplectic structure ω which are compatible, $\mathcal{L}_Q \omega = 0$.*

The conclusion is that if

$$\iota_Q \omega = \mathrm{d}S$$

for some function S, then Q is a Hamiltonian vector field and S is a Hamiltonian. This function S is even and fulfils $(S, S) = 0$, so every solution to the classical master equation determines a QP-structure and vice versa. The main point is that the geometrical structure, the QP-manifold, produces solutions of the master equation, which does not have to be solved in the usual way. This geometrical construction includes the algebraic structures and is a natural generalization of the concept of Poisson manifolds and Lie algebroids to describe gauge theory [18, 21].

4.3. The Tangent and Cotangent Bundle as QP-Manifolds

The most important examples for Q- and P-manifolds are the tangent and cotangent bundle with reversed parity in the fibers, ΠTN and ΠT^*N. The base manifold N is allowed to be a supermanifold.

Starting with a ordinary manifold N, the bundle ΠTN is a supermanifold with a natural Q-structure

$$\hat{Q}_{\Pi TN} = \eta^a \frac{\partial}{\partial x^a} \quad \text{with the coordinates } x^a \text{ in } N$$

$$\text{and the coordinates } \eta^a \text{ in the fibers of } \Pi TN.$$

The action of $\hat{Q}$ on the coordinates is as follows

$$\hat{Q}_{\Pi TN} x^a = \eta^b \frac{\partial}{\partial x^b} x^a = \eta^b \delta_b^a = \eta^a$$

$$\hat{Q}_{\Pi TN}(\hat{Q}_{\Pi TN} x^a) = \hat{Q}_{\Pi TN} \eta^a = \eta^b \frac{\partial}{\partial x^b} \eta^a = 0 \quad \Longrightarrow \quad \hat{Q}_{\Pi TN}^2 = 0$$

so $\hat{Q}_{\Pi TN}$ is a nilpotent operator.

On the cotangent bundle ΠT^*N a natural P-structure can be defined in an analog way

$$\omega_{\Pi T^*N} = \mathrm{d}x^a \mathrm{d}x_a^* \quad \text{with the coordinates } x^a \text{ in } N$$

$$\text{and the coordinates } x_a^* \text{ in the fibers of } \Pi T^*N.$$

The matrix elements ω_{ab} can be taken as the matrix elements of the canonical symplectic form, such a coordinate system can always defined on the cotangent bundle [17].

These fundamental structures will play an important role later on, because the representation is the same when N is a supermanifold. Moreover, the Q- and P-structures can in general be made compatible (see section 4.6), so the tangent and the cotangent bundle can always be viewed as a QP-manifold.

The functions on ΠTN can be identified with the differential forms on N, so the Q-structure is the deRham differential operator, $\hat{Q}_{\Pi TN} = d_{\text{deRham}}$.

$$\mathbb{H}(\Pi TN, \hat{Q}) = \frac{\ker \hat{Q}}{\operatorname{im} \hat{Q}} \simeq \mathbb{H}(N, d_{\text{deRham}}) = \frac{\ker d}{\operatorname{im} d}$$

A submanifold Y, which is a restriction to the fixed points of Q, can be identified with the original base manifold N

$$Y = \{m \in \Pi TN; \ Q(m) = 0\} \simeq N \subset \Pi TN$$

so the homology groups $\mathbb{H}_m$ of Y are trivial.

Let M be an arbitrary Q-manifold and let $\mathbb{H}_m = 0, \forall m \in Y$, then one can find a local coordinate system such that $\hat{Q} = \eta^a \frac{\partial}{\partial x^a}$ in a neighborhood of every $m \in Y$. In a neighborhood of Y one can identify $M \simeq \Pi TN$. Moreover, if M is a manifold the identification can be done globally [1]. The main point is that under certain conditions an arbitrary Q-manifold is endowed with the fundamental Q-structure in the convenient coordinate representation.

4.4. Actions on Q-Manifold

If a group G acts freely on a Q-manifold M

$$\sigma : G \times M \longrightarrow M$$

preserving the Q-structure, then one can define a Q-structure on the quotient space M/G, where functions on the quotient space can be identified with G-invariant functions on the original Q-manifold M. The map

$$\hat{Q} : G\text{-invariant functions on } M \longrightarrow G\text{-invariant functions on } M/G$$

specifies a Q-structure on M/G.

Let $M = \Pi TG$, where G is a Lie group which acts in a natural way on ΠTG

$$\sigma : G \times \Pi TG \longrightarrow \Pi TG.$$

Functions on the quotient space $\Pi TG/G \simeq \Pi\mathfrak{g}$ can be identified with G-invariant functions on ΠTG and one gets in this way a Q-structure on $\Pi\mathfrak{g}$, with

$$\hat{Q}_{\Pi\mathfrak{g}} = f^\alpha_{\beta\gamma} c^\beta c^\gamma \frac{\partial}{\partial c^\alpha}$$

where the $f^\alpha_{\beta\gamma}$ are the structure constants of the Lie algebra $\mathfrak{g}$. They fulfil:

$$f^\gamma_{\alpha\beta} = -f^\gamma_{\beta\alpha} \quad \text{and} \quad f^\gamma_{[\alpha\beta} f^\varepsilon_{\delta]\gamma} = 0 \quad \text{for the Jacobi identity.}$$

The explicit form for the Lie bracket is not necessary here, only the general properties are needed. To show that $\hat{Q}_{\Pi\mathfrak{g}}$ is a nilpotent operator, we need the graded Leibniz identity

$$\frac{\partial}{\partial c^\eta}c^\beta c^\gamma = \frac{\partial c^\beta}{\partial c^\eta}c^\gamma + (-1)^{\epsilon_\beta \epsilon_\gamma}c^\beta\frac{\partial c^\gamma}{\partial c^\eta}$$

which gives rise to a minus sign, since the parity of the c's are odd:

$$\hat{Q}_{\Pi\mathfrak{g}}c^\delta = f^\alpha_{\beta\gamma}c^\beta c^\gamma\frac{\partial}{\partial c^\alpha}c^\delta = f^\delta_{\beta\gamma}c^\beta c^\gamma$$

$$\hat{Q}_{\Pi\mathfrak{g}}(\hat{Q}_{\Pi\mathfrak{g}}c^\delta) = f^\eta_{\xi\varepsilon}c^\xi c^\varepsilon\frac{\partial}{\partial c^\eta}(f^\delta_{\beta\gamma}c^\beta c^\gamma)$$

$$= f^\eta_{\xi\varepsilon}c^\xi c^\varepsilon f^\delta_{\beta\gamma}\delta^\beta_\eta c^\gamma - f^\eta_{\xi\varepsilon}c^\xi c^\varepsilon f^\delta_{\beta\gamma}c^\beta \delta^\gamma_\eta$$

$$= f^\eta_{\xi\varepsilon}c^\xi c^\varepsilon f^\delta_{\eta\gamma}c^\gamma - f^\eta_{\xi\varepsilon}c^\xi c^\varepsilon f^\delta_{\beta\eta}c^\beta$$

$$= 2f^\eta_{\xi\varepsilon}f^\delta_{\eta\gamma}c^\xi c^\varepsilon c^\gamma = 0.$$

This equation vanishes by an antisymmetrization of the relevant indices because of the Jacobi identity for the structure constants

$$f^\eta_{[\xi\varepsilon}f^\delta_{\gamma]\eta} = 0.$$

In this way $\hat{Q}$ is a nilpotent operator

$$\hat{Q}^2_{\Pi\mathfrak{g}} = 0.$$

If we define a G-manifold X as a manifold with an action of a Lie group

$$\sigma : G \times X \longrightarrow X$$

then the product manifold $X \times \Pi TG$ has a natural Q-structure, since X is endowed with a trivial Q-structure. Then G acts on $X \times \Pi TG$

$$\sigma : G \times \underbrace{X \times \Pi TG}_{\text{product } Q\text{-mf.}} \longrightarrow X \times \Pi TG.$$

Now we are able to introduce a Q-structure on the quotient space of the product space, i.e. $X \times \Pi TG/G \simeq X \times \Pi\mathfrak{g}$

$$\hat{Q}_{X\times\Pi\mathfrak{g}} = T^a_\alpha c^\alpha\frac{\partial}{\partial x^a} + f^\alpha_{\beta\gamma}c^\beta c^\gamma\frac{\partial}{\partial c^\alpha}, \quad \text{where} \quad \hat{Q}^2 = 0 \quad \text{is to be proved.}$$

The first term comes from the trivial Q-structure on X, keeping in mind the general form $\hat{Q} = \eta^a\frac{\partial}{\partial x^a}$ and take $T^a_\alpha(x)$ as vector fields on X corresponding to the generators of the Lie algebra $t_\alpha \in \mathfrak{g}$. The second term is the well-known Q-structure of $\Pi\mathfrak{g}$:

$$\hat{Q}_{X\times\Pi\mathfrak{g}}c^\delta = \hat{Q}_{\Pi\mathfrak{g}}c^\delta \quad \Longrightarrow \quad \hat{Q}^2_{X\times\Pi\mathfrak{g}}c^\delta = 0.$$

So $\hat{Q}^2_{X \times \Pi\mathfrak{g}}$ with respect to c^δ is equal to zero:

$$\hat{Q}_{X \times \Pi\mathfrak{g}} x^b = T^a_\alpha c^\alpha \frac{\partial}{\partial x^a} x^b = T^b_\alpha c^\alpha$$

$$\hat{Q}_{X \times \Pi\mathfrak{g}} (\hat{Q}_{X \times \Pi\mathfrak{g}} x^b) = T^c_\delta c^\delta \frac{\partial}{\partial x^c} (T^b_\alpha c^\alpha) + f^\delta_{\eta\xi} c^\eta c^\xi \frac{\partial}{\partial c^\delta} (T^b_\alpha c^\alpha)$$

$$= T^c_\delta c^\delta \frac{\partial T^b_\alpha}{\partial x^c} c^\alpha + f^\delta_{\eta\xi} c^\eta c^\xi T^b_\delta.$$

With the Lie algebra homomorphism from Proposition 1 one gets a graded Lie bracket for odd vector fields, thereby follows a commutator relation for the T^a_α

$$f^\delta_{\eta\xi} T^b_\delta = \frac{1}{2} \left(T^c_\xi \frac{\partial T^b_\eta}{\partial x^c} - (-1)^{\epsilon_\xi \epsilon_\eta + 1} T^c_\eta \frac{\partial T^b_\xi}{\partial x^c} \right) \tag{4}$$

and the above formula turns with the substitution $\eta\xi \to \delta\alpha$ into

$$\hat{Q}_{X \times \Pi\mathfrak{g}} (\hat{Q}_{X \times \Pi\mathfrak{g}} x^b) = \left(T^c_\delta \frac{\partial T^b_\alpha}{\partial x^c} + \frac{1}{2} \left(T^c_\alpha \frac{\partial T^b_\delta}{\partial x^c} - T^c_\delta \frac{\partial T^b_\alpha}{\partial x^c} \right) \right) c^\alpha c^\delta$$

$$= \frac{1}{2} \left(T^c_\delta \frac{\partial T^b_\alpha}{\partial x^c} + T^c_\alpha \frac{\partial T^b_\delta}{\partial x^c} + \underbrace{T^c_\delta \frac{\partial T^b_\alpha}{\partial x^c} - T^c_\delta \frac{\partial T^b_\alpha}{\partial x^c}}_{=0} \right) c^\alpha c^\delta$$

$$= \frac{1}{2} \left(-T^c_\alpha \frac{\partial T^b_\delta}{\partial x^c} + T^c_\alpha \frac{\partial T^b_\delta}{\partial x^c} \right) c^\alpha c^\delta = 0$$

$$\implies \hat{Q}^2_{X \times \Pi\mathfrak{g}} x^b = 0.$$

Comparing the relation (4) for vector fields with the bracket relation (2) for the generators of the gauge algebra, one sees that the Q-structure is only nilpotent when the Noether identities are satisfied, or the Noether identities are satisfied when the Q-structure is nilpotent.

4.5. Construction of the Extended Action

Start with a Q-manifold M and a P-manifold $\Pi T^* M$, keeping in mind, that every supermanifold N has a natural P-structure on $\Pi T^* N$. The Q-structure on M induces a Q-structure on $\Pi T^* M$, while the P-structure on $\Pi T^* M$ is Q-invariant, $\mathcal{L}_Q \omega = 0$, so $\Pi T^* M$ is a QP-manifold. It holds that every compact P-manifold is of the form $\Pi T^* N$, [1].

Apply the construction to the Q-manifold $M = X \times \Pi\mathfrak{g}$, where X is a G-manifold and $\mathfrak{g}$ its Lie algebra. On

$$\Pi T^* M = \Pi T^* (X \times \Pi\mathfrak{g}) = \Pi T^* X \times \Pi\mathfrak{g} \times \mathfrak{g}^* \equiv E$$

one obtains a QP-structure

$$\hat{Q}_E = T_\alpha^a c^\alpha \frac{\partial}{\partial x^a} - x_a^* \frac{\partial T_\alpha^a}{\partial x^b} c^\alpha \frac{\partial}{\partial x_b^*} + f_{\beta\gamma}^\alpha c^\beta c^\gamma \frac{\partial}{\partial c^\alpha} - (x_a^* T_\beta^a + 2 c_\alpha^* f_{\beta\gamma}^\alpha c^\gamma) \frac{\partial}{\partial c_\beta^*} \quad (5)$$

$$\omega = \mathrm{d}z^a \omega_{ab} dz^b, \text{ with } z^a = (x^a, c^\alpha; x_a^*, c_\alpha^*) \quad (6)$$

with the coordinates x^a in X, the base G-manifold; x_a^* in the fibers of $\Pi T^* X$; c^α in the Lie algebra with reversed parity $\Pi\mathfrak{g}$ and c_α^* in the dual of the Lie algebra $\mathfrak{g}$. This $\hat{Q}_E$ is also nilpotent.

$$\hat{Q}_E x^b = T_\alpha^b c^\alpha$$

$$\hat{Q}_E(\hat{Q}_E x^b) = \hat{Q}_E(\hat{Q}_{X \times \Pi\mathfrak{g}} x^b) = \hat{Q}_{X \times \Pi\mathfrak{g}}^2 x^b = 0$$

$$\text{where} \quad \hat{Q}_{X \times \Pi\mathfrak{g}}^2 = 0 \quad \text{implies} \quad \hat{Q}_E^2 x^b = 0$$

$$\hat{Q}_E c^\delta = f_{\beta\gamma}^\alpha c^\alpha c^\delta$$

$$\hat{Q}_E(\hat{Q}_E c^\delta) = \hat{Q}_E(\hat{Q}_{\Pi\mathfrak{g}} c^\delta) = \hat{Q}_{\Pi\mathfrak{g}}^2 c^\delta = 0$$

$$\text{where} \quad \hat{Q}_{\Pi\mathfrak{g}}^2 = 0 \quad \text{implies} \quad \hat{Q}_E^2 c^\delta = 0$$

$$\hat{Q}_E x_c^* = -x_a^* \frac{\partial T_\alpha^a}{\partial x^b} c^\alpha \delta_b^c = -x_a^* \frac{\partial T_\alpha^a}{\partial x^c} c^\alpha \equiv -x_a^* \frac{\partial T_\delta^a}{\partial x^c} c^\delta$$

$$\hat{Q}_E(\hat{Q}_E x_c^*) = -T_\alpha^d c^\alpha x_a^* \frac{\partial^2 T_\delta^a}{\partial x^a \partial x^c} c^\delta + x_d^* \frac{\partial T_\alpha^d}{\partial x^b} c^\alpha \delta_b^a \frac{\partial T_\delta^a}{\partial x^c} c^\delta - f_{\beta\gamma}^\alpha c^\beta c^\gamma \delta_\alpha^\delta x_a^* \frac{\partial T_\delta^a}{\partial x^c}$$

$$= -T_\alpha^d c^\alpha x_a^* \frac{\partial^2 T_\delta^a}{\partial x^a \partial x^c} c^\delta + x_d^* \frac{\partial T_\alpha^d}{\partial x^a} c^\alpha \frac{\partial T_\delta^a}{\partial x^c} c^\delta - f_{\beta\gamma}^\alpha c^\beta c^\gamma x_a^* \frac{\partial T_\alpha^a}{\partial x^c}$$

$$= \frac{\partial}{\partial x^c} \left(T_\alpha^d \frac{\partial T_\delta^a}{\partial x^a} c^\alpha c^\delta x_a^* + f_{\beta\gamma}^\alpha T_\alpha^a c^\beta c^\gamma x_a^* \right)$$

$$= \frac{\partial}{\partial x^c} (\hat{Q}_{X \times \Pi\mathfrak{g}}^2 x^b) x_a^*$$

where $\hat{Q}_{X \times \Pi\mathfrak{g}}^2 = 0$ implies $\hat{Q}_E^2 x_c^* = 0$.

For the last contribution one gets

$$\hat{Q}_E c_\delta^* = -(x_a^* T_\beta^a + 2 c_\alpha^* f_{\beta\gamma}^\alpha c^\gamma) \delta_\delta^\beta = -(x_a^* T_\delta^a + 2 c_\alpha^* f_{\delta\gamma}^\alpha c^\gamma)$$

$$\hat{Q}_E(\hat{Q}_E c_\delta^*) = -T_\alpha^a c^\alpha x_b^* \frac{\partial T_\delta^b}{\partial x^a} + x_a^* \frac{\partial T_\alpha^a}{\partial x^b} c^\alpha \delta_c^b T_\delta^c - 2 f_{\beta\gamma}^\alpha c^\beta c^\gamma \delta_\alpha^\varepsilon c_\xi^* f_{\delta\varepsilon}^\xi$$

$$+ 2(x_a^* T_\beta^a + 2 c_\alpha^* f_{\beta\gamma}^\alpha c^\gamma) \delta_\beta^\xi f_{\delta\varepsilon}^\xi c^\varepsilon$$

$$= -T_\alpha^a c^\alpha x_b^* \frac{\partial T_\delta^b}{\partial x^a} + x_a^* \frac{\partial T_\alpha^a}{\partial x^b} c^\alpha T_\delta^b - 2 f_{\beta\gamma}^\alpha c^\beta c^\gamma c_\xi^* f_{\delta\varepsilon}^\xi + 2 x_a^* T_\xi^a f_{\delta\varepsilon}^\xi c^\varepsilon$$

$$+ 4 c_\alpha^* f_{\beta\gamma}^\alpha c^\gamma f_{\delta\varepsilon}^\beta c^\varepsilon$$

$$= \underbrace{\left(-T_\alpha^a \frac{\partial T_\delta^b}{\partial x^a} + \frac{\partial T_\alpha^b}{\partial x^b} T_\delta^a \right) c^\alpha x_c^* + 2 x_a^* T_\xi^a f_{\delta\varepsilon}^\xi c^\varepsilon - 2 f_{\beta\gamma}^\alpha c^\beta c^\gamma c_\xi^* f_{\delta\varepsilon}^\xi}_{-2 f_{\alpha\delta}^\xi T_\xi^c}$$

$$+ 4 c_\alpha^* f_{\beta\gamma}^\alpha c^\gamma f_{\delta\varepsilon}^\beta c^\varepsilon .$$

With the commutator relation (4) the first term cancels against the second; the last two terms cancel against each other by antisymmetrization and use of the Jacobi identity for the structure constants. Then

$$\hat{Q}_E^2 c^\delta = 0 \quad \text{and} \quad \hat{Q}_E^2 = 0.$$

This $\hat{Q}$ on $E = \Pi T^* X \times \Pi \mathfrak{g} \times \mathfrak{g}^*$ leads by $\iota_Q \omega = \mathrm{d} S_0$ to an action

$$S_0 = x_a^* T_\alpha^a c^\alpha + c_\alpha^* f_{\beta\gamma}^\alpha c^\beta c^\gamma \tag{7}$$

with ghost number zero, $\mathrm{gh}[S_0] = 0$ and even Grassmann parity, $\epsilon_{S_0} = 0$, which fulfils the master equation $(S_0, S_0) = 0$!

For the calculation of $\iota_Q \omega = \mathrm{d} S_0$ we need the the product ι which is a pairing between the tangent space and its dual, respectively between a vector field and p-forms

$$\iota : \chi(M) \times \Omega^p(M) \longrightarrow \Omega^{p-1}(M)$$
$$(X, \omega) \longmapsto \iota_X \omega \equiv X \lrcorner \omega \equiv \omega(X)$$

defined by

$$(\iota_X \omega)(X_1, \ldots, X_{p-1}) \equiv (X \lrcorner \omega)(X_1, \ldots, X_{p-1})$$
$$\equiv \omega(X, X_1, \ldots, X_{p-1}).$$

The coupling between a vector field and an one-form is defined by

$$\langle \ , \ \rangle : T_x^* M \otimes T_x M \longrightarrow \mathbb{R}$$
$$(\omega, X) \longmapsto \langle \omega, X \rangle$$

which reads in the usual basis

$$\langle \omega, X \rangle \equiv \omega(X)$$
$$= \omega_\mu X^\nu \left\langle \mathrm{d} x^\mu, \frac{\partial}{\partial x^\nu} \right\rangle$$
$$= \omega_\mu X^\nu \delta_\nu^\mu = \omega_\mu X^\mu \in \mathbb{R}.$$

For the case of a coupling between a two-form and two vector fields, there is a determinant to calculate

$$(\omega_1 \wedge \omega_2)(X_1, X_2) = \det \begin{pmatrix} \omega_1(X_1) & \omega_1(X_2) \\ \omega_2(X_1) & \omega_2(X_2) \end{pmatrix}$$
$$= \omega_1(X_1)\omega_2(X_2) - \omega_1(X_2)\omega_2(X_1).$$

So the P-structure ω (6) which is a two-form, has to couple with two vector fields, one of which is our Q-structure (5), the other one is arbitrary, let us call it X. The only terms which contribute come from

$$\omega = \mathrm{d}x^a \mathrm{d}x_a^* + \mathrm{d}c^\alpha \mathrm{d}c_\alpha^*$$

$$(\iota_Q\omega)(X) = (\mathrm{d}x^a \mathrm{d}x_a^*)(T_\alpha^a c^\alpha \frac{\partial}{\partial x^a} - x_a^* \frac{\partial T_\alpha^a}{\partial x^b} c^\alpha \frac{\partial}{\partial x_b^*}, X)$$

$$+ (\mathrm{d}c^\alpha \mathrm{d}c_\alpha^*)(f_{\beta\gamma}^\alpha c^\beta c^\gamma \frac{\partial}{\partial c^\alpha} - (x_a^* T_\beta^a + 2c_\alpha^* f_{\beta\gamma}^\alpha c^\gamma)\frac{\partial}{\partial c_\beta^*}, X).$$

The first term leads to

$$T_\alpha^a c^\alpha \det \begin{pmatrix} \mathrm{d}x^a \left(\frac{\partial}{\partial x^a}\right) & \mathrm{d}x^a(X) \\ \mathrm{d}x_a^* \left(\frac{\partial}{\partial x^a}\right) & \mathrm{d}x_a^*(X) \end{pmatrix} - x_a^* \frac{\partial T_\alpha^a}{\partial x^b} c^\alpha \det \begin{pmatrix} \mathrm{d}x^a \left(\frac{\partial}{\partial x_b^*}\right) & \mathrm{d}x^a(X) \\ \mathrm{d}x_a^* \left(\frac{\partial}{\partial x_b^*}\right) & \mathrm{d}x_a^*(X) \end{pmatrix}$$

$$= T_\alpha^a c^\alpha \mathrm{d}x_a^*(X) - x_a^* \frac{\partial T_\alpha^a}{\partial x^b} c^\alpha (-\delta_b^a) \mathrm{d}x_a(X) = T_\alpha^a c^\alpha \mathrm{d}x_a^*(X) + x_b^* \frac{\partial T_\alpha^a}{\partial x^b} c^\alpha \mathrm{d}x_a(X).$$

The second one gives a contribution

$$f_{\beta\gamma}^\alpha c^\beta c^\gamma \det \begin{pmatrix} \mathrm{d}c^\alpha \left(\frac{\partial}{\partial c^\alpha}\right) & \mathrm{d}c^\alpha(X) \\ \mathrm{d}c_\alpha^* \left(\frac{\partial}{\partial c^\alpha}\right) & \mathrm{d}c_\alpha^*(X) \end{pmatrix}$$

$$- (x_a^* T_\beta^a + 2c_\alpha^* f_{\beta\gamma}^\alpha c^\gamma) \det \begin{pmatrix} \mathrm{d}c^\alpha \left(\frac{\partial}{\partial c_\beta^*}\right) & \mathrm{d}c^\alpha(X) \\ \mathrm{d}c_\alpha^* \left(\frac{\partial}{\partial c_\beta^*}\right) & \mathrm{d}c_\alpha^*(X) \end{pmatrix}$$

$$= f_{\beta\gamma}^\alpha c^\beta c^\gamma \mathrm{d}c_\alpha^*(X) - (x_a^* T_\beta^a + 2c_\alpha^* f_{\beta\gamma}^\alpha c^\gamma)(-\delta_\beta^\alpha) \mathrm{d}c^\alpha(X)$$

$$= f_{\beta\gamma}^\alpha c^\beta c^\gamma \mathrm{d}c_\alpha^*(X) + (x_a^* T_\alpha^a + 2c_\alpha^* f_{\beta\gamma}^\alpha c^\gamma) \mathrm{d}c^\beta(X).$$

In summary we have

$$\iota_Q\omega = T_\alpha^a c^\alpha \mathrm{d}x_a^* + x_b^* \frac{\partial T_\alpha^a}{\partial x^b} c^\alpha \mathrm{d}x_a + f_{\beta\gamma}^\alpha c^\beta c^\gamma \mathrm{d}c_\alpha^* + (x_a^* T_\alpha^a + 2c_\beta^* f_{\beta\gamma}^\alpha c^\gamma) \mathrm{d}c^\alpha = \mathrm{d}S_0$$

where

$$\mathrm{d}S_0 = \frac{\partial S_0}{\partial x_a^*} \mathrm{d}x_a^* + \frac{\partial S_0}{\partial x^a} \mathrm{d}x^a + \frac{\partial S_0}{\partial c_\alpha^*} \mathrm{d}c_\alpha^* + \frac{\partial S_0}{\partial x^\alpha} \mathrm{d}c^\alpha$$

with the partial derivatives from (7)

$$\frac{\partial S_0}{\partial x_a^*} = T_\alpha^a c^\alpha, \qquad \frac{\partial S_0}{\partial x^b} = x_a^* \frac{\partial T_\alpha^a}{\partial x^a} c^\alpha,$$

$$\frac{\partial S_0}{\partial c_\alpha^*} = f_{\beta\gamma}^\alpha c^\beta c^\gamma, \qquad \frac{\partial S_0}{\partial c^\alpha} = x_a^* T_\alpha^a + 2 c_\beta^* f_{\beta\gamma}^\alpha c^\gamma.$$

So an action of the form (3),

$$S[\Phi, \Phi^*] = S_0[\phi] + \phi_i^* T_\alpha^a C^\alpha + C_\alpha^* f_{\beta\gamma}^\alpha C^\beta C^\gamma + \cdots$$

the well-known Batalin-Vilkovisky action was derived. This was done by the construction of the space

$$E = \Pi T^* X \times \Pi \mathfrak{g} \times \mathfrak{g}^*$$

as a QP-manifold, which leads to an action

$$S_0 = x_a^* T_\alpha^a c^\alpha + c_\alpha^* f_{\beta\gamma}^\alpha c^\beta c^\gamma$$

which can be identified as the Batalin-Vilkovisky action with the comparison of the coordinates and fields with the respective Grassmann parities and ghost numbers.

Coordinates	Fields		Ghost number	Parity
$x^a \in X$	field	ϕ	0	$\epsilon_a = 0$
$x_a^* \in \Pi T^* X$	antifield	ϕ^*	-1	$\epsilon_a^* = \epsilon_a + 1 = 1$
$c^\alpha \in \Pi \mathfrak{g}$	ghost	C	1	$\epsilon_\alpha = 1$
$c_\alpha^* \in \mathfrak{g}^*$	antighost	C^*	-2	$\epsilon_\alpha^* = \epsilon_\alpha + 1 = 0$

So all the familiar structures of a gauge theory are achieved. The QP-manifolds can be viewed as a fiber bundle for gauge theories, where the algebraic structure reflects the Noether identities, which generate the symmetries.

4.6. Outlook

A more general solution can be created with an arbitrary extension s of the above action

$$S = s + S_0 = s(x) + x_a^* T_\alpha^a c^\alpha + c_\alpha^* f_{\beta\gamma}^\alpha c^\beta c^\gamma$$

where s is an G-invariant solution on $\Pi T X$, which leads to a G-invariant solution on X with $s = s(x)$. This is the classical part, a solution of the master equation. All in all the BV-action functional for an irreducible theory with m_0 gauge invariances was created in a geometrical way with fields of the form $\Phi^A = \{x^a, c^\alpha\}$ and the corresponding antifields $\Phi_A^* = \{x_a^*, c_\alpha^*\}$.

Given a manifold N with an even symplectic structure σ, then $\Pi T N$ is a Q-manifold and $\Pi T^* N$ a P-manifold. The P-structure on $M = \Pi T N = \Pi T^* N$ is Q-invariant, so M is a QP-manifold.

On $M = \Pi T^* N$ one has the standard P-structure

$$\omega_{\Pi T^* N} = \mathrm{d}x^a \mathrm{d}x_a^*$$

and the non-standard Q-structure

$$\hat{Q}_{\Pi T^* N} = -2\sigma^{ab}(x)x_b^* \frac{\partial}{\partial x^a} - x_a^* \frac{\partial \sigma^{ab}}{\partial x^c} x_b^* \frac{\partial}{\partial x_c^*}$$

which leads to an action

$$S = x_a \sigma^{ab}(x) x_b^*$$

where the x^a are the coordinates on N and the x_a^* are the coordinates in the fibers of $\Pi T^* N$.

On the other hand, one has on $M = \Pi T N$ the standard Q-structure

$$\hat{Q}_{\Pi T N} = \xi^a \frac{\partial}{\partial x^a}$$

and a non-standard P-structure

$$\omega_{\Pi T N} = \mathrm{d}x^a \mathrm{d}(\sigma_{ab}\xi^b)$$

with the coordinates x^a on N and $\xi^a = \sigma^{ab}x_b^*$ on $\Pi T N$. In general the P-structure can be expressed as

$$\omega = \mathrm{d}\left(\sum_{n=1}^{N} \Omega_{i_1 \cdots i_n}(x)\eta^{i_2} \cdots \eta^{i_n}\mathrm{d}x^{i_1}\right)$$

where Ω is a symplectic structure.

Small deformations of QP-manifolds of the form $M = \Pi T N = \Pi T^* N$ lead to a description of all structures. On $M = \Pi T N$ one has the advantage of a standard Q-structure, but the P-structure has to be deformed into

$$\omega \longrightarrow \omega_t = \omega - t\mathcal{L}_Q \mathrm{d}\sigma.$$

On $M = \Pi T^* N$ one has a standard P-structure, which leads to the BV-bracket and the master equation $(S, S) = 0$, with infinitesimal deformations s of S one obeys

$$\hat{Q}s = 0 \Longleftrightarrow s = (S, f) = \hat{Q}f.$$

In this way one can deform the algebraic structure due to [10, 19], different approaches lead to a more general quantization [4, 16] or quantization schemes for special models [7, 13, 14].

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