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PREQUANTIZATION OF SYMPLECTIC SUPERMANIFOLDS

GIJS M. TUYNMAN

UFR de Mathématiques, Université de Lille I
F-59655 Villeneuve d'Ascq Cedex, France

Abstract. This paper presents the formalism of symplectic supermanifolds with a non-homogeneous symplectic form and their prequantization.

1. Supermanifolds

The idea behind a supermanifold is that one wants to have *anticommuting* variables, i.e., a kind of "numbers" such that $\xi\eta = -\eta\xi$. In this context it is customary to denote ordinary/commuting (real) "numbers" by Latin characters and the anticommuting kind by Greek characters. One of the ideas to create such "numbers" is to replace the standard real line $\mathbb{R}$ by a graded commutative ring $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$ and to take the commuting "numbers" x_i in the even part: $x_i \in \mathcal{A}_0$ and to take the anticommuting variety ξ_j in the odd part: $\xi_j \in \mathcal{A}_1$. The basic example of such a ring is the exterior algebra of an infinite dimensional (real) vector space E

$$\mathcal{A} = \bigwedge E = \left(\bigoplus_{k=0}^{\infty} \bigwedge^{2k} E \right) \oplus \left(\bigoplus_{k=0}^{\infty} \bigwedge^{2k+1} E \right).$$

The first step in creating a theory of differential geometry based on these commuting and anticommuting "numbers," usually called even and odd coordinates, is to define what smooth functions are. When one tries to define the derivative of a function, one encounters immediately two problems: i) the most natural topology on the graded ring $\mathcal{A}$ is not Hausdorff making uniqueness of limits questionable and ii) due to nilpotent elements in $\mathcal{A}$ even a difference quotient is problematic. The solution adopted in [3] is based on the following two observations.

Lemma 1. *Let $U \subset \mathbb{R}^p$ be a convex open set and let $f : U \to \mathbb{R}^d$ be a function of class C^1. Then the function $g : U^2 \to \mathrm{End}(\mathbb{R}^p, \mathbb{R}^d) \cong \mathbb{R}^{pd}$ defined by*

$$g(x, y) = \int_0^1 f'\big(sx + (1 - s)y\big)\, \mathrm{d}s$$

is continuous and satisfies

$$\text{for all } x, y \in U : f(x) - f(y) = g(x,y)\cdot(x-y)\ .$$

Theorem 1. *Let $U \subset \mathbb{R}^p$ be a convex open set and let $f : U \to \mathbb{R}^d$ be any function. If there exists a continuous function $g : U^2 \to \mathrm{End}(\mathbb{R}^p, \mathbb{R}^d) \cong \mathbb{R}^{pd}$ satisfying*

$$f(x) - f(y) = g(x,y)\cdot(x-y) \quad \text{for all} \quad x, y \in U$$

then f is of class C^1 with $f'(x) = g(x,x)$.

It follows that functions of class C^1 can be defined without talking about difference quotients or limits. Only the topology is needed because g need to be continuous. Note that it is never required that g is unique (in general it will not be), but that the diagonal $g(x,x)$ is unique. Using this idea, one can show that smooth functions in p even coordinates and q odd coordinates $(x_1, \ldots, x_p, \xi_1, \ldots, \xi_q) \in \mathcal{A}_0^p \times \mathcal{A}_1^q$ are given as ordinary smooth functions of real variables $(x_1, \ldots, x_p)$ and polynomials of degree at most one in each of the ξ_j

$$C^\infty(\mathcal{A}_0^p \times \mathcal{A}_1^q) \cong C^\infty(\mathbb{R}^p) \otimes \bigwedge \mathbb{R}^q\ .$$

Once one has the notion of smooth functions, it is fairly straightforward to develop differential geometry: manifolds, tangent bundles, differential forms and so forth. The "only" precaution one has to take is that when one interchanges to items, an additional sign appears when it contains two odd variables. For instance, if ξ and η are two odd coordinates, we have the one-forms $\mathrm{d}\xi$ and $\mathrm{d}\eta$, and a two-form $\mathrm{d}\xi\wedge\mathrm{d}\eta$, but also the two-forms $\mathrm{d}\xi \wedge \mathrm{d}\xi$ and $\mathrm{d}\eta \wedge \mathrm{d}\eta$, simply because when we interchange $\mathrm{d}\xi$ with $\mathrm{d}\xi$ in this wedge product, *two* signs appear: one from interchanging two one-forms and one from interchanging the odd coordinates ξ and ξ. On the other hand, if x is an even coordinate, we still have $\mathrm{d}x \wedge \mathrm{d}\xi = -\mathrm{d}\xi \wedge \mathrm{d}x$.

2. Symplectic Supermanifolds

If we try to mimick the symplectic geometry, it seems reasonable to look for closed non-degenerate two-forms, where non-degenerate means that the induced map from the tangent space to the cotangent space is a bijection. So let us look at an example.

Example 1. On the manifold $M = \mathcal{A}_0^2 \times \mathcal{A}_1^2$ with two even and two odd coordinates: $(x, y, \xi, \eta) \in \mathcal{A}_0^2 \times \mathcal{A}_1^2$ we consider the two-form

$$\omega = \mathrm{d}x \wedge \mathrm{d}y + \mathrm{d}\xi \wedge \mathrm{d}\eta + \mathrm{d}x \wedge \mathrm{d}\xi\ .$$

It is straightforward to check that this is a closed and non-degenerate two-form. We also consider the two vector fields X and Y defined as

$$X = 2y\,\frac{\partial}{\partial x} - 2y\,\frac{\partial}{\partial \eta} \qquad \text{and} \qquad Y = -\xi\,\frac{\partial}{\partial \xi} + \eta\,\frac{\partial}{\partial \eta} + \xi\,\frac{\partial}{\partial y}\cdot$$

These two vector fields satisfy the relations

$$\iota(X)\omega = \mathrm{d}(y^2) \qquad \text{and} \qquad \iota(Y)\omega = \mathrm{d}(\eta\xi)$$

which means that these two vector fields are (globally) hamiltonian. A simple computation shows that their commutator is given as

$$[X, Y] = -2\xi \, \frac{\partial}{\partial x} - 2y \, \frac{\partial}{\partial \eta} - 2\xi \, \frac{\partial}{\partial \eta} \cdot$$

But $\iota([X, Y])\omega = \mathrm{d}(y\xi) + 2\xi \, \mathrm{d}\xi$ is not even closed. It follows that the commutator of two globally hamiltonian vector fields is not even a locally hamiltonian vector field.

In view of the above example, the question is: should we change the definition of (globally/locally) hamiltonian vector field, should we require that the closed and non-degenerate two-form is also homogeneous or ...? The solution I adopt in [2] is that we should change the definition of non-degeneracy! Any differential k-form α splits into an even (α_0) and odd (α_1) part: $\alpha = \alpha_0 + \alpha_1$. To see whether a form is even or odd, one just adds the number of odd coordinates in α to the degree k and takes the result modulo 2. For instance $\mathrm{d}\xi \wedge \mathrm{d}\xi$ is even, but $\mathrm{d}x \wedge \mathrm{d}\xi$ is odd.

Definition 1. *A closed two-form $\omega = \omega_0 + \omega_1$ on M is said to be* ***symplectic*** *if for all $m \in M$ we have*

$$\ker(\omega_0 : T_m \to T_m^*) \cap \ker(\omega_1 : T_m \to T_m^*) = \{0\} \, .$$

Remark 1. If a closed two-form ω satisfies the condition $\ker(\omega : T_m \to T_m^*) = \{0\}$, i.e., if it is non-degenerate in the classical sense, then it is symplectic in the sense of the above definition.

Definition 2. *A vector field X on a symplectic manifold (M, ω) is* ***locally/globally Hamiltonian*** *if the two one-forms*

$$\iota(X)\omega_0 \qquad \textit{and} \qquad \iota(X)\omega_1$$

are closed/exact.

Theorem 2. *The commutator of two locally hamiltonian vector fields is globally hamiltonian.*

Definition 3. *The Poisson algebra $\mathcal{P}$ of a symplectic manifold is given by*

$$\mathcal{P} = \{(f_0, f_1) \in C^\infty(M)^2 \, ; \, \exists X : \iota(X)\omega_0 = -\mathrm{d}f_0 \textit{ and } \iota(X)\omega_1 = -\mathrm{d}f_1 \, \} \, .$$

Remark 2. With this definition we abandon the idea that the Poisson algebra is/should be the set of smooth functions on the (symplectic) manifold. It becomes a subset of the set of all pairs of smooth functions. And indeed, it is fairly easy to construct symplectic manifolds for which $\mathcal{P}$ is a proper subset the set of all pairs of smooth functions. Even the idea that the map $\mathcal{P} \to C^\infty$, $(f_0, f_1) \mapsto f_0 + f_1$

is surjective turns out to be false in general. However, in the case the symplectic form is homogeneous (either ω_0 or ω_1 zero), the condition of being symplectic becomes the classical one of non-degeneracy and for the Poisson algebra we can "forget" about the component (in the pair) corresponding to the zero two-form. For homogeneous symplectic forms we thus recover the classical Poisson algebra.

Definition 4. *For an element $f = (f_0, f_1) \in \mathcal{P}$, the unique vector field X satisfying $\iota(X)\omega_\alpha = -\mathrm{d}f_\alpha$ is called the* ***Hamiltonian vector field*** *of f and denoted as X_f. The Poisson bracket $\{f, g\}$ of two elements $f, g \in \mathcal{P}$ is defined as $\{f, g\}_\alpha = X_f g_\alpha$.*

Theorem 3. *The Poisson bracket satisfies the conditions of a super Lie algebra structure and the map $f \mapsto X_f$ is an even homomorphism of (super) Lie algebras.*

Definition 5. *A* ***symmetry group*** *of a symplectic (super)manifold is a smooth left-action $\Phi : G \times M \to M$ of a Lie supergroup G on M preserving the symplectic form*

$$\Phi_g^*\omega = \omega, \qquad g \in G$$

where $\Phi_g : M \to M$ denotes the map $m \mapsto \Phi(g, m)$.

Definition 6. *A* ***momentum map*** *for a symmetry group G with Lie algebra $\mathfrak{g}$ of a symplectic manifold (M, ω) is a map $J : M \to \mathfrak{g}^*$ satisfying the condition*

$$\iota(v^M)\omega_\alpha = -\mathrm{d}\langle v \,|\, J_\alpha\rangle, \quad v \in \mathfrak{g}$$

where v^M denotes the fundamental vector field on M associated to the Lie algebra element v.

If there exists a momentum map for the symmetry group G, one says that the action is (weakly) hamiltonian. The action is said to be strongly hamiltonian if the map $\mathfrak{g} \to \mathcal{P}$, $v \mapsto \langle v \,|\, J\rangle$ is a Lie algebra morphism.

Theorem 4. *Let G be a Lie group with Lie algebra $\mathfrak{g}$, let $\mu_o \in \mathfrak{g}^*$ be a fixed dual element, let $\mathcal{O}_{\mu_o}$ be its coadjoint orbit and let ω^{KKS} be the Kirillov-Kostant-Souriau two-form on $\mathcal{O}_{\mu_o}$ defined by*

$$-\iota(v^*)\iota(w^*)\omega_\mu^{KKS} = \langle\, [v, w] \,|\mu\rangle \ .$$

Then ω^{KKS} is symplectic but not necessarily non-degenerate and the identity map $J : \mathcal{O}_{\mu_o} \to \mathfrak{g}^*$, $J(\mu) = \mu$ is a strongly hamiltonian momentum map.

3. Non-Super Prequantization

For non-super symplectic manifolds (M, ω) the notion of prequantization comes in two flavors, one by B. Kostant and one by J.-M. Souriau. According to Kostant one should look for a complex line bundle $\pi : L \to M$ over M with a (linear) connection ∇ whose curvature is the symplectic form

$$\mathrm{curv}(\nabla) = -\mathrm{i}\hbar^{-1} \cdot \omega$$

and a compatible hermitian inner product (on L). I inserted the physical constant $\hbar$ (without referring to it in the text) to show how it enters in the formalism because one application of geometric (pre)quantization is in physics (where the name quantization comes from).

According to Souriau one should look for a principal $\mathbb{S}^1$-bundle $\pi : Y \to M$ over M equipped with a one-form α satisfying three conditions

1. α is invariant under the $\mathbb{S}^1$-action
2. $\mathrm{d}\alpha = \pi^*\omega$
3. $\int_{\mathbb{S}^1\text{-orbit}} \alpha = 2\pi\hbar$.

Souriau's formulation of prequantization can be cast in the form of the quest for a principal $\mathbb{S}^1$-bundle with connection, but some care has to be taken to do so. To better understand the difficulty, we start with a simple observation. Let $d > 0$ be a fixed positive real number, let M be a manifold, let θ be a one-form on M, let $G = \mathbb{R}/d\mathbb{Z}$ be the real line modulo d and let x be a coordinate (modulo d) on G. Then ∂_x is a left-invariant vector field on G and

$$\widehat{\alpha} = (\theta + dx) \otimes \partial_x$$

is a connection one-form on the principal G-bundle $M \times G \to M$. Its curvature is given by

$$\mathrm{curv}(\widehat{\alpha}) = \mathrm{d}\theta \otimes \partial_x \ .$$

This situation can be generalized in the sense that one can specify (to a certain extent) the curvature form as stated in the next theorem.

Theorem 5. *Let ω be a closed two-form on M whose group of periods $\mathrm{Per}(\omega)$ is defined as*

$$\mathrm{Per}(\omega) = \left\{ \int_\gamma \omega \,;\ \gamma \textit{ a two-cycle on } M \right\} .$$

Then there exists a principal $\mathbb{R}/d\mathbb{Z}$-bundle $\pi : Y \to M$ with one-form α such that $\widehat{\alpha} = \alpha \otimes \partial_x$ is a connection one-form whose curvature is $\omega \otimes \partial_x$ if and only if the group of periods $\mathrm{Per}(\omega)$ is contained in $d\mathbb{Z}$. If Y exists, we have $\int_{\mathbb{S}^1\text{-orbit}} \alpha = d$.

Corollary 1. *A couple (Y, α) is a prequantization according to Souriau if and only if $\widehat{\alpha} = \hbar^{-1} \cdot \alpha \otimes \partial_\varphi$ is a connection one-form on Y whose curvature is $\hbar^{-1} \cdot \omega \otimes \partial_\varphi$, where φ is an angle coordinate modulo 2π on the circle $\mathbb{S}^1$ (i.e., $\mathrm{e}^{\mathrm{i}\varphi} \in \mathbb{S}^1$).*

With this corollary it is easy to see the link between the two flavors of prequantization: the complex line bundle $L \to M$ is the line bundle associated to the principal $\mathbb{S}^1$-bundle $Y \to M$ by the (anti-)tautological representation of $\mathbb{S}^1 \subset \mathbb{C}$ on $\mathbb{C}$ defined as $\rho(\mathrm{e}^{\mathrm{i}\varphi})z = \mathrm{e}^{-\mathrm{i}\varphi} \cdot z$. Taking the infinitesimal form of this representation gives us the curvature form of the induced linear connection ∇ from the principal connection $\widehat{\alpha}$ as

$$\mathrm{curv}(\nabla) = T_e\rho\,\mathrm{curv}(\widehat{\alpha}) = -\mathrm{i} \cdot \hbar^{-1} \cdot \omega\ .$$

According to Theorem 5 a prequantization does not always exist, the condition for existence being that the group of periods $\mathrm{Per}(\omega)$ is included in $2\pi\hbar\mathbb{Z}$, or equivalently, that $\mathrm{Per}(\hbar^{-1} \cdot \omega)$ is included in $2\pi\mathbb{Z}$. But this is exactly the condition that $\hbar^{-1} \cdot \omega$ represents an integral class in cohomology, the condition well known from Kostant's flavor of prequantization.

In [1] I have argued that it is a good idea to consider prequantization as being part of classical mechanics, provided one relaxes condition 3 of Souriau to

3′. $\int_{\mathbb{S}^1\text{-orbit}} \alpha = \mathrm{const} \neq 0$.

The reasons for doing so are beyond the scope of this talk, but I will adopt this viewpoint in the next section when prequantizing symplectic supermanifolds.

4. Prequantization of Symplectic Supermanifolds

Definition 7. *A prequantization of a symplectic (super)manifold (M, ω) is a principal $(\mathcal{A}_0/d\mathbb{Z}) \times \mathcal{A}_1$-bundle $\pi : Y \to M$ with a one-form α satisfying the following three conditions:*

1. *α is invariant under the $(\mathcal{A}_0/d\mathbb{Z}) \times \mathcal{A}_1$-action*
2. $\mathrm{d}\alpha = \pi^*\omega$
3. $\int_{\mathcal{A}_0/d\mathbb{Z}\textit{-orbit}} \alpha = d = \mathrm{const} \neq 0$.

A necessary and sufficient condition for the existence of a prequantization is that the group $\mathrm{Per}(\omega_0)$ of periods of ω_0 (the even part of the symplectic form) is contained in $d\mathbb{Z}$. If we do not specify the positive real number $d > 0$ in advance, the necessary and sufficient condition for existence is that the group of periods is a discrete subgroup of $\mathbb{R}$.

Remark 3. If e_0, e_1 is the appropriate basis for the Lie algebra of $(\mathcal{A}_0/d\mathbb{Z}) \times \mathcal{A}_1$, then $\alpha_0 \otimes e_0 + \alpha_1 \otimes e_1$ is a connection one-form with curvature $\omega_0 \otimes e_0 + \omega_1 \otimes e_1$.

Theorem 6. *Let $\pi : Y \to M$ be a prequantization of the symplectic manifold (M, ω) and let α be the "connection" one-form. For each $f = (f_0, f_1) \in \mathcal{P}$ there exists a unique vector field η_f on Y satisfying the following conditions (some of which are consequences of others):*

1. *η_f projects to the hamiltonian vector field X_f: $\pi_*\eta_f = X_f$*
2. *η_f preserves the one-form α: $\mathcal{L}_{\eta_f}\alpha = 0$*

3. *contraction of* η_f *with* α *yields* f: $\iota(\eta_f)\alpha_{0/1} = \pi^* f_{0/1}$.

Furthermore, the map $f \mapsto \eta_f$ *is an isomorphism of super Lie algebras from the Poisson algebra* $\mathcal{P}$ *to the* α*-preserving vector fields on* Y $($*meaning in particular that every vector field on* Y *preserving* α *is necessarily of the form* η_f *for some* f *in the Poisson algebra* $\mathcal{P})$.

Question 1. I have presented prequantization of a symplectic supermanifold in the flavor of Souriau, so what about Kostant's version? The problem is that there is no representation of $(\mathcal{A}_0/d\mathbb{Z}) \times \mathcal{A}_1$ on a finite dimensional super vector space that is not trivial on the $\mathcal{A}_1$-part. This means that the best we can do is to look for a "line" bundle $L \to M$ with a connection ∇ whose curvature is $-\mathrm{i}\hbar^{-1} \cdot \omega_0$.

But if we do so, then the obvious question becomes whether the absence of the odd part of the symplectic form in line bundle prequantization has serious repercussions. The answer is that I don't know!

References

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