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SOME SUMMATION FORMULAE OVER THE SET OF PARTITIONS

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ABSTRACT. Some algebraic identities with independent variables are established by means of the calculus on formal power series. Applications to special functions and classical polynomials are also demonstrated.

1. Introduction

Let $\sigma(n)$ denote the set of partitions of n (a nonnegative integer) usually denoted by $1^{k_1}2^{k_2}\dots n^{k_n}$ with $\sum ik_i = n$; k_i is, of course, the number of partitions of size i. If the number of parts for the partition set of n is restricted to k, i.e., $\sum k_i = k$; then the corresponding subset of $\sigma(n)$ is denoted by $\sigma(n,k)$.

For nonnegative integral vector $\bar{k} = (k_1,\dots,k_n)$, the multinomial coefficient $\binom{x}{\bar{k}}$ as usual, is defined by

$$\binom{x}{\bar{k}} = \frac{(x)_{|\bar{k}|}}{\prod (k_i)!}, \tag{1.1}$$

where the finite product $\prod$ runs over i from 1 to n, $(x)_k$ stands for the all factorial notation, and $|\bar{k}|$ represents the coordinate sum for the vector $\bar{k} = (k_1,\dots,k_n)$.

In his recent paper, Chu [**3**] obtained a useful algebraic identity in the form

$$\binom{xy}{n} = \sum_{\sigma(n)} \binom{x}{\bar{k}} \prod \binom{y}{i}^{k_i}, \tag{1.2}$$

where x and y are assumed as two complex numbers. This formula contains numerous similar binomial summations over the set of partitions as special cases.

The purpose of the present paper is to generalize (1.2) involving certain sequences which are generated by the function $f(t)\,[g(t)]^x$ (see Carlitz [**1,** p. 521]). The algebraic identity to be obtained may also be viewed as the means of variable-separation for the sequences involved. The usefulness of our results is depicted by considering some applications which yields summations formulas over the set of partitions for special functions and classical polynomials.

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2. Main Theorem

For formal power series $f(t)$ and $g(t)$, with $f(0) = g(0) = 1$, consider the formal expansions

$$[f(t)]^x = \sum_{n\geq 0} \lambda_n(x) t^n; \quad \lambda_n(1) = \lambda_n; \tag{2.1}$$

$$[g(t)]^x = \sum_{n\geq 0} A_n(x) t^n, \tag{2.2}$$

and

$$f(t)\,[g(t)]^x = \sum_{n\geq 0} c_n(x) t^n, \tag{2.3}$$

where x is an arbitrary complex number independent of t.

It follows from (2.2) and (2.3) that

$$C_n(x) = \sum_{m=0}^{n} \lambda_{n-m} A_m(x). \tag{2.4}$$

We propose to establish the following:

Theorem. *For arbitrary complex numbers x and y,*

$$C_n(xy) = \sum_{m=0}^{n} \lambda_{n-m} \sum_{\sigma(m)} \binom{x}{\bar{k}} \prod [A_i(y)]^{k_i}, \tag{2.5}$$

and

$$C_n(xy) = \sum_{m=0}^{n} \lambda_{n-m}(1-x) \sum_{\sigma(m)} \binom{x}{\bar{k}} \prod [C_i(y)]^{k_i}, \tag{2.6.}$$

Proof. In view of the defining equations (2.2) and (2.3), and the exponential law $a^{xy} = (a^y)^x$, and performing simple calculus on formal power series, we have

$$\begin{aligned} f(t)\,[g(t)]^{xy} &= f(t)\left\{1 + ([g(t)]^y - 1)\right\}^x \\ &= f(t) \sum_{k\geq 0} \binom{x}{k} \left\{ \sum_{i\geq 1} A_i(y) t^i \right\}^k \\ &= f(t) \sum_{k\geq 0} \binom{x}{k} \sum_{\sum k_i = k} \binom{k}{\bar{k}} \prod \left\{ [A_i(y)]^{k_i}\, t^{ik_i} \right\}^k \\ &= f(t) \sum_{k\geq 0} t^k \sum_{\sigma(k)} \binom{x}{k} \prod [A_i(y)]^{k_i}. \end{aligned}$$

The assertion (2.5) follows now on comparing the coefficient of t^n on both the sides.

By writing

$$f(t)\,[g(t)]^{xy} = [f(t)]^{1-x}\left\{1 + (f(t)\,[g(t)]^{y} - 1)\right\}^{x},$$

and proceeding on the same lines as indicated above, the desired result (2.6) is easily arrived at. □

Corollary. *For arbitrary complex numbers x and y,*

$$A_n(xy) = \sum_{\sigma(n)} \binom{x}{\bar{k}} \prod [A_i(y)]^{k_i}\,. \tag{2.7}$$

It follows easily from (2.5) (or (2.6)) in the case when $f(t) \equiv 1$.

3. Applications

Since most of the classical special functions and orthogonal polynomials can be identified with the sequence $\{C_n\}$ defined by (2.3), therefore, our theorem would widely be applicable. To illustrate, we consider the following examples.

Example 1. Let us set

$$f(t) = \frac{1+u-au}{1+u-(a+b)u}, \quad \text{and} \quad g(t) = 1+u,$$

where u is implicitly defined in terms of t by $u = t(1+u)^{a+b}$. By appealing to the results of Gould [**4**]:

$$\sum_{k\geq 0} \frac{x+bk}{x+(a+b)k}\binom{x+(a+b)k}{k} t^k = (1+u)^x \frac{1+u-au}{1+u-(a+b)u}, \tag{3.1}$$

the following combinatorial identities emerge from (2.5) and (2.7):

$$\begin{aligned} \frac{xy+bn}{xy+(a+b)n}\binom{xy+(a+b)n}{n} &= \sum_{m=0}^{n} \frac{b}{a+b}\binom{(a+b)(n-m)}{n-m} \\ &\cdot \sum_{\sigma(m)} \binom{x}{\bar{k}} \prod \left\{ \frac{y}{y+(a+b)i}\binom{y+(a+b)i}{i} \right\}^{k_i} \end{aligned} \tag{3.2}$$

and

$$\frac{xy}{xy+bn}\binom{xy+bn}{n} = \sum_{\sigma(n)} \binom{x}{\bar{k}} \prod \left\{ \frac{y}{y+bi}\binom{y+bi}{i} \right\}^{k_i}. \tag{3.3}$$

It may be observed that for $a = b = 0$, both (3.2) and (3.3) correspond to the formula (1.2).

Example 2. Next let

$$f(t) = \frac{1-av}{1-(a+b)v}, \quad g(t) = e^v,$$

where $v = te^{(a+b)v}$, then (2.5) and (2.7) in conjuction with the known formula (Gould [**4**])

$$\sum_{k\geq 0} \frac{x+bk}{x+(a+b)k}\,[x+(a+b)k]^k\,\frac{t^k}{k\,!} = \frac{(1-av)e^{vx}}{1-(a+b)v}, \tag{3.4}$$

yield the following identities on Abel coefficients:

$$\begin{aligned}\frac{xy+bn}{xy+(a+b)n}\,\frac{[xy+(a+b)n]^n}{n\,!} &= \sum_{m=0}^{n} \frac{b}{a+b}\,\frac{[(a+b)(n-m)]^{n-m}}{(n-m)\,!} \\ &\quad \cdot \sum_{\sigma(m)} \binom{x}{\bar{k}} \prod \left\{ \frac{y}{y+(a+b)i}\,\frac{[y+(a+b)i]^i}{i\,!} \right\}^{k_i}\end{aligned} \tag{3.5}$$

and

$$\frac{xy}{xy+bn}\,\frac{[xy+bn]^n}{n\,!} = \sum_{\sigma(n)} \binom{x}{\bar{k}} \prod \left\{ \frac{y}{y+bi}\,\frac{(y+bi)^i}{i\,!} \right\}^{k_i}. \tag{3.6}$$

Incidentally, Gould's formula (3.1) has also been used by Chu [**2**] to derive new partition identities.

Example 3. For the Laguerre polynomials, we have the generating function [**5,** p. 84, Eqn. (15)]

$$\sum_{n=0}^{\infty} L_n^{(\alpha-n)}(x)t^n = (1+t)^{\alpha} e^{-xt}. \tag{3.7}$$

On comparing it with (2.3), we find that (2.5) yields the result

$$L_n^{(\alpha-n)}(xy) = \sum_{m=0}^{n} \frac{(-1)^{n-m}(-\alpha)_{n-m}}{(n-m)\,!} \sum_{\sigma(m)} \binom{x}{\bar{k}} \prod \left\{ \frac{(-y)^i}{i\,!} \right\}^{k_i}. \tag{3.8}$$

Example 4. By considering the generating function [**5,** p. 85] for the Bernoulli polynomials

$$\sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} = \frac{te^{xt}}{e^t-1}, \tag{3.9}$$

then from (2.3) and (2.5) we have

$$\frac{B_n(xy)}{n!} = \sum_{m=0}^{n} \frac{B_{n-m}}{(n-m)!} \sum_{\sigma(m)} \binom{x}{\bar{k}} \prod \left\{\frac{y^i}{i!}\right\}^{k_i}, \tag{3.10}$$

where B_n denotes the Bernoulli numbers.

Example 5. If we consider the Lagrange polynomials [**5,** p. 85] defined by the generating function

$$\sum_{n=0}^{\infty} g_n^{(x,z)}(u,v)t^n = (1-ut)^{-x}(1-vt)^{-z}, \tag{3.11}$$

then from (2.3) and (2.5) we get

$$g_n^{(xy,z)}(u,v) = \sum_{m=0}^{n} \frac{v^{n-m}}{(n-m)!}(z)_{n-m} \sum_{\sigma(m)} \binom{x}{\bar{k}} \prod \left\{\frac{(y)_i u^i}{i!}\right\}^{k_i}. \tag{3.12}$$

Example 6. For the generalized hypergeometric function we have the generating function [**5,** p. 139, Eqn. (10)]

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!}\, {}_{p+1}F_{q+1}\left[\begin{matrix} -n\ ;\ (a_p)\ ; \\ 1-\lambda-n\ ;\ (b_q)\ ; \end{matrix}\ x\right] t^n = (1-t)^{-\lambda}\, {}_pF_q\left[\begin{matrix} (a_p)\ ; \\ (b_q)\ ; \end{matrix}\ xt\right]. \tag{3.13}$$

Comparing it with (2.3), then (2.5) gives

$$\begin{aligned} &(xy)_n\, {}_{p+1}F_{q+1}\left[\begin{matrix} -n\ ;\ (a_p)\ ; \\ 1-x-n\ ;\ (b_q)\ ; \end{matrix}\ z\right] \\ &\quad = \sum_{m=0}^{n} \binom{n}{m} \frac{\prod_{j=1}^{p}(a_j)_{n-m}}{\prod_{j=1}^{q}(b_j)_{n-m}} m!\, z^{n-m} \sum_{\sigma(n)} \binom{x}{\bar{k}} \prod \left\{\frac{(y)_i}{i!}\right\}^{k_i}, \end{aligned} \tag{3.14}$$

where (a_p) denotes the array of p-parameters $a_1,\dots,a_p$, and here in (3.12) to (3.14), $(x)_n$ stands for the Pochhammer symbol $(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)}$.

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