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COMMON k–TRANSVERSAL OF FINITE FAMILIES

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1. Introduction

Let $\mathcal{A} = (A_i, i \in I)$ be a family of subsets of a set E, let k be a natural number. Suppose that there exists a mapping $f\colon M \to E$, $M \subseteq I \times \{1, 2, \dots, k\}$, such that $f(i, j) \in A_i$ for each $(i, j) \in M$ and f is injective in both coordinates, i.e. $f(i, j_1) \neq f(i, j_2)$ for $j_1 \neq j_2$, $f(i_1, j) \neq f(i_2, j)$ for $i_1 \neq i_2$. Then the bag (= multiset) $\{f(i, j), (i, j) \in M\}$ is called partial k-transversal of cardinality $|M|$ or of defect $k \cdot |I| - |M|$. In the case $M = I \times \{1, 2, \dots, k\}$ the bag $\{f(i, j), (i, j) \in M\}$ is referred to as (full) k-transversal.

Clearly, for $k = 1$, the concept of a partial k-transversal coincides with the concept of a partial transversal defined in the usual manner.

k-transversals have turned out to play an important role in transversal theory. There are two fundamental results concerning both transversal and matroids. Rado (Theorem 6.2.1 of [**4**]) established a necessary and sufficient condition for a finite family of sets to possess a transversal which is independent in a given matroid. The second result, stated by Edmonds and Fulkerson (Theorem 6.5.2 of [**4**]), says that the partial transversals of a finite family of sets form a matroid. The two theorems lie at the very heart of transversal theory and therefore there are many variations and generalizations of them. Yet, the generalization of these milestone results seem to go in different directions.

In [**2**], the first "parallel" generalization of them has been obtained. The results might be of great help when proving other generalizations in transversal theory. A possible utilization is presented here.

In transversal theory there exist plenty of results concerning various aspects of common transversal. A survey of this field can be found in [**4**], for later ones see e.g. [**1**], [**3**]. In this paper we determine a necessary and sufficient condition for two finite families $\mathcal{A}$, $\mathcal{B}$ of the sets to possess a common partial k-transversal of cardinality d, that is the criterion for the existence of a bag of cardinality d which is a partial k-transversal of both $\mathcal{A}$ and $\mathcal{B}$. This result is a generalization of several theorems among others also of a classical theorem of Ford and Fulkerson

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(Corollary 9.3.4 in [**4**]).

2. Preliminaries

For the sake of compact formulations, besides the concept of a set, we will also make use of the concept of a bag. Sometimes bags are also called multisets. A bag is a collection of elements over some domain allowing multiple occurrences of elements. For a bag B the $|(x,B)|$ function defines the number of occurrences of an element x in a bag B. The cardinality $|B|$ of a bag B is defined by $|B| = \sum_x |(x,B)|$. A bag A is a subbag of a bag B, denoted by $A \subseteq B$, if $|(x,A)| \leq |(x,B)|$ for each x. For the bag union $A \cup B$, the bag intersection $A \cap B$, the bag difference $A - B$ we have $|(x, A \cup B)| = \max(|(x,A)|, |(x,B)|)$, $|(x, A \cap B)| = \min(|(x,A)|, |(x,B)|)$, $|(x, A-B)| = \max(0, |(x,A) - |(x,B)|)$. Let E be a set and d be a natural number. Then the bag space $\overline{E}_d$ is the collection of all bags B satisfying, for any $x \in E$, $0 \leq |(x,B)| \leq d$.

To avoid misunderstanding any time we treat bags we will refer to it explicitly.

Let $r\colon \overline{E}_d \to N = \{0,1,2,\dots\}$ be a function with the following properties:

(i) $r(A) \leq |A|$,
(ii) if $A \subset B$, then $r(A) \leq r(B)$,
(iii) r is submodular.

The pair $(\overline{E}_d, r)$ is called a bag matroid over E and the bags B of $\overline{E}_d$ with the property $r(B) = |B|$ are called independent. When not specified, the subscript d will be dropped. It obvious that the bag matroids are a special case of supermatroids [cf. 5].

Further, let $\mathcal{A} = (A_i, i \in I)$ be a family of sets, let k be a natural number and $J \subseteq I$. Then by $A(J)_k$ we will denote the bag B such that $|(x,B)| = \max(k, |\{i, i \in J, x \in A_i\}|)$. For $k = 1$, $A(J)_1 = A(J) = \cup_{i \in J} A_i$ which is the standard notation in transversal theory.

Using powerful matroid techniques, Mirsky and Perfect presented in [**4**] a "concise" proof of the necessary and sufficient condition for families $\mathcal{A}$ and $\mathcal{B}$ to possess a common partial transversal (i.e. for $k = 1$). To be able to make use of their idea for general k we state two theorems of [**2**].

Theorem 1([2]). *Let $\mathcal{A} = (A_i, i \in I)$ be a finite family of subsets of a set E, $\mathcal{M} = (\overline{E}, r)$ be a bag matroid over E, and let k be a natural number. Then $\mathcal{A}$ possesses a partial k-transversal of detect m, which is independent in $\mathcal{M}$, if and only if, for each $J \subseteq I$, $r(A(J)_k) \geq k \cdot |J| - m$.*

Theorem 2([2]). *Let $\mathcal{A} = (A_i, i \in I)$ be a finite family of subsets of a set E, and let k be a natural number. Then there exists a bag matroid over E such that collection of its independent bags coincides with the collection of all partial k-transversals of $\mathcal{A}$.*

3. Results

We start this section with the main result of the paper.

Theorem 3. *Let k, p be natural numbers. Then finite families $\mathcal{A} = (A_1, \ldots, A_m)$, $\mathcal{B} = (B_1, \ldots, B_n)$ of subsets of a set E possess a common partial k-transversal of cardinality p if and only if $|A(J)_k \cap B(I)_k| \geq k(|J| + |I| - m - n) + p$ for all $J \subseteq \{1, \ldots, m\}$, $I \subseteq \{1, \ldots, n\}$.*

Proof. First of all we prove an auxiliary statement: Let Y be a bag over E. Then Y contains a subbag which is a partial k-transversal of detect d of $\mathcal{B} = (B_1, \ldots, B_n)$ if and only if $|B(I)_k \cap Y| \geq k \cdot |I| - d$ for all $I \subseteq \{1, \ldots, n\}$.The left to right implication is an immediate consequence of Theorem 1 for the free bag matroid. The converse will be proved by induction with the governing index n. The first step of induction, for $n = 1$, is straightforward. Now let X be a minimum subbag of Y satisfying the condition $|B(I)_k \cap X| \geq k \cdot |I| - d$ for all $I \subseteq \{1, \ldots, n\}$. By virtue of minimality of X there exists $I_0 \subseteq \{1, \ldots, n\}$ such that

$$|B(I_0)_k \cap X| = k \cdot |I_0| - d. \tag{1}$$

Consider two cases.

a) There exists a proper subset I_0 of $\{1, \ldots, n\}$ with property (1). Put $X_1 = B(I_0)_k \cap X$ and $X_2 = X - X_1$. Then, for all $I_1 \subseteq I_0$, $|B(I_1)_k \cap X_1| = |B(I_1)_k \cap X| \geq k \cdot |I_1| - d$ holds. Further, for all $I_2 \in \{1, \ldots, n\} - I_0$, we have $k(|I_0| + |I_2|) - d \leq |B(I_0 \cup I_2)_k \cap X| \leq |B(I_0)_k \cap X| + |B(I_2)_k \cap X_2| = k \cdot |I_0| - d + |B(I_2)_k \cap X_2|$, i.e. $|B(I_2)_k \cap X_2| \geq k \cdot |I_2|$. Thus, from the induction hypothesis, X_1 contains a subbag Z_1 which is a partial k-transversal of $(B_i, i \in I_0)$ of defect d and X_2 contains a subbag Z_2 which is a full k-transversal of $(B_i, i \in \{1, \ldots, n\} - I_0)$. However, from the minimality of X, we have $|(x, X)| \leq k$ for each $x \in E$. Thus, $Z = Z_1 \cup Z_2 \subseteq X \subseteq Y$ is a partial k-transversal of $\mathcal{B}$ of detect d.

b) For every proper subset I of $\{1, \ldots, n\}$ it holds $|B(I)_k \cap X| > k \cdot |I| - d$ and only for $I = \{1, \ldots, n\}$ the property (1) is satisfied. Then, on the basis of the minimality of X, $|X| = k \cdot n - d$. If $\mathcal{B}$ has the property

$$\sum_{1 \leq i \leq n} |B_i| = k \cdot n - d \tag{2}$$

then X is the required partial k-transversal. If not, then there exists an element in some of B's whose omitting does not violate the condition $|B(I)_k \cap X| \geq k|I| - d$, $I \subseteq \{1, \ldots, n\}$. Repeatedly using the above procedure we have to arrive at a family $\mathcal{B}' = (B'_1, \ldots, B'_n)$, $B'_i \subseteq B_i$, $1 \leq i \leq n$, which either satisfies (1) for a proper subset I_0 of $\{1, \ldots, n\}$ or satisfies (2). The proof of the statement follows.

Now let $\mathcal{A} = (A_1, \ldots, A_m)$, $\mathcal{B} = (B_1, \ldots, B_n)$ be families of subsets of a set E. According to Theorem 2 the set of all partial k-transversals of $\mathcal{B}$ coincides with the collection of independent bags of a bag matroid $\mathcal{M} = (\overline{E}, r)$. From Theorem 1 the family $\mathcal{A}$ possesses a partial k-transversals of cardinality p, i.e. with defect $d = m \cdot k - p$, which is independent in $\mathcal{M}$ (and therefore, at the same time, is a partial k-transversal of $\mathcal{B}$) if and only if $r(A(J)_k) \geq k \cdot |J| - d = k \cdot |J| - (m \cdot k - p)$ for each $J \subseteq \{1, \dots, m\}$. Like in the case of matroids we have, for any bag D, $r(D) = \max\{|C|, C \subseteq B,\ C$ is an independent bag$\}$. Thus, $R(A(J)_k) \geq k \cdot |J| - (m \cdot k - p)$ precisely when $A(J)_k$ contains as a subbag a partial k-transversals of $\mathcal{B}$ of cardinality $k \cdot |J| - (m \cdot k - p)$, that is of defect $d' = n \cdot k + m \cdot k - k \cdot |J| - p$. In accordance with our auxiliary result this happens if and only if $|B(I)_k \cap A(J)_k| \geq k \cdot |I| - d' = k \cdot |I| + k \cdot |J| - k \cdot m - k \cdot n + p$. The proof is complete. □

As an immediate consequence of Theorem 3 we obtain

Corollary 1. *The maximum cardinality t of common partial k-transversals of the families $\mathcal{A} = \{A_1, \dots, A_m\}$, $\mathcal{B} = \{B_1, \dots, B_n\}$ is given by*

$$t = k(m+n) + \min\{A(J)_k \cap B(I)_k - k|J| - k|I|\},$$

where the minimum runs over all $J \subseteq \{1, \dots, m\}$, $I \subseteq \{1, \dots, n\}$.

Putting $k = 1$ in Theorem 3 we get Theorem 9.3.2 of [**4**]. For $n = m$, $k = 1$ we obtain the following generalization of the criterion of Ford and Fulkerson (Corollary 9.3.4 of [**4**]).

Corollary 2. *Let $\mathcal{A} = (A_1, \dots, A_n)$, $\mathcal{B} = (B_1, \dots, B_n)$ be finite families of subsets of a set E, let k be a natural number. Then $\mathcal{A}$ and $\mathcal{B}$ possess a common k-transversal if and only if, for all $I, J \subseteq \{1, \dots, n\}$, $|A(I_k) \cap B(J)_k| \geq k(|I| + |J| - n)$.*

References

1. Brown T. C., *Common transversals for partitions of a finite set*, Discrete Math. **51** (1984), 119–124.
2. Horák P., *Transversals and matroids, Topics in Combinatorics and Graph Theory* (R. Bodendiek, R. Henn, eds.), Physica-Verlag, Heidelberg, 1990, pp. 381–389.
3. Livingston M. C., *Common transversals for families of partitions*, Cong. Numer. **40** (1983), 181–187.
4. Mirsky L., *Transversals Theory*, Academic Press, 1971.
5. Welsh D. J. A., *Matroid Theory*, Academic Press, 1976.

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