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PARTIAL SUMS OF CERTAIN ANALYTIC AND UNIVALENT FUNCTIONS

B.A. FRASIN

Abstract. In this paper, we study the ratio of a function of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

to its sequence of partial sums of the form $f_n(z) = z + \sum_{k=2}^{n} a_k z^k$. Also, we will determine sharp lower bounds for $\operatorname{Re}\{f(z)/f_n(z)\}$, $\operatorname{Re}\{f_n(z)/f(z)\}$, $\operatorname{Re}\{f'(z)/f_n'(z)\}$ and $\operatorname{Re}\left\{f_n'(z)/f'(z)\right\}$.

1. Introduction and definitions

Let $\mathcal{A}$ denote the class of functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic in the open unit disk $\mathcal{U} = \{z : |z| < 1\}$. Further, by $\mathcal{S}$ we shall denote the class of all functions in $\mathcal{A}$ which are univalent in $\mathcal{U}$. Then a function $f(z)$ belonging to $\mathcal{A}$ is said to be starlike of order α if it satisfies

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \qquad (z \in \mathcal{U}) \tag{1.2}$$

for some $\alpha(0 \leq \alpha < 1)$. We denote by $\mathcal{S}_\alpha^*$ the subclass of $\mathcal{A}$ consisting of functions which are starlike of order α in $\mathcal{U}$. Also, a function $f(z)$ belonging to $\mathcal{A}$ is said to be convex of order α if it satisfies

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha \qquad (z \in \mathcal{U}) \tag{1.3}$$

for some $\alpha(0 \leq \alpha < 1)$. We denote by $\mathcal{K}_\alpha$ the subclass of $\mathcal{A}$ consisting of functions which are convex of order α in $\mathcal{U}$. A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{P}_\alpha$ iff

$$\operatorname{Re}(f'(z)) > \alpha, \qquad (z \in \mathcal{U}). \tag{1.4}$$

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It is well known that $\mathcal{K}_{\alpha} \subset \mathcal{S}^*_{\alpha} \subset \mathcal{S}$. Given two functions $f, g \in \mathcal{A}$, where $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ and $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$, their *Hadamard product* or *convolution* $f(z) * g(z)$ is defined by

$$f(z) * g(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k, \qquad (z \in \mathcal{U}). \tag{1.5}$$

Ruschewehy [2] using the convolution techniques, introduced and studied the class of prestarlike functions of order α, which is denoted by $\mathcal{R}_{\alpha}$. Thus $f \in \mathcal{A}$ is said to be prestarlike functions of order $\alpha (0 \leq \alpha < 1)$ if $f * s_{\alpha}(z) \in \mathcal{S}^*_{\alpha}$ where $s_{\alpha}(z) = z/(1-z)^{2(1-\alpha)}$. It may be noted that $\mathcal{R}_0 \equiv \mathcal{K}_0$ and $\mathcal{R}_{1/2} \equiv \mathcal{S}^*_{1/2}$.

In our present paper we shall make use of the following definition due to Juneja et al. [2].

Definition. Given the analytic functions

$$\Phi(z) = z + \sum_{k=2}^{\infty} \lambda_k z^k \text{ and } \quad \Psi(z) = z + \sum_{k=2}^{\infty} \mu_k z^k, \qquad (0 \leq \alpha < 1;\ z \in \mathcal{U}\ ), \tag{1.6}$$

where $\lambda_k \geq 0, \mu_k \geq 0$ and $\lambda_k \geq \mu_k$ for $k \geq 2$, we say that $f \in \mathcal{A}$ is in $\mathcal{E}(\Phi, \Psi; \alpha)$ if $f(z) * \Psi(z) \neq 0$ and

$$\text{Re} \left\{ \frac{f(z) * \Phi(z)}{f(z) * \Psi(z)} \right\} > \alpha \qquad (z \in \mathcal{U}). \tag{1.7}$$

It is easy to check that various subclasses of $\mathcal{S}$ referred to above can be represented as $\mathcal{E}(\Phi, \Psi; \alpha)$ for suitable choices of Φ, Ψ. For example

(i) $\mathcal{E}\left(\frac{z}{(1-z)^2}, \frac{z}{1-z}; \alpha\right) = \mathcal{S}^*_{\alpha}$;

(ii) $\mathcal{E}\left(\frac{z+z^2}{(1-z)^3}, \frac{z}{(1-z)^2}; \alpha\right) = \mathcal{K}_{\alpha}$;

(iii) $\mathcal{E}\left(\frac{z}{(1-z)^2}, z; \alpha\right) = \mathcal{P}_{\alpha}$;

(iv) $\mathcal{E}\left(\frac{z+(1-2\alpha)z^2}{(1-z)^{3-2\alpha}}, \frac{z}{(1-z)^{2-2\alpha}}; \alpha\right) = \mathcal{R}_{\alpha}$.

In fact many new subclasses of $\mathcal{S}$ can be defined and studied by suitably choosing $\Phi(z)$ and $\Psi(z)$. Thus

(v) $\mathcal{E}\left(\frac{z+z^2}{(1-z)^3}, z; \alpha\right) = \{f \in S\text{: } \text{Re}\left((zf'(z))'\right) > \alpha\}$

(vi) $\mathcal{E}\left((1-\delta)\frac{z}{(1-z)^2} + \delta\frac{z+z^2}{(1-z)^3}, z; \alpha\right)$
$= \{f \in S\text{: } \text{Re}\left((1-\delta)f'(z) + \delta(zf'(z))'\right) > \alpha\}$ and so on.

A sufficient condition for a function of the form (1.1) to be in $\mathcal{E}(\Phi, \Psi; \alpha)$ is that

$$\sum_{k=2}^{\infty} (\lambda_k - \alpha\mu_k) |a_k| \leq 1 - \alpha. \tag{1.8}$$

For the functions of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k, \qquad a_k \geq 0 \tag{1.9}$$

the sufficient condition (1.8) is also necessary (see [1]).

In the present paper and by following the earlier works by Silverman [3] on partial sums of analytic functions, we study the ratio of a function of the form (1.1) to its sequence of partial sums of the form

$$f_n(z) = z + \sum_{k=2}^{n} a_k z^k. \tag{1.10}$$

when the coefficients of $f(z)$ are satisfy the condition (1.8). We will determine sharp lower bounds for $\operatorname{Re}\{f(z)/f_n(z)\}$, $\operatorname{Re}\{f_n(z)/f(z)\}$, $\operatorname{Re}\{f'(z)/f'_n(z)\}$ and $\operatorname{Re}\left\{f'_n(z)/f'(z)\right\}$. It is seen that this study not only gives as a particular case, the results of Silverman [3] but also give rise to several new results.

2. Main results

Theorem 1. *If $f(z)$ of the form (1.1) satisfies the condition (1.8), and*

$$\lambda_{k+1} - \alpha\mu_{k+1} \geq \begin{cases} 1-\alpha, & k = 2,3,\ldots,n \\ \lambda_{n+1} + \alpha\mu_{n+1}, & k = n+1, n+2, \ldots \end{cases}$$

then

$$\operatorname{Re}\left\{\frac{f(z)}{f_n(z)}\right\} \geq \frac{\lambda_{n+1} - \alpha\mu_{n+1} - 1 + \alpha}{\lambda_{n+1} - \alpha\mu_{n+1}} \qquad (z \in \mathcal{U}) \tag{2.1}$$

and

$$\operatorname{Re}\left\{\frac{f_n(z)}{f(z)}\right\} \geq \frac{\lambda_{n+1} - \alpha\mu_{n+1}}{1-\alpha+\lambda_{n+1} - \alpha\mu_{n+1}} \qquad (z \in \mathcal{U}). \tag{2.2}$$

The results (2.1) and (2.2) are sharp with the function given by

$$f(z) = z + \frac{1-\alpha}{\lambda_{n+1} - \alpha\mu_{n+1}} z^{n+1}. \tag{2.3}$$

Proof. Define the function $w(z)$ by

$$\begin{aligned} \frac{1+w(z)}{1-w(z)} &= \frac{\lambda_{n+1} - \alpha\mu_{n+1}}{1-\alpha}\left[\frac{f(z)}{f_n(z)} - \left(\frac{\lambda_{n+1} - \alpha\mu_{n+1} - 1 + \alpha}{\lambda_{n+1} - \alpha\mu_{n+1}}\right)\right] \\ &= \frac{1 + \sum\limits_{k=2}^{n} a_k z^{k-1} + \left(\frac{\lambda_{n+1} - \alpha\mu_{n+1}}{1-\alpha}\right)\sum\limits_{k=n+1}^{\infty} a_k z^{k-1}}{1 + \sum\limits_{k=2}^{n} a_k z^{k-1}}. \end{aligned} \tag{2.4}$$

It suffices to show that $|w(z)| \leq 1$. Now, from (2.4) we can write

$$w(z) = \frac{\left(\frac{\lambda_{n+1} - \alpha\mu_{n+1}}{1-\alpha}\right)\sum\limits_{k=n+1}^{\infty} a_k z^{k-1}}{2 + 2\sum\limits_{k=2}^{n} a_k z^{k-1} + \left(\frac{\lambda_{n+1} - \alpha\mu_{n+1}}{1-\alpha}\right)\sum\limits_{k=n+1}^{\infty} a_k z^{k-1}}$$

to find that

$$|w(z)| \leq \frac{\left(\frac{\lambda_{n+1} - \alpha\mu_{n+1}}{1-\alpha}\right)\sum\limits_{k=n+1}^{\infty} |a_k|}{2 - 2\sum\limits_{k=2}^{n} |a_k| - \left(\frac{\lambda_{n+1} - \alpha\mu_{n+1}}{1-\alpha}\right)\sum\limits_{k=n+1}^{\infty} |a_k|}$$

Now $|w(z)| \leq 1$ if

$$2\left(\frac{\lambda_{n+1}-\alpha\mu_{n+1}}{1-\alpha}\right)\sum_{k=n+1}^{\infty}|a_k| \leq 2-2\sum_{k=2}^{n}|a_k|$$

or, equivalently

$$\sum_{k=2}^{n}|a_k|+\sum_{k=n+1}^{\infty}\frac{\lambda_{n+1}-\alpha\mu_{n+1}}{1-\alpha}|a_k| \leq 1.$$

From the condition (1.8), it is sufficient to show that

$$\sum_{k=2}^{n}|a_k|+\sum_{k=n+1}^{\infty}\frac{\lambda_{n+1}-\alpha\mu_{n+1}}{1-\alpha}|a_k| \leq \sum_{k=2}^{\infty}\frac{\lambda_k-\alpha\mu_k}{1-\alpha}|a_k|$$

which is equivalent to

$$\sum_{k=2}^{n}\left(\frac{\lambda_k-\alpha\mu_k-1+\alpha}{1-\alpha}\right)|a_k| + \sum_{k=n+1}^{\infty}\left(\frac{\lambda_k-\alpha\mu_k-\lambda_{n+1}+\alpha\mu_{n+1}}{1-\alpha}\right)|a_k| \geq 0. \tag{2.5}$$

To see that the function given by (2.3) gives the sharp result, we observe that

$$\frac{f(z)}{f_n(z)} = 1+\frac{1-\alpha}{\lambda_{n+1}-\alpha\mu_{n+1}}z^n \to 1-\frac{1-\alpha}{\lambda_{n+1}-\alpha\mu_{n+1}} = \frac{\lambda_{n+1}-\alpha\mu_{n+1}-1+\alpha}{\lambda_{n+1}-\alpha\mu_{n+1}} \quad \text{when } r\to 1^-.$$

To prove the second part of this theorem, we write

$$\frac{1+w(z)}{1-w(z)} = \frac{1-\alpha+\lambda_{n+1}-\alpha\mu_{n+1}}{1-\alpha}\left[\frac{f_n(z)}{f(z)}-\left(\frac{\lambda_{n+1}-\alpha\mu_{n+1}}{1-\alpha+\lambda_{n+1}-\alpha\mu_{n+1}}\right)\right] = \frac{1+\sum_{k=2}^{n}a_kz^{k-1}-\left(\frac{\lambda_{n+1}-\alpha\mu_{n+1}}{1-\alpha}\right)\sum_{k=n+1}^{\infty}a_kz^{k-1}}{1+\sum_{k=2}^{n}a_kz^{k-1}}$$

where

$$|w(z)| \leq \frac{\left(\frac{1-\alpha+\lambda_{n+1}-\alpha\mu_{n+1}}{1-\alpha}\right)\sum_{k=n+1}^{\infty}|a_k|}{2-2\sum_{k=2}^{n}|a_k|-\left(\frac{1-\alpha-\lambda_{n+1}+\alpha\mu_{n+1}}{1-\alpha}\right)\sum_{k=n+1}^{\infty}|a_k|} \leq 1.$$

This last inequality is equivalent to

$$\sum_{k=2}^{n}|a_k|+\sum_{k=n+1}^{\infty}\frac{\lambda_{n+1}-\alpha\mu_{n+1}}{1-\alpha}|a_k| \leq 1.$$

Making use of (1.8) to get (2.5). Finally, equality holds in (2.2) for the extremal function $f(z)$ given by (2.3). □

Taking $\Phi(z) = z/(1-z)^2$ and $\Psi(z) = z/(1-z)$ in Theorem 1, we obtain

Corollary 1 ([3]). *Let the function $f(z)$ be defined by (1.1). If*

$$\sum_{k=2}^{\infty}(k-\alpha)\,|a_k| \leq 1-\alpha \tag{2.6}$$

then

$$\operatorname{Re}\left\{\frac{f(z)}{f_n(z)}\right\} \geq \frac{n}{n+1-\alpha} \qquad (z \in \mathcal{U}) \tag{2.7}$$

and

$$\operatorname{Re}\left\{\frac{f_n(z)}{f(z)}\right\} \geq \frac{n+1-\alpha}{n+2-2\alpha} \qquad (z \in \mathcal{U}). \tag{2.8}$$

The results are sharp with the function given by

$$f(z) = z + \frac{1-\alpha}{n+1-\alpha}z^{n+1}. \tag{2.9}$$

Taking $\Phi(z) = (z+z^2)/(1-z)^3$ and $\Psi(z) = z/(1-z)^2$ in Theorem 1, we obtain

Corollary 2 ([5]). *Let the function $f(z)$ be defined by (1.1). If*

$$\sum_{k=2}^{\infty}k(k-\alpha)\,|a_k| \leq 1-\alpha \tag{2.10}$$

then

$$\operatorname{Re}\left\{\frac{f(z)}{f_n(z)}\right\} \geq \frac{n(n+2-\alpha)}{(n+1)(n+1-\alpha)} \qquad (z \in \mathcal{U}) \tag{2.11}$$

and

$$\operatorname{Re}\left\{\frac{f_n(z)}{f(z)}\right\} \geq \frac{(n+1)(n+1-\alpha)}{(n+1)[(n+1)-\alpha]+1-\alpha} \qquad (z \in \mathcal{U}). \tag{2.12}$$

The results are sharp with the function given by

$$f(z) = z + \frac{1-\alpha}{(n+1)^2-\alpha(n+1)}z^{n+1}. \tag{2.13}$$

Taking $\Phi(z) = z/(1-z)$ and $\Psi(z) = z$ in Theorem 1, we obtain

Corollary 3. *Let the function $f(z)$ be defined by (1.1).If*

$$\sum_{k=2}^{\infty}|a_k| \leq 1-\alpha \tag{2.14}$$

then

$$\operatorname{Re}\left\{\frac{f(z)}{f_n(z)}\right\} \geq \alpha \qquad (z \in \mathcal{U}) \tag{2.15}$$

and

$$\operatorname{Re}\left\{\frac{f_n(z)}{f(z)}\right\} \geq \frac{1}{2-\alpha} \qquad (z \in \mathcal{U}). \tag{2.16}$$

The results are sharp with the function given by

$$f(z) = z + (1-\alpha)z^{n+1}. \tag{2.17}$$

Taking $\Phi(z) = z/(1-z)^2$ and $\Psi(z) = z$ in Theorem 1, we obtain

Corollary 4. *Let the function $f(z)$ be defined by (1.1). If*

$$\sum_{k=2}^{\infty} k\,|a_k| \leq 1-\alpha \tag{2.18}$$

then

$$\operatorname{Re}\left\{\frac{f(z)}{f_n(z)}\right\} \geq \frac{n+\alpha}{n+1} \qquad (z \in \mathcal{U}) \tag{2.19}$$

and

$$\operatorname{Re}\left\{\frac{f_n(z)}{f(z)}\right\} \geq \frac{n+1}{n+2-\alpha} \qquad (z \in \mathcal{U}). \tag{2.20}$$

The results are sharp with the function given by

$$f(z) = z + \frac{1-\alpha}{n+1} z^{n+1}. \tag{2.21}$$

Taking $\Phi(z) = (z+(1-2\alpha)z^2)/(1-z)^{3-2\alpha}$ and $\Psi(z) = z/(1-z)^{2-2\alpha}$ in Theorem 1, we obtain

Corollary 5. *Let the function $f(z)$ be defined by (1.1). If*

$$\sum_{k=2}^{\infty} C(\alpha,k)(k-\alpha)\,|a_k| \leq 1-\alpha \tag{2.22}$$

then

$$\operatorname{Re}\left\{\frac{f(z)}{f_n(z)}\right\} \geq \frac{C(\alpha,n+1)(n+1-\alpha)-1+\alpha}{C(\alpha,n+1)(n+1-\alpha)} \qquad (z \in \mathcal{U}) \tag{2.23}$$

and

$$\operatorname{Re}\left\{\frac{f_n(z)}{f(z)}\right\} \geq \frac{C(\alpha,n+1)(n+1-\alpha)}{1-\alpha+C(\alpha,n+1)(n+1-\alpha)} \qquad (z \in \mathcal{U}). \tag{2.24}$$

where $C(\alpha,k) = \prod_{i=2}^{k}(i-2\alpha)/(i-1)!$.

The results are sharp with the function given by

$$f(z) = z + \frac{1-\alpha}{C(\alpha,n+1)(n+1-\alpha)} z^{n+1}. \tag{2.25}$$

Taking $\Phi(z) = (z+z^2)/(1-z)^3$ and $\Psi(z) = z$ in Theorem 1, we obtain

Corollary 6. *Let the function $f(z)$ be defined by (1.1). If*

$$\sum_{k=2}^{\infty} k^2\,|a_k| \leq 1-\alpha \tag{2.26}$$

then

$$\operatorname{Re}\left\{\frac{f(z)}{f_n(z)}\right\} \geq \frac{n^2+2n+\alpha}{(n+1)^2} \qquad (z \in \mathcal{U}) \tag{2.27}$$

and

$$\operatorname{Re}\left\{\frac{f_n(z)}{f(z)}\right\} \geq \frac{(n+1)^2}{n^2+2n+2-\alpha} \qquad (z \in \mathcal{U}). \tag{2.28}$$

The results are sharp with the function given by

$$f(z) = z + \frac{1-\alpha}{(n+1)^2} z^{n+1}. \tag{2.29}$$

Taking $\Phi(z) = (1-\delta)z/(1-z)^2 + \delta(z+z^2)/(1-z)^3$ and $\Psi(z) = z$ in Theorem 1, we obtain

Corollary 7. *Let the function $f(z)$ be defined by (1.1). If*

$$\sum_{k=2}^{\infty} [(1-\delta)k + \delta k^2] |a_k| \leq 1-\alpha \tag{2.30}$$

then

$$\mathrm{Re}\left\{\frac{f(z)}{f_n(z)}\right\} \geq \frac{(n+1)(1+n\delta)-1+\alpha}{(n+1)(1+n\delta)} \qquad (z \in \mathcal{U}) \tag{2.31}$$

and

$$\mathrm{Re}\left\{\frac{f_n(z)}{f(z)}\right\} \geq \frac{(n+1)(1+n\delta)}{1-\alpha+(n+1)(1+n\delta)} \qquad (z \in \mathcal{U}). \tag{2.32}$$

The results are sharp with the function given by

$$f(z) = z + \frac{1-\alpha}{(n+1)(1+n\delta)} z^{n+1}. \tag{2.33}$$

We next turns to ratios involving derivatives.

Theorem 2. *If $f(z)$ of the form (1.1) satisfies the condition (1.8), and*

$$\lambda_{k+1} - \alpha\mu_{k+1} \geq \begin{cases} k(1-\alpha), & k = 2,3,\ldots,n \\ k(1-\alpha) + \frac{(\lambda_{n+1}-\alpha\mu_{n+1})k}{(n+1)}, & k = n+1, n+2, \ldots. \end{cases}$$

then

$$\mathrm{Re}\left\{\frac{f'(z)}{f'_n(z)}\right\} \geq \frac{\lambda_{n+1}-\alpha\mu_{n+1}-(n+1)(1-\alpha)}{\lambda_{n+1}-\alpha\mu_{n+1}} \qquad (z \in \mathcal{U}) \tag{2.34}$$

and

$$\mathrm{Re}\left\{\frac{f'_n(z)}{f'(z)}\right\} \geq \frac{\lambda_{n+1}-\alpha\mu_{n+1}}{(n+1)(1-\alpha)+\lambda_{n+1}-\alpha\mu_{n+1}} \qquad (z \in \mathcal{U}). \tag{2.35}$$

The results are sharp with the function given by (2.3).

Proof. We write

$$\frac{1+w(z)}{1-w(z)} = \frac{\lambda_{n+1}-\alpha\mu_{n+1}}{(n+1)(1-\alpha)}\left[\frac{f'(z)}{f'_n(z)} - \left(\frac{\lambda_{n+1}-\alpha\mu_{n+1}-(n+1)(1-\alpha)}{\lambda_{n+1}-\alpha\mu_{n+1}}\right)\right]$$

where

$$w(z) = \frac{\left(\frac{\lambda_{n+1}-\alpha\mu_{n+1}}{(n+1)(1-\alpha)}\right)\sum\limits_{k=n+1}^{\infty} k a_k z^{k-1}}{2 + 2\sum\limits_{k=2}^{n} k a_k z^{k-1} + \left(\frac{\lambda_{n+1}-\alpha\mu_{n+1}}{(n+1)(1-\alpha)}\right)\sum\limits_{k=n+1}^{\infty} k a_k z^{k-1}}$$

Now $|w(z)| \leq 1$ if

$$\sum_{k=2}^{n} k|a_k| + \frac{\lambda_{n+1}-\alpha\mu_{n+1}}{(n+1)(1-\alpha)} \sum_{k=n+1}^{\infty} k|a_k| \leq 1.$$

From the condition (1.8), it is sufficient to show that

$$\sum_{k=2}^{n} k\,|a_k| + \frac{\lambda_{n+1}-\alpha\mu_{n+1}}{(n+1)(1-\alpha)} \sum_{k=n+1}^{\infty} k\,|a_k| \le \sum_{k=2}^{\infty} \frac{\lambda_k-\alpha\mu_k}{1-\alpha}\,|a_k|$$

which is equivalent to

$$\sum_{k=2}^{n}\left(\frac{\lambda_k-\alpha\mu_k-(1-\alpha)k}{1-\alpha}\right)|a_k| + \sum_{k=n+1}^{\infty}\frac{(n+1)\left(\lambda_k-\alpha\mu_k\right)-(\lambda_{n+1}-\alpha\mu_{n+1})k}{(1-\alpha)(n+1)}\,|a_k| \ge 0.$$

To prove the result (2.32), define the function $w(z)$ by

$$\frac{1+w(z)}{1-w(z)} = \frac{(n+1)(1-\alpha)+\lambda_{n+1}-\alpha\mu_{n+1}}{1-\alpha} \times \left[\frac{f_n'(z)}{f'(z)} - \left(\frac{\lambda_{n+1}-\alpha\mu_{n+1}}{(n+1)(1-\alpha)+\lambda_{n+1}-\alpha\mu_{n+1}}\right)\right]$$

where

$$w(z) = \frac{\left(1+\frac{\lambda_{n+1}-\alpha\mu_{n+1}}{(n+1)(1-\alpha)}\right)\sum\limits_{k=n+1}^{\infty} k a_k z^{k-1}}{2+2\sum\limits_{k=2}^{n} k a_k z^{k-1} + \left(1+\frac{\lambda_{n+1}-\alpha\mu_{n+1}}{(n+1)(1-\alpha)}\right)\sum\limits_{k=n+1}^{\infty} k a_k z^{k-1}}.$$

Now $|w(z)| \le 1$ if

$$\sum_{k=2}^{n} k\,|a_k| + \left(1+\frac{\lambda_{n+1}-\alpha\mu_{n+1}}{(n+1)(1-\alpha)}\right)\sum_{k=n+1}^{\infty} k\,|a_k| \le 1. \tag{2.36}$$

It suffices to show that the left hand side of (2.36) is bounded above by the condition $\sum\limits_{k=2}^{\infty}\left((\lambda_k-\alpha\mu_k)/(1-\alpha)\right)|a_k|$, which is equivalent to

$$\sum_{k=2}^{n}\left(\frac{\lambda_k-\alpha\mu_k}{1-\alpha}-k\right)|a_k| + \sum_{k=n+1}^{\infty}\left(\frac{\lambda_k-\alpha\mu_k}{1-\alpha} - \left(1+\frac{\lambda_{n+1}-\alpha\mu_{n+1}}{(n+1)(1-\alpha)}\right)k\right)|a_k| \ge 0.$$

□

Taking $\Phi(z) = z/(1-z)^2$ and $\Psi(z) = z/(1-z)$ in Theorem 2, we obtain

Corollary 8 ([3])**.** *Let the function $f(z)$ be defined by (1.1). If*

$$\sum_{k=2}^{\infty}(k-\alpha)\,|a_k| \le 1-\alpha \tag{2.37}$$

then

$$\operatorname{Re}\left\{\frac{f'(z)}{f_n'(z)}\right\} \ge \frac{n\alpha}{n+1-\alpha} \qquad (z \in \mathcal{U}) \tag{2.38}$$

and

$$\operatorname{Re}\left\{\frac{f_n'(z)}{f'(z)}\right\} \ge \frac{n+1-\alpha}{(n+1)(2-\alpha)-\alpha} \qquad (z \in \mathcal{U}). \tag{2.39}$$

The results are sharp with the function given by

$$f(z) = z + \frac{1-\alpha}{n+1-\alpha} z^{n+1}. \tag{2.40}$$

Taking $\Phi(z) = (z+z^2)/(1-z)^3$ and $\Psi(z) = z/(1-z)^2$ in Theorem 2, we obtain

Corollary 9 ([3]). *Let the function $f(z)$ be defined by (1.1). If*

$$\sum_{k=2}^{\infty} k(k-\alpha)\,|a_k| \leq 1-\alpha \tag{2.41}$$

then

$$\operatorname{Re}\left\{\frac{f'(z)}{f_n'(z)}\right\} \geq \frac{n}{n+1-\alpha} \qquad (z \in \mathcal{U}) \tag{2.42}$$

and

$$\operatorname{Re}\left\{\frac{f_n'(z)}{f'(z)}\right\} \geq \frac{n+1-\alpha}{n+2-2\alpha} \qquad (z \in \mathcal{U}). \tag{2.43}$$

The results are sharp with the function given by

$$f(z) = z + \frac{1-\alpha}{(n+1)^2 - \alpha(n+1)} z^{n+1}. \tag{2.44}$$

Taking $\Phi(z) = z/(1-z)$ and $\Psi(z) = z$ in Theorem 2, we obtain

Corollary 10. *Let the function $f(z)$ be defined by (1.1). If*

$$\sum_{k=2}^{\infty} |a_k| \leq 1-\alpha \tag{2.45}$$

then

$$\operatorname{Re}\left\{\frac{f'(z)}{f_n'(z)}\right\} \geq 1-(n+1)(1-\alpha) \qquad (z \in \mathcal{U}) \tag{2.46}$$

and

$$\operatorname{Re}\left\{\frac{f_n'(z)}{f'(z)}\right\} \geq \frac{1}{(n+1)(1-\alpha)+1} \qquad (z \in \mathcal{U}). \tag{2.47}$$

The results are sharp with the function given by

$$f(z) = z + (1-\alpha) z^{n+1}. \tag{2.48}$$

Taking $\Phi(z) = z/(1-z)^2$ and $\Psi(z) = z$ in Theorem 2, we obtain

Corollary 11. *Let the function $f(z)$ be defined by (1.1). If*

$$\sum_{k=2}^{\infty} k\,|a_k| \leq 1-\alpha \tag{2.49}$$

then

$$\operatorname{Re}\left\{\frac{f'(z)}{f_n'(z)}\right\} \geq \frac{(n+1)\alpha}{n+1} \qquad (z \in \mathcal{U}) \tag{2.50}$$

and

$$\operatorname{Re}\left\{\frac{f_n'(z)}{f'(z)}\right\} \geq \frac{n+1}{(n+1)(2-\alpha)} \qquad (z \in \mathcal{U}). \tag{2.51}$$

The results are sharp with the function given by

$$f(z) = z + \frac{1-\alpha}{n+1} z^{n+1} \tag{2.52}$$

Taking $\Phi(z) = (z+(1-2\alpha)z^2)/(1-z)^{3-2\alpha}$ and $\Psi(z) = z/(1-z)^{2-2\alpha}$ in Theorem 2, we obtain

Corollary 12. *Let the function $f(z)$ be defined by (1.1). If*

$$\sum_{k=2}^{\infty} C(\alpha,k)(k-\alpha)\,|a_k| \leq 1-\alpha \tag{2.53}$$

then

$$\operatorname{Re}\left\{\frac{f'(z)}{f_n'(z)}\right\} \geq \frac{C(\alpha,n+1)(n+1-\alpha)-(n+1)(1-\alpha)}{C(\alpha,n+1)(n+1-\alpha)} \qquad (z\in\mathcal{U}) \tag{2.54}$$

and

$$\operatorname{Re}\left\{\frac{f_n'(z)}{f'(z)}\right\} \geq \frac{C(\alpha,n+1)(n+1-\alpha)}{(n+1)(1-\alpha)+C(\alpha,n+1)(n+1-\alpha)} \qquad (z\in\mathcal{U}). \tag{2.55}$$

The results are sharp with the function given by

$$f(z) = z + \frac{1-\alpha}{C(\alpha,n+1)(n+1-\alpha)} z^{n+1}. \tag{2.56}$$

Taking $\Phi(z) = (z+z^2)/(1-z)^3$ and $\Psi(z) = z$ in Theorem 2, we obtain

Corollary 13. *Let the function $f(z)$ be defined by (1.1). If*

$$\sum_{k=2}^{\infty} k^2\,|a_k| \leq 1-\alpha \tag{2.57}$$

then

$$\operatorname{Re}\left\{\frac{f'(z)}{f_n'(z)}\right\} \geq \frac{n+\alpha}{n+1} \qquad (z\in\mathcal{U}) \tag{2.58}$$

and

$$\operatorname{Re}\left\{\frac{f_n'(z)}{f'(z)}\right\} \geq \frac{n+1}{n+2-\alpha} \qquad (z\in\mathcal{U}). \tag{2.59}$$

The results are sharp with the function given by

$$f(z) = z + \frac{1-\alpha}{(n+1)^2} z^{n+1}. \tag{2.60}$$

Taking $\Phi(z) = (1-\delta)z/(1-z)^2 + \delta(z+z^2)/(1-z)^3$ and $\Psi(z) = z$ in Theorem 2, we obtain

Corollary 14. *Let the function $f(z)$ be defined by (1.1). If*

$$\sum_{k=2}^{\infty} [(1-\delta)k + \delta k^2]\,|a_k| \leq 1-\alpha \tag{2.61}$$

then

$$\operatorname{Re}\left\{\frac{f'(z)}{f_n'(z)}\right\} \geq \frac{n\delta+\alpha}{1+n\delta} \qquad (z\in\mathcal{U}) \tag{2.62}$$

and

$$\operatorname{Re}\left\{\frac{f_n'(z)}{f'(z)}\right\} \geq \frac{1+n\delta}{2+n\delta-\alpha} \qquad (z \in \mathcal{U}). \tag{2.63}$$

The results are sharp with the function given by

$$f(z) = z + \frac{1-\alpha}{(n+1)(1+n\delta)} z^{n+1}. \tag{2.64}$$

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Department of Mathematics,
Al al-Bayt University,
Mafraq, Jordan
E-mail address: bafrasin@yahoo.com.