Acta Math. Univ. Comenianae
Vol. LX, 1(1991), pp. 31–34

ON QUASI–CONTINUOUS BIJECTIONS

Z. GRANDE AND T. NATKANIEC

In memory of Prof. T. Neubrunn

Abstract. The well-known classical theorem ascertains that if f is a one-to-one continuous function from I onto I, then the inverse function f^{-1} is continuous too (i.e. f is a homeomorphism). The purpose of this paper is to state that the analogous result does not hold for quasi-continuous functions.

Let us establish some terminology to be used later. $\mathbb{R}$ denotes the set of all reals and I denotes the closed unit interval $[0,1]$. For topological spaces X and Y a function $f\colon X \to Y$ is said to be quasi-continuous at a point $x \in X$ if for every open neighbourhoods U of x and V of $f(x)$ there exists a non-empty open set $W \subset U \cap f^{-1}(V)$. If Y is a metric space then a function $f\colon X \to Y$ is said to be cliquish at a point $x \in X$ if for every open neighbourhood U of x and for every $\varepsilon > 0$ there exists a non-empty set $W \subset U$ with $\operatorname{osc}_W f < \varepsilon$ ([**3**] and [**1**]). It is well-known (and easy to see) that a real function f defined on $\mathbb{R}$ is cliquish iff it is pointwise dicontinuous. Additionally, each quasi-continuous function $f\colon \mathbb{R} \to \mathbb{R}$ is cliquish and therefore it has the Baire property (see e.g. [**4**]). A function $f\colon \mathbb{R} \to \mathbb{R}$ is said to be left (right) hand sided quasi-continuous at a point $x \in \mathbb{R}$ if for every $\varepsilon > 0$ and for every open neighbourhood V of $f(x)$ there exists a non-empty open set $W \subset (x-\varepsilon, x) \cap f^{-1}(V)$ ($W \subset (x, x+\varepsilon) \cap f^{-1}(V)$). f is bilaterally quasi-continuous at x if it is both left and right hand sided quasi-continuous at this point. Every set which is homeomorphic with the Cantor ternary set $C \subset I$ is called a Cantor set.

Lemma 1. *For a closed interval $J = [a,b]$ and a Cantor set K there exists a strictly increasing quasi-continuous function from J into K.*

Proof. Let $(q_n)_{n=1}^{\infty}$ be a one-to-one sequence of all rationals from I. Let $g\colon I \to I$ be the function defined as $g(x) = \sum_{q_n \le x} 2^{-n}$. It is obvious that g is a strictly increasing and right hand sided continuous (hence quasi-continuous) function. Moreover the set

$$I \backslash g(I) = \cup_{n=1}^{\infty} \left[\sum_{q_m < q_n} 2^{-m}, \sum_{q_m \le q_n} 2^{-m} \right)$$

Received February 19, 1991; revised March 18, 1991.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 26A15.

is dense in I. Thus the set $g(I)$ is nonwhere dense and dense in itself. Consequently, $\overline{g(I)}$ is a Cantor set. Let h_1 and h_2 be increasing homeomorphisms from J onto I and from $\overline{g(I)}$ onto K, respectively. Then the composition $f = h_2 \circ g \circ h_1$ satisfies all conditions of Lemma 1. □

Proposition 1. *There exists a measurable and quasi-continuous bijection f from I onto I such that the function f^{-1} does not have the Baire property (hence f^{-1} is not quasi-continuous) and is non-measurable.*

Proof. Let C be the Cantor ternary set and let $(J_{k,n})_{k,n=1}^{\infty}$ be a one-to-one sequence of all components of the set $I \setminus C$ such that for each $n \in N$ the set $\cup_{k=1}^{\infty} J_{k,n}$ is dense in C. Let $(I_n)_{n=1}^{\infty}$ be a basis of I consisting of intervals and let $(C_{k,n})_{k,n=1}^{\infty}$ be a sequence of pairwise disjoint Cantor sets having Lebesgue measure zero and such that $C_{k,n} \subset I_n$ for each $k \in N$. For any $k, n \in N$ let $f_{n,k} \colon \overline{J_{k,n}} \to C_{k,n}$ be a function as in Lemma 1. Then the union $\cup_{k,n=1}^{\infty} f_{k,n}(\overline{J_{k,n}})$ is a set of the first category and measure zero. Let $\{A, B\}$ be a decomposition of $I \setminus \cup_{k,n=1}^{\infty} f_{k,n}(\overline{J_{k,n}})$ into two non-measurable sets of the cardinality of the continuum and without the Baire property. Let C_0 be the set consisting of $0, 1$, and all bilaterally accumulation points of C and let $f_{0,0}$, $f_{0,1}$ be bijections from $C_0 \cap [0, 1/2]$ onto A and from $C_0 \cap [0, 1/2]$ onto B, respectively. Let us put $f = f_{0,0} \cup f_{0,1} \cup \bigcup_{k,n=1}^{\infty} f_{k,n}$. It is obvious that f is measurable bijection from I onto I. Since f is quasi-continuous on the open set $\cup_{k,n=1}^{\infty} J_{k,n}$, we shall focus on points $x \in C$ and verify that f is even bilaterally quasi-continuous at any of these points. According to the quasi-continuity of $f_{k,n}$ on the whole $\overline{J_{k,n}}$, for x being a right- (left-) hand sided accumulation point of C we need to show only quasi-continuity from the right (left). Let us assume that x is a righ-hand sided accumulation point of C, V is an open neighbourhood of $f(x)$ and ε is a fixed positive number. Then $I_n \subset V$ for some $n \in N$ and $\overline{J_{k,n}} \subset (x, x + \varepsilon)$ for some $k \in N$. Therefore $f(J_{k,n}) = f_{k,n}(J_{k,n}) \subset C_{k,n} \subset I_n \subset V$. On the other hand, $C_0 \cap [0, 1/2]$ is a Borel measurable set and $f(C_0 \cap [0, 1/2]) = A$ is non-measurable and without the Baire property. Thus f^{-1} is non-measurable and does not have the Baire property. □

Theorem 1. *Let us suppose that X and Y are topological spaces and f is a quasi-continuous bijection from X onto Y. If $\operatorname{int}_Y(f(V))$ is non-empty for each non-empty open set $V \subset X$, then f^{-1} is quasi-continuous.*

Proof. If y is an isolated point of Y then f^{-1} is continuous at y. Let y be an accumulation point of Y. Let us fix open neighbourhoods U of y and V of $x = f^{-1}(y)$. Since f is quasi-continuous at x, there is a non-empty open subset W of V such that $f(W) \subset U$. So $U_0 = \operatorname{int} f(W)$ is a non-empty open subset of U and $f^{-1}(U_0) \subset V$. Thus f^{-1} is quasi-continuous at y. □

Let us recall that a function $f \colon X \to Y$ is said to be somewhat continuous if $\operatorname{int}(f^{-1}(V))$ is non-empty for any open set $V \subset Y$ with $f^{-1}(V) \neq \emptyset$ [**2**]. It is known that quasi-continuity implies somewhat continuity but there exist somewhat continuous functions which are not quasi-continuous [**4**]. Thus from Theorem 1 it follows that for a quasi-continuous bijection f the quasi-continuity and the somewhat continuity of f^{-1} are equivalent. The analogous theorem does not hold for cliquish functions.

Proposition 2. *There exists a cliquish, measurable bijection f from I onto I such that* $\operatorname{int} f(V)$ *is non-empty for every non-empty open set $V \subset I$, and f^{-1} is non-measurable and without the Baire property.*

Proof. Let $C \subset I$ be the Cantor ternary set. For each $n \in N$ let I_n denote the open interval $(1/(n+1), 1/n)$. Let $\{A_0, A_1\}$ be a decomposition of I_1 into non-measurable sets of the cardinality of the continuum and without the Baire property,

$$f_0\colon C \cap [0, 1/2] \to A_0 \cup \{0, 1, 1/2, \dots\} \quad \text{and} \quad f_{-1}\colon C \cap [1/2, 1] \to A_1$$

be bijections. Let $(J_n)_{n=1}^{\infty}$ be a one-to-one sequence of all components of the set $I \backslash C$. For each $n \in N$ let f_n be a linear function from J_n onto I_{n+1} and let $f = \cup_{n=-1}^{\infty} f_n$. Evidently the function f is measurable bijection from I onto I. Since f is continuous at each point of $I \backslash C$, it is cliquish. Moreover, $\operatorname{int} f(V) \supset \operatorname{int} f(V \backslash C) \neq \emptyset$ for each non-empty, open set $V \subset I$. Finally, the set $C \cap [1/2, 1]$ is closed and $f(C \cap [1/2, 1])$ is non-measurable and without the Baire property (hence f^{-1} is not cliquish). □

Let us remark that from Propositions 1 and 2 it follows that for a cliquish bijection f from I onto I neither of the conditions:

1. $\operatorname{int} f^{-1}(V) \neq \emptyset$ for non-empty open sets V,
2. $\operatorname{int} f(V) \neq \emptyset$ for non-empty open sets V,

is not sufficient for cliquishness of the inverse function f^{-1}. On the other hand we have the following result.

Theorem 2. *Let us suppose that X and Y are metric spaces and f is a bijection from X onto Y. If* $\operatorname{int}_Y f(U)$ *and* $\operatorname{int}_X f^{-1}(V)$ *are non-empty for each non-empty open sets U in X and V in Y, then the functions f and f^{-1} are cliquish.*

Proof. It is enough to prove that f is cliquish. Let us fix a point x of X, an open neighbourhood U of x and $\varepsilon > 0$. Since $\operatorname{int} f(U)$ is non-empty, we can find a non-empty, open set $V \subset f(U)$ with the diameter less than ε. Since $f^{-1}(V) \neq \emptyset$, there exists a non-empty open set $W \subset f^{-1}(V) \subset U$. Evidently, $\operatorname{osc}_W(f) \leq \varepsilon$, what finishes the proof. □

The following example proves that the assumptions of Theorem 2 do not imply the quasi-continuity of f. Let $f\colon I \to I$ be the function defined by

$$f(x) = x \text{ for } x \in (0,1) \quad \text{and} \quad f(x) = 1 - x \text{ for } x \in \{0,1\}.$$

Then f satisfies all assumptions of Theorem 2 but f (and f^{-1}) are not quasi-continuous at $x \in \{0,1\}$.

Let us assume that $f\colon X \to Y$ is a bijection and f, f^{-1} are quasi-continuous. It is natural to ask whether f is a homeomorphism. It is not difficult to find an example a bijection from I onto I which answers this question in the negative (e.g. $f(x) = x$ for $x \in [0, 1/2)$ and $f(x) = 3/2 - x$ for $x \in [1/2, 1]$). But this example leads in a natural way to the question whether a bijection f from I onto I is a homeomorphism if f and f^{-1} are bilaterally quasi-continuous.

Proposition 3. *There exists a bilaterally quasi-continuous bijection f from I onto I for which the inverse function f^{-1} is equal to f and which is not continuous (hence f is not a homeomorphism).*

Proof. Let $C \subset I$ be the Cantor ternary set and let C_0 be the set consisting of $0, 1$, and all bilaterally accumulation points of C. We can arrange all connected components of $I \setminus C$ in a one-to-one sequence $(I_n)_{n=1}^{\infty}$ such that both sets $A_0 = \cup_{k=1}^{\infty} \overline{I_{2k}}$ and $A_1 = \cup_{k=1}^{\infty} \overline{I_{2k-1}}$ are dense in C and that for any $x \in [0, 1]$, $j \in \{0, 1\}$, $x \in A_j$ iff $1 - x \in A_j$. Let f be the function defined by

$$f(x) = x \text{ for } x \in A_0 \cup C_0 \text{ and } f(x) = 1 - x \text{ for } x \in A_1.$$

Obviously f is a bijection from I onto I and f is discontinuous e.g. at $x = 0$. Moreover, it is easy to see that f is bilaterally quasi-continuous and the inverse function f^{-1} is equal to f, so f satisfies all conditions of Proposition 3. □

References

1. Bledsoe W. W., *Neighbourly functions*, Proc. Amer. Math. Soc. **3** (1972), 114–115.

2. Gentry K. R. and Hoyle H. B., *Somewhat continuous functions*, Czech. Math. J. **21** (1971), 5–12.

3. Kempisty S., *Sur les fonctions quasicontinues*, Fund. Math. **19** (1932), 184–197.

4. Neubrunn T., *Quasi-continuity*, Real Analysis Exchange **14** (1988-89), 259–306.

Z. Grande, Institut Matematiky UMK, ul. Chopina 12/18, 87-100 Toruń, Poland

T. Natkaniec, Institut Matematyki WSP, ul. Chodkiewicza 30, 85-064 Bydgoszcz, Poland