Acta Mathematica Academiae Paedagogicae Nyíregyháziensis
17 (2001), 155–159
www.emis.de/journals

ON THE SEPARATION OF DIFFERENCE EQUATIONS

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Abstract. In this paper, we have developed necessary conditions for separating two coupled linear difference equations extracted from particular type of Schrodinger equation.

1. Introduction

In order to solve a system of coupled difference equations one must, first of all, decouple the system, as one of the techniques. In fact there are many different approaches of decoupling operation, namely, increasing the order of the difference equation, approximation methods, diagonalization of the transformed system, etc. These types of approaches depend on the nature of the structure of the system one deals with. In many-channel problems one frequently deals with a finite system of coupled differential equations of the Schrodinger type of the form

$$\nabla y_i(x) = \sum_j \phi_{ij}(x)\, y_j(x)$$

from which the wave function for each channel can be extracted. In which ∇ is the usual Laplace operator, ϕ_{ij} represents the interaction term connecting channel i to other channels j, see [2]. Our work is developing a method of separation the coupled difference equations

$$\Lambda y_i(x_n) = \sum_{j=1,2} \phi_{ij}(x_n)\, y_j(x_n) + \mu_i(x_n), \qquad i = 1,2$$

where $\Lambda = \sum_{s=1}^{N} r_s(x_n)\, \Delta^s$, Δ is the difference operator i.e.

$$\Delta^s y_n = \sum_{i=0}^{s} (-1)^i \binom{s}{i} y_{n+s-i},$$

r_s's, ϕ_{ij}'s and $\mu_i(x_n)$'s are functions of x_n where

$$\binom{s}{i} = \frac{s!}{i!\,(s-i)!}.$$

For more explanation we give the following details

$$\Delta^1 y_n = \sum_{i=0,1} (-1)^i \binom{1}{i} y_{n+1-i} = \binom{1}{0} y_{n+1} + (-1)\binom{1}{1} y_n = y_{n+1} - y_n$$

2000 *Mathematics Subject Classification.* 39A05.
Key words and phrases. Difference Equations, separation, coupled difference equations.

and

$$\Delta^2 y_n = \sum_{i=0}^{2} (-1)^i \binom{2}{i} y_{n+2-i} = \binom{2}{0} y_{n+2} - \binom{2}{1} y_{n+1} + \binom{2}{2} y_n$$
$$= y_{n+2} - 2y_{n+1} + y_n$$

and so on.

This is extracted from Schrodinger equations. We aim to diagonalize a transformed system of the system

$$\Lambda Y = \Phi Y + \mu \tag{1.1}$$

where $\Phi = (\phi_{ij})$ and μ is the column vector of μ_i's.

To this end, apply a suitable transformation $A = (a_{kl})$, such that $AT = Z$ on system (1.1) to get the form

$$RZ = A\mu$$

where $R = A\Gamma A^{-1}$ is a difference operator which is intended to be diagonalized and $\Gamma = \Lambda I - \Phi$.

We have developed conditions under which one of the variable z_ℓ's can be separated, and the conditions under which both z_1 and z_2 can be separated where $Z = \binom{z_1}{z_2}$. In the first case the separated variable can be found out from the separated equation then replacing it in the second coupled equation from which we determine the second variable. The hypothesis of Theorems 1 and 2 are necessary conditions for the separation.

2. Separation of the equations

Consider the following system of two coupled difference equations

$$\Lambda y_i (x_n) = \sum_{j=1,2} \phi_{ij} (x_n)\, y_j (x_n) + \mu_i\ (x_n) \qquad i = 1, 2 \tag{2.1}$$

where $\Lambda = \sum\limits_{s=1}^{N} r_s (x_n)\, \Delta^s$, Δ is the usual difference operator defined by $\Delta y_i (x_n) = y_i (x_{n+1}) - y_i (x_n)$, r_s's, ϕ_{ij}'s, and μ_i's are functions of x_n, provided that none of the coupling functions ϕ_{12} and ϕ_{21} is zero. We aim to decouple this system. The technique is to choose a suitable transformation $A = (a_{k\ell})$ of size 2×2 (where $a_{k\ell}$'s are independent of x_n to be determined), and make use of this transformation in system (2.1). The transformed system will be in the matrix form

$$A\Gamma A^{-1} A Y = A\mu \tag{2.2}$$

where $\Gamma = \Lambda I - \Phi$, I is the identity matrix, $\Phi = (\phi_{ij})$, $\mu = \binom{\mu_1}{\mu_2}$, and Y is the column $\binom{y_1}{y_2}$. The required transformation A is that one which makes the matrix $A\Gamma A^{-1}$ of the left side of equation (2.2) to be either triangular or diagonal matrix according to the aim that we are interested in.

Suppose $A\Gamma A^{-1} = B = (b_{ij})$, so the crossing terms $b_{k\ell}$, $k \neq \ell$ will be of the form

$$b_{k\ell} = a_{kk} a_{k\ell} (\phi_{kk} - \phi_{\ell\ell}) - a_{kk}^2 \phi_{k\ell} + a_{k\ell}^2 \phi_{\ell k} - a_{kk} \Lambda a_{k\ell} + a_{k\ell} \Lambda a_{kk}$$

since $a_{k\ell}$'s are independent of x_n, hence

$$-a_{kk} \Lambda a_{k\ell} + a_{k\ell} \Lambda a_{kk} = -a_{kk} a_{k\ell} \Lambda + a_{kk} a_{k\ell} \Lambda = 0.$$

Therefore

$$b_{k\ell} = -a_{kk}^2 \phi_{k\ell} + (\phi_{kk} - \phi_{\ell\ell})\, a_{k\ell} a_{kk} + \phi_{\ell k} a_{k\ell}^2.$$

If we assume $a_{k\ell}$'s , $k \neq \ell$ are non zero constants then,

$$b_{k\ell} = \left[\phi_{k\ell}\left(\frac{a_{kk}}{a_{k\ell}}\right)^2 - (\phi_{kk} - \phi_{\ell\ell})\frac{a_{kk}}{a_{k\ell}} - \phi_{\ell k}\right]\left(-a_{k\ell}^2\right). \tag{2.3}$$

In order to make the matrix B triangular or diagonal, we have to search for the roots of the equations

$$b_{k\ell} = 0, \qquad k \neq \ell.$$

This implies

$$\frac{a_{kk}}{a_{k\ell}} = \frac{\phi_{kk} - \phi_{\ell\ell}}{2\phi_{k\ell}} \pm \left[\left(\frac{\phi_{kk} - \phi_{\ell\ell}}{2\phi_{k\ell}}\right)^2 + \frac{\phi_{\ell k}}{\phi_{k\ell}}\right]^{\frac{1}{2}}.$$

If we insert the character

$$\Omega_{k\ell}^t = \frac{\phi_{kk} - \phi_{\ell\ell}}{2\phi_{k\ell}} + (-1)^t\left[\left(\frac{\phi_{kk} - \phi_{\ell\ell}}{2\phi_{k\ell}}\right)^2 + \frac{\phi_{\ell k}}{\phi_{k\ell}}\right]^{\frac{1}{2}}, \ \ell \neq k \tag{2.4}$$

we obtain

$$\frac{a_{kk}}{a_{k\ell}} = \Omega_{k\ell}^t$$

i.e.

$$a_{kk} = \Omega_{k\ell}^t a_{k\ell}, \tag{2.5}$$

where t may be one or zero.

In order to outmatch some important properties of the quantity $\Omega_{k\ell}^t$ we justify the following lemmas

Lemma 1. *$\Omega_{k\ell}^1 \neq 0$, and $\Omega_{k\ell}^0 \neq 0$.*

Proof. Since none of the coupling functions ϕ_{12} and ϕ_{21} is zero. So the second term $\frac{\phi_{\ell k}}{\phi_{k\ell}}$ in the bracket under the square root of the formula (2.4) could not be zero. This implies the required result. □

Lemma 2. *$\Omega_{k\ell}^t \cdot \Omega_{\ell k}^t = 1$.*

Proof. The proof can be done by using the formula (2.4) and direct computation. □

Lemma 3. *$\Omega_{12}^0 = \Omega_{12}^1$ if and only if $\Omega_{21}^0 = \Omega_{21}^1$.*

Proof. $\Omega_{12}^0 = \Omega_{12}^1$ if and only if $\frac{1}{\Omega_{21}^0} = \frac{1}{\Omega_{21}^1}$ if and only if $\Omega_{21}^0 = \Omega_{21}^1$ □

Lemma 4. *$\Omega_{k\ell}^0 = \Omega_{k\ell}^1$ if and only if $\Omega_{\ell k}^0 \cdot \Omega_{k\ell}^1 = 1$.*

Proof. From lemma 2 we get

$$\Omega_{k\ell}^0 \cdot \Omega_{\ell k}^1 = \frac{\Omega_{k\ell}^0}{\Omega_{k\ell}^1}.$$

Hence if $\Omega_{k\ell}^0 = \Omega_{k\ell}^1$ then $\Omega_{k\ell}^0 \cdot \Omega_{\ell k}^1 = 1$; if $\Omega_{k\ell}^0 \cdot \Omega_{\ell k}^1 = 1$ then $\frac{\Omega_{k\ell}^0}{\Omega_{k\ell}^1} = 1$, so $\Omega_{k\ell}^0 = \Omega_{k\ell}^1$. □

Returning back to the transformation A concerning the relation (2.5) the transformation will be in the form

$$A = \begin{pmatrix} \Omega_{12}^{t_1} a_{12} & a_{12} \\ a_{21} & \Omega_{21}^{t_2} a_{21} \end{pmatrix}$$

where $t_1 = 0,1$ and $t_2 = 0,1$ and a_{12}, a_{21}may be any nonzero numbers. For simplicity choose $a_{12} = a_{21} = 1$, therefore

$$A = \begin{pmatrix} \Omega_{12}^{t_1} & 1 \\ 1 & \Omega_{21}^{t_2} \end{pmatrix}.$$

In order to avoid the singularity of this transformation, we may choose either $(t_1 = 0, t_2 = 1)$ or $(t_1 = 1, t_2 = 0)$, with a necessary condition $\Omega_{12}^0 \neq \Omega_{12}^1$.

Now we are ready to introduce our first theorem on the separation of system (2.1).

Theorem 1. *If both of the quantities Ω_{12}^0 and Ω_{12}^1(and hence Ω_{21}^0 and Ω_{21}^1) are distinct real and independent of x_n, then system (2.1) can be decoupled in the form $A_m z_m = \Omega_{m,3-m}^{m-1}\mu_m + \mu_{3-m}$ where*

$$\begin{aligned} A_m = {} & \Lambda - \phi_{mm} \\ & + \frac{1}{\Omega_{12}^0\Omega_{21}^1}\left\{\phi_{3-m,3-m} + \phi_{mm} + \Omega_{m,3-m}^{m-1}\phi_{m,3-m} - \Omega_{3-m,m}^{2-m}\phi_{3-m,m}\right\}, \\ & \qquad m = 1,2. \end{aligned}$$

Proof. Consider the transformation

$$A = \begin{pmatrix} \Omega_{12}^{0} & 1 \\ 1 & \Omega_{21}^{1} \end{pmatrix}.$$

This matrix is nonsingular since its determinant is $1 - \Omega_{12}^0\Omega_{21}^1$and this different from zero see Lemma 4. Make use of this transformation in system (2.1) we get the matrix equation of the form

$$A\Gamma A^{-1}YY = A\mu,$$

i.e.

$$Bz = A\mu$$

where $B = (b_{ij}) = A\Gamma A^{-1}$, $\quad z = AY$, and Γ as defined before. Therefor

$$b_{k\ell} = -\phi_{k\ell}\left(\Omega_{k\ell}^{k-1}\right)^2 + (\phi_{kk} - \phi_{\ell\ell})\,\Omega_{k\ell}^{k-1} + \phi_{\ell k} - \Omega_{k\ell}^{k-1}\Lambda + \Lambda\Omega_{k\ell}^{k-1}.$$

Since $\Omega_{k\ell}^{k-1}$ is independent of x_n, the last two terms of this equation is zero. Hence

$$b_{k\ell} = -\phi_{k\ell}\left(\Omega_{k\ell}^{k-1}\right)^2 + (\phi_{kk} - \phi_{\ell\ell})\,\Omega_{k\ell}^{k-1} + \phi_{\ell k}$$

but $\Omega_{k\ell}^{k-1}$ is nothing but the zero root of this equation which implies $b_{k\ell} = 0$, in other words $b_{12} = b_{21} = 0$. Therefore the matrix B is now diagonal matrix and the diagonal terms are

$$b_{11} = \Lambda + \frac{1}{\Omega_{12}^0\Omega_{21}^1 - 1}\left\{-\Omega_{12}^0\Omega_{21}^1\phi_{11} + \Omega_{12}^0\phi_{12} - \Omega_{21}^1\phi_{21} + \phi_{22}\right\}$$

$$b_{22} = \Lambda + \frac{1}{\Omega_{12}^0\Omega_{21}^1 - 1}\left\{-\Omega_{12}^0\Omega_{21}^1\phi_{22} - \Omega_{12}^0\phi_{12} + \Omega_{21}^1\phi_{21} + \phi_{11}\right\}$$

Simplifying these equations we get

$$b_{11} = \Lambda - \phi_{11} + \frac{1}{\Omega_{12}^0\Omega_{21}^1 - 1}\left\{\phi_{22} - \phi_{11} + \Omega_{12}^0\phi_{12} - \Omega_{21}^1\phi_{21}\right\}$$

$$b_{22} = \Lambda - \phi_{22} + \frac{1}{\Omega_{12}^0\Omega_{21}^1 - 1}\left\{\phi_{11} - \phi_{22} + \Omega_{21}^1\phi_{21} - \Omega_{12}^0\phi_{12}\right\}.$$

Unifying these two equations in one formula assuming $A_1 = b_{11}$ and $A_2 = b_{22}$, the required form in the statement of the theorem will be obtained. $\square$

Theorem 2. *If one of the quantities Ω_{12}^0 or Ω_{12}^1 is independent of x_n (and hence Ω_{21}^0 or Ω_{21}^1) and denoted by Ω^{t_0}, then system (2.1) can be decoupled in the form*

$$A_m z_m = (m-1)\left[(\phi_{22}-\Lambda)\, z_1 + \mu_1\right] + (2-m)\left[\Omega^{t_0}\mu_1 + \mu_2\right]$$

where

$$A_m = (m-1)\left[\phi_{22}\Omega^{t_0} - \Lambda\Omega^{t_0} - \phi_{21}\right] + (2-m)\left[\Lambda - \Omega^{t_0}\phi_{12} - \phi_{22}\right], \quad m = 1,2.$$

Proof. Consider the nonsingular matrix transformation

$$A = \begin{pmatrix} \Omega^{t_0} & 1 \\ 1 & 0 \end{pmatrix}.$$

Make use of this transformation on system (2.1) we get a matrix equation of the form

$$Bz = A\mu \tag{2.6}$$

where $B = (b_{ij}) = A\Gamma A^{-1}$, $z = AY$, Γ as defined before. Hence the entry b_{12} is

$$b_{12} = \phi_{12}\left(\Omega^{t_0}\right)^2 - (\phi_{11}-\phi_{12})\,\Omega^{t_0} - \phi_{21} + \Omega^{t_0}\Lambda - \Lambda\Omega^{t_0}.$$

Since Ω^{t_0} is independent of x_n, of $\Omega^{t_0}\Lambda - \Lambda\Omega^{t_0} = 0$. Therefore

$$b_{12} = \phi_{12}\left(\Omega^{t_0}\right)^2 - (\phi_{11}-\phi_{12})\,\Omega^{t_0} - \phi_{21}.$$

Recall that Ω^{t_0} is nothing but a zero of the right hand side of this equation, hence $b_{12} = 0$. Therefore the equations of the system (2.6) will be in the form

$$b_{11}z_1 = \Omega^{t_0}\mu_1 + \mu_2,$$
$$b_{21}z_1 + b_{22}z_2 = \mu_1$$

where

$$b_{11} = \Lambda - \Omega^{t_0}\phi_{12} - \phi_{22},$$
$$b_{21} = \Lambda - \phi_{22},$$
$$b_{22} = -\Omega^{t_0}\Lambda + \Omega^{t_0}\phi_{22} - \phi_{21}.$$

Using the notation $A_1 = b_{11}$, $A_2 = b_{22}$ we get the required form stated in the theorem. □

Case of $\Omega_{k\ell}^0 = \Omega_{k\ell}^1$ is included in Theorem 2 as a case of the identity

$$(\phi_{kk} - \phi_{\ell\ell})^2 + 4\phi_{\ell k}\phi_{k\ell} = 0.$$

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Received December 13, 2000; February 21, 2001 in revised form.

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