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ERROR ESTIMATES OF A LINEAR APPROXIMATION SCHEME FOR NONLINEAR DIFFUSION PROBLEMS

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1. Introduction

The aim of this paper is to analyze the accuracy of a linear approximation scheme for nonlinear parabolic problems of the type:

$$u_t(t,x) - \Delta\beta(u(t,x)) = f(t,x,\beta(u(t,x))) \qquad (t,x) \in Q := (0,T] \times \Omega, \tag{1.1}$$
$$\beta(u(0,x)) = \beta(u_0(x)) \qquad \text{on } \Omega, \tag{1.2}$$
$$-\partial_\nu \beta(u(t,x)) = \gamma\beta(u(t,x)) + \varphi(t,x) \qquad \text{on } (0,T] \times \Gamma, \tag{1.3}$$

where $u\colon (0,T] \times \Omega \to \mathbb{R}$ is the unknown function, $\Omega \subset \mathbb{R}^N$ is a bounded domain with a Lipschitz continuous boundary Γ, $0 < T < \infty$ and $\partial_\nu\beta(u)$ denotes outside normal derivative. Function $\beta\colon \mathbb{R} \to \mathbb{R}$ is nondecreasing Lipschitz continuous function satisfying (2.1). Function $f\colon (0,T] \times \Omega \times \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous with Lipschitz constant L_f and satisfies (2.2). We denote $g(t,x,s)\colon = \gamma s + \varphi(t,x)$, where γ and φ satisfy (2.3).

These problems have been of great interest in recent years both from theoretical and numerical point of view. Method based on the so-called nonlinear Chernoff's formula has been studied by Rogers, Berger and Brezis in [**1**]. Their linear approximation scheme corresponding to (1.1)–(1.3) reads as follows:

$$\theta_i - \frac{\tau}{\mu}\,\Delta\theta_i = \beta(u_{i-1}) + \frac{\tau}{\mu} f(t_i, x, \beta(u_{i-1})) \qquad \text{in } \Omega, \tag{1.4}$$
$$-\partial_\nu \theta_i = g(t_i, x, \theta_{i-1}) \qquad \text{on } \Gamma, \tag{1.5}$$
$$u_i = u_{i-1} + \mu(\theta_i - \beta(u_{i-1})) \qquad \theta_0 = \beta(u_0(x)), \tag{1.6}$$

where $\tau = \frac{T}{n}$, $t_i = i\tau$, $(i = 1, ..., n)$ and $0 < \mu \le L_\beta^{-1}$ (L_β is Lipschitz constant of β, μ is a relaxation parameter).

Solving (1.4)–(1.5) we first obtain aproximation θ_i of $\beta(u(t_i,x))$ in time steps t_i and then from algebraic equation (1.6) we can compute approximation u_i of

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$u(t_i, x)$. The error estimates of this scheme are analyzed by Magenes, Nochetto and Verdi in [**7**] for Dirichlet boundary conditions. Nonlinear Neumann type conditions are treated by Nochetto in [**9**], [**10**] Verdi in [**11**] and Kačurová in [**5**].

However, the results of numerical experiments obtained using this scheme are not satisfactory, esspecially in porous medium type problems with interface. The main argument for this is that θ_i represents an approxiamtion of $\beta(u_i)$ and μ represents an approximation of $\frac{1}{\beta'(u)}$ which is unbounded in the neighbourhood of the interface (i.e. where $\beta'(u) = 0$). Thus, the fact that μ attains small values ($\mu \le L_\beta^{-1}$) is a source of errors in the neighbourhood of the interface.

Another approximation scheme for problems of the type (1.1)–(1.3) was treated by Jager and Kačur in [**2**] and by Kačur, Handlovičová and Kačurová in [**4**]. They used the following scheme:

$$\mu_i(\theta_i - \beta(u_{i-1})) - \tau\Delta\theta_i = \tau f(t_i, x, \beta(u_{i-1})) \quad \text{in } \Omega, \tag{1.7}$$

$$-\partial_\nu\theta_i = g(t_i, x, \theta_{i-1}) \quad \text{on } \Gamma, \tag{1.8}$$

$$u_i = u_{i-1} + \mu_i(\theta_i - \beta(u_{i-1})), \quad i = 1, \dots, n \quad \theta_0 = \beta(u_0), \tag{1.9}$$

with convergence test

$$|\beta(u_{i-1} + \mu_i(\theta_i - \beta(u_{i-1}))) - \beta(u_{i-1})| \le \alpha|\theta_i - \beta(u_{i-1})| + o\left(\frac{1}{n}\right), \tag{1.10}$$

where $\mu_i \in \mathrm{L}_\infty(\Omega)$, $0 < \delta \le \mu_i \le \mathrm{K}$ for every $i = 1, \dots, n$ and K^{-1}, $1 - \alpha$ and δ are sufficiently small constants.

The main difference between these two schemes is the fact that so called relaxation parameter of scheme (1.4)–(1.6) is substituted by function in scheme (1.7)–(1.10) The latter scheme has the disadvantage of not being explicit with respect to θ_i, μ_i. We can determine μ_i and θ_i by an iterative method. For further details see [**2**] or [**4**].

Let denote

$$\begin{aligned} e_\theta(t, x) &= \beta(u(t, x)) - \theta_i(x) \\ e_u(t, x) &= u(t, x) - u_i(x) \;\; \text{for} \;\; t \in [t_{i-1}, t_i]. \end{aligned} \tag{1.11}$$

The main results of this paper are the following error estimates for the scheme (1.7)–(1.10):

$$\|e_\theta\|^2_{L_2(Q)} + \|e_u\|^2_{L_2(0,T,H^{-1}(\Omega))} + \operatorname{ess}\sup_{0\le t\le T}\left\|\nabla\int_0^t e_\theta\, ds\right\|^2_{L_2(\Omega)} \le C\tau \tag{1.12}$$

for strictly increasing β and

$$\|e_\theta\|^2_{L_2(Q)} + \|e_u\|^2_{L_2(0,T,H^{-1}(\Omega))} + \operatorname{ess}\sup_{0\le t\le T} \left\|\nabla \int_0^t e_\theta\, ds\right\|^2_{L_2(\Omega))} \le C\tau^{\frac{1}{2}} \tag{1.13}$$

for β nondecreasing.

The paper is organized as follows:

In Section 2 we summarize the results of [**2**] and [**4**] and we give some notations and assumptions.

In Section 3 we prove the error estimates of numerical solution of scheme (1.7)–(1.10) for both strictly increasing and nondecreasing nonlinearities of β.

2. Basic Assumptions, Notations and the Previous Convergence and Existence Results

We denote $I = [0, T]$ for $T < \infty$ and $\tau = \frac{T}{n}$, $t_i = i\tau$ for $i = 1, \dots, n$, $I_i = [t_{i-1}, t_i]$ for $i = 1, \dots, n$.

For simplicity let us denote $f(t, s)\colon = f(t, x, s)$, $g(t, s)\colon = g(t, x, s)$, $(u, v)\colon = \int_\Omega u(x)v(x)\, d\Omega$ and $(u, v)_\Gamma := \int_\Gamma u(s)v(s)\, ds$. Further we use functional spaces and their symbols as in [**6**]. By $\|.\|$, $|.|_\Omega$ and $|.|_\Gamma$ we denote the norms in the functional spaces $H(\Omega)\colon = W^1_2(\Omega)$, $L_2(\Omega)$ and $L_2(\Gamma)$, respectively. By $\langle ., . \rangle$ we denote the duality between $H^*(\Omega)$ (dual space to space $W^1_2(\Omega)$) and $H(\Omega)$. By C we denote a generic positive constant.

We shall assume that

(2.1) $\beta\colon \mathbb{R} \to \mathbb{R}$ is nondecreasing Lipschitz continuous function satisfying $0 \le l_\beta \le \beta'(s) \le L_\beta < \infty$ for a.e. $s \in \mathbb{R}$; $|\beta(s)| \ge c_1|s| - c_2 \;\; \forall s \in \mathbb{R}$ and some constants $c_1, c_2 \in \mathbb{R}$ and $\beta(0) = 0$.

Function

(2.2) $f\colon I \times \Omega \times \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous with Lipschitz constant L_f and $|f(t, x, s)| \le C(1 + |s|)$, where $C \in \mathbb{R}$.

We denote by

(2.3) $g\colon I \times \Gamma \times \mathbb{R} \to \mathbb{R}$ function $g(t, x, s) = \gamma s + \varphi(t, x)$ where $\gamma \ge 0$ and φ is Lipschitz continuous function with Lipschitz constant L_φ.

For initial condition due to iterative method used in (1.7)–(1.10) we must assume

$$\beta(u_0) \in C^{0.\delta}(\bar{\Omega}), \quad \delta > 0. \tag{2.4}$$

We shall denote

$$\theta^{(n)} := \theta_{i-1} + \frac{t - t_{i-1}}{\tau}(\theta_i - \theta_{i-1}) \text{ for } t \in I_i, \quad i = 1, \dots, n. \tag{2.5}$$

Similarly we define $u^{(n)}$. By $\bar{\theta}^{(n)}$ we denote the corresponding step function

$$\bar{\theta}^{(n)} := \theta_i \text{ for } t \in I_i, \quad i = 1, \ldots, n, \quad \bar{\theta}^{(n)}(0) := \beta(u_0). \tag{2.6}$$

and similarly we define $\bar{u}^{(n)}, \bar{\mu}^{(n)}$.

Definition. Function $u \in L_2(I, L_2(\Omega))$ such that $\partial_t u \in L_2(I, H^*(\Omega))$ is a variational solution of (1.1)–(1.3) iff $\beta(u) \in L_2(I, H(\Omega))$ and

$$\int_I \langle \partial_t u, \varphi \rangle \, dt + \int_I (\nabla \beta(u), \nabla \varphi) \, dt + \int_I (g(t, \beta(u)), \varphi)_\Gamma \, dt = \int_I (f(t, \beta(u)), \varphi) \, dt \tag{2.7}$$

for all $\varphi \in L_2(I, H(\Omega))$.

Let us denote

$$u^i := u(t_i, x), \text{ where } u \text{ is a variational solution of (1.1)–(1.3)} \tag{2.8}$$

$$\partial_t u^i := \frac{u^i - u^{i-1}}{\tau} \text{ for } i = 1, \ldots, n, \tag{2.9}$$

$$\bar{u}^i := \frac{1}{\tau} \int_{I_i} u(., t) \, dt \text{ and similarly } \bar{\beta}^i, \bar{g}^i, \bar{f}^i, \tag{2.10}$$

$$\delta u_i := \frac{u_i - u_{i-1}}{\tau}, \text{ where } u_i \text{ is obtained from the aproximation scheme.} \tag{2.11}$$

In [**2**] and [**4**] the convergence of the approximation scheme (1.7)–(1.10) has been studied for β strictly increasing Lipschitz continuous function and for β nondecreasing function too. Their main results are existence of a variational solution u of (1.1)–(1.3) and the existence of subsequences $\{\theta^{(n_k)}\}, \{u^{(n_k)}\}$ of sequences $\{\theta^{(n)}\}, \{u^{(n)}\}$ such that $u^{(n_k)} \rightharpoonup u$ and $\theta^{(n_k)} \to \theta = \beta(u)$ in $L_2(Q_T)$ and $\theta^{(n_k)} \to \beta(u)$ in $L_2(I, H)$ for $n \to \infty$. Moreover, the estimates:

$$\max_{1 \le i \le n} |\beta(u_i)|_\Omega + \sum_{i=1}^{n} \|\theta_i\|^2 \tau + \sum_{i=1}^{n} |u_i - u_{i-1}|_\Omega^2 \le C \tag{2.12}$$

hold unifornly for $n \ge n_0 > 0$ and

$$\begin{aligned} |\theta_i|_\Omega &\le C, \\ |u_i|_\Omega &\le C \end{aligned} \tag{2.13}$$

for $i = 1, \ldots, n$ uniformly for n.

3. Error Estimates

Putting $\varphi \chi_{[t_{i-1}, t_i]}$ for all $\varphi \in H(\Omega)$ and $i = 1, \ldots, n$ as a test function in (2.7) yields

$$\left\langle \partial_t u^i, \varphi \right\rangle + (\nabla \bar{\beta}^i, \nabla \varphi) + (\bar{g}^i, \varphi)_\Gamma = (\bar{f}^i, \varphi) \tag{3.1}$$

The variational solution of (1.7)–(1.10) satisfies

$$(\delta u_i, \varphi) + (\nabla\theta_i, \nabla\varphi) + (g(t_i, \theta_{i-1}), \varphi)_\Gamma = (f(t_i, \beta(u_{i-1})), \varphi) \tag{3.2}$$

for all $\varphi \in H(\Omega)$ $i = 1, \ldots, n$. Subtracting (3.2) from (3.1) and sum up for $i = 1, \ldots, k$ multiplying by τ we get

$$\begin{aligned} &\langle u^k - u_k, \varphi\rangle + \tau\left(\nabla\sum_{i=1}^{k}(\bar\beta^i - \theta_i), \nabla\varphi\right) \\ &\quad + \tau\left(\sum_{i=1}^{k}(\bar g^i - g(t_i, \theta_{i-1})), \varphi\right)_\Gamma = \tau\left(\sum_{i=1}^{k}(\bar f^i - f(t_i, \beta(u_{i-1}))), \varphi\right). \end{aligned} \tag{3.3}$$

We put $\varphi = \tau(\bar\beta^k - \theta_k) = \int_{I_k} e_\theta\, ds$. We know that $\varphi \in H(\Omega)$ becuase of properties of β and θ_k (see [**4**]). We sum up (3.3) for $k = 1, \ldots, m$ and easily obtain the inequality:

$$\begin{aligned} &\sum_{k=1}^{m}\int_{I_k}\langle e_u, e_\theta\rangle\, dt + \sum_{k=1}^{m}\int_{I_k}\langle u^k - u, e_\theta\rangle\, dt + \sum_{k=1}^{m}\tau^2(\nabla\sum_{i=1}^{k}(\bar\beta^i - \theta_i), \nabla(\bar\beta^k - \theta_k)) \\ &\qquad + \sum_{k=1}^{m}\tau^2(\sum_{i=1}^{k}(\bar g^i - g(t_i, \theta_{i-1})), \bar\beta^k - \theta_k)_\Gamma \\ &\quad = \sum_{k=1}^{m}\int_{I_k}\tau\left(\sum_{i=1}^{k}(\bar f^i - f(t_i, \beta(u_{i-1}))), e_\theta\right) dt\ . \end{aligned} \tag{3.4}$$

We will estimate separately each term in (3.4) as we want to obtain the inequality (3.13), which is the most important point for the proof of Theorem 1 and Theorem 2. We denote (3.4) formally as I + II + III + IV = V.

First we define the function

$$h(s) = s - \frac{\beta(s)}{H}\ ,$$

where

$$H = \max\left\{\frac{1}{\delta}\ ; L_\beta\right\}$$

(L_β is Lipschitz constant for β and δ has the property $0 < \delta \le \mu_i \le K$ for all $i = 1, \ldots, n$ see [**4**]). The function $h(s)$ satisfies :

$$0 \le h'(s) \le 1 \quad \text{for a.e.} \quad s \in \mathbb{R}\ . \tag{3.5}$$

Using (1.9) we have for $t \in I_k$

$$e_\theta = \beta(u) - \theta_k = \beta(u) - \beta(u_{k-1}) - \frac{u_k - u_{k-1}}{\mu_k}$$

Similarly,

$$e_u - \frac{e_\theta}{H} = h(u) - h(u_{k-1}) - \left(1 - \frac{1}{H\mu_k}\right)(u_k - u_{k-1})$$

for every $k = 1, \dots, n$.

Now we will estimate first term I as follows:

$$\begin{aligned} \mathrm{I} &= \sum_{k=1}^{m} \int_{I_k} \langle e_u, e_\theta \rangle \, dt \\ &= \frac{1}{H} \sum_{k=1}^{m} \int_{I_k} |e_\theta|_\Omega^2 \, dt + \sum_{k=1}^{m} \int_{I_k} \langle e_u - \frac{e_\theta}{H}, e_\theta \rangle \, dt = \frac{1}{H} I_1^m + I_2 \ , \end{aligned}$$

where $\mathrm{I}_1^m = \|e_\theta\|^2_{L_2(0,t_m;L_2(\Omega))}$ and

$$\begin{aligned} \mathrm{I}_2 = &\sum_{k=1}^{m} \int_{I_k} \langle h(u) - h(u_{k-1}), \beta(u) - \beta(u_{k-1}) \rangle \, dt \\ &- \sum_{k=1}^{m} \int_{I_k} \left\langle h(u) - h(u_{k-1}), \frac{u_k - u_{k-1}}{\mu_k} \right\rangle dt \\ &- \sum_{k=1}^{m} \int_{I_k} (\left(1 - \frac{1}{H\mu_k}\right)(u_k - u_{k-1}), e_\theta) \, dt \\ \geq\ & l_\beta \sum_{k=1}^{m} \int_{I_k} |h(u) - h(u_{k-1})|_\Omega^2 \, dt - \sum_{k=1}^{m} \int_{I_k} \left\langle h(u) - h(u_{k-1}), \frac{u_k - u_{k-1}}{\mu_k} \right\rangle dt - \\ &- \sum_{k=1}^{m} \int_{I_k} \left(\left(1 - \frac{1}{H\mu_k}\right)(u_k - u_{k-1}), e_\theta\right) dt \ . \end{aligned}$$

Finally :

$$\begin{aligned} \mathrm{I} \geq\ & l_\beta \|h(u) - h(u_{k-1})\|^2_{L_2(0,t_m,L_2(\Omega))} + \frac{\mathrm{I}_1^m}{H} \\ &- \sum_{k=1}^{m} \int_{I_k} \left\langle h(u) - h(u_{k-1}), \frac{u_k - u_{k-1}}{\mu_k} \right\rangle dt \\ &- \sum_{k=1}^{m} \int_{I_k} \left(\left(1 - \frac{1}{H\mu_k}\right)(u_k - u_{k-1}), e_\theta\right) dt \ . \end{aligned} \tag{3.6}$$

It is easily to check that

$$
\begin{aligned}
|\mathrm{II}| &= \left|\sum_{k=1}^{m} \int_{I_k} \langle u^k - u, e_\theta\rangle \, dt\right| \\
&= \left|\sum_{k=1}^{m} \int_{I_k} \left\langle \int_t^{t_k} \partial_t u \, ds, e_\theta(t)\right\rangle dt\right| \\
&\le \tau \|\partial_t u \,\|_{L_2(0,t_m,H^*(\Omega))} \|e_\theta\|_{L_2(0,t_m,H(\Omega))} \ .
\end{aligned}
\tag{3.7}
$$

For treating the term III we use the elementary identity:

$$
2\sum_{k=1}^{m} a_k \left(\sum_{i=1}^{k} a_i\right) = \left(\sum_{k=1}^{m} a_k\right)^2 + \sum_{k=1}^{m} a_k^2 \quad \text{for } a_k \in \mathbb{R} \ . \tag{3.8}
$$

Thus we have

$$
2\mathrm{III} = \left|\nabla \int_0^{t_m} e_\theta(t) \, dt\right|_\Omega^2 + \tau^2 \sum_{k=1}^{m} \left|\nabla(\bar\beta^k - \theta_k)\right|_\Omega^2 \ . \tag{3.9}
$$

The last term on the left hand-side of (3.4) we estimate as follows:

$$
\begin{aligned}
\mathrm{IV} &= \sum_{k=1}^{m} \tau^2 \left(\sum_{i=1}^{k} (\bar g^i - g(t_i, \theta_{i-1})), \bar\beta^k - \theta_k\right)_\Gamma \\
&= \sum_{k=1}^{m} \tau^2 \left(\sum_{i=1}^{k} (\bar g^i - g(t_i, \theta_i)), \bar\beta^k - \theta_k\right)_\Gamma \\
&\quad + \tau^2 \sum_{k=1}^{m} \left(\sum_{i=1}^{k} (g(t_i,\theta_i) - g(t_i, \theta_{i-1})), \bar\beta^k - \theta_k\right)_\Gamma \\
&= \gamma\tau^2 \sum_{k=1}^{m} \left(\sum_{i=1}^{k} (\bar\beta^i - \theta_i), \bar\beta^k - \theta_k\right)_\Gamma \\
&\quad + \tau^2 \sum_{k=1}^{m} \left(\sum_{i=1}^{k} (g(t_i,\theta_i) - g(t_i, \theta_{i-1})), \bar\beta^k - \theta_k\right)_\Gamma \\
&\quad + \tau^2 \sum_{k=1}^{m} \left(\sum_{i=1}^{k} (\bar\varphi^i - \varphi_i), \bar\beta^k - \theta_k\right)_\Gamma = \mathrm{IV}_1 + \mathrm{IV}_2 + \mathrm{IV}_3 \ .
\end{aligned}
$$

Using (2.3) we get after aplying (3.8)

$$\text{IV}_1 = \gamma\tau^2 \sum_{k=1}^{m} \left(\sum_{i=1}^{k} (\bar{\beta}^i - \theta_i), \bar{\beta}^k - \theta_k \right)_\Gamma \tag{3.10}$$

$$= \frac{\gamma}{2}\tau^2 \sum_{k=1}^{m} \left|\bar{\beta}^k - \theta_k\right|^2_{L_2(\Gamma)} + \frac{\gamma}{2} \left| \int_0^{t_m} e_\theta \, dt \right|^2_{L_2(\Gamma)}$$

$$= \frac{\gamma\tau^2}{2}\text{IV}_{11} + \frac{\gamma}{2} \left| \int_0^{t_m} e_\theta \, dt \right|^2_{L_2(\Gamma)} .$$

IV_2 we estiamte as follows:

$$|\text{IV}_2| \le \left| \tau^2 \sum_{k=1}^{m} \left(\sum_{i=1}^{k} (g(t_i, \theta_i) - g(t_i, \theta_{i-1})), \bar{\beta}^k - \theta_k \right)_\Gamma \right|$$

$$\le \gamma\tau^2 \sum_{k=1}^{m} \left(\left| \sum_{i=1}^{k} (\theta_i - \theta_{i-1}) \right|, |\bar{\beta}^k - \theta_k| \right)_\Gamma$$

$$\le \frac{\gamma\tau^2\varepsilon}{2} \sum_{k=1}^{m} |\bar{\beta}^k - \theta_k|^2_\Gamma + \frac{\gamma\tau^2}{2\varepsilon} \sum_{k=1}^{m} \left| \sum_{i=1}^{k} (\theta_i - \theta_{i-1}) \right|^2_\Gamma$$

$$\le \frac{\gamma\tau^2\varepsilon}{2}\text{IV}_{11} + \frac{\gamma\tau^2}{2\varepsilon} C \left(\sum_{k=1}^{m} |\nabla\theta_k|^2_\Omega + |\theta_k|^2_\Omega \right) + \frac{\gamma\tau^2}{2\varepsilon}|\theta_0|^2_\Gamma ,$$

where we have used the facts that:

$$|v|^2_\Gamma \le C \left(\varepsilon |\nabla v \;|^2_\Omega + \frac{1}{\varepsilon}|v|^2_\Omega \right) \qquad \text{(see [\textbf{8,} page 15])},$$

$$ab \le \frac{a^2}{2\varepsilon} + \frac{\varepsilon b^2}{2} \quad \forall a, b \in R; \quad \varepsilon > 0 .$$

Finally we have

$$|\text{IV}_2| \le \frac{\gamma\tau^2}{2}\varepsilon\text{IV}_{11} + C\tau. \tag{3.11}$$

because of (2.12), (2.13) and (2.4).

We can estimate the term IV_3 as follows:

$$|\text{IV}_3| \le \frac{\tau^2\varepsilon}{2} \sum_{k=1}^{m} |\bar{\beta}_k - \theta_k|^2_\Gamma + \frac{\tau^2}{2\varepsilon} \sum_{k=1}^{m} \left| \sum_{i=1}^{k} \bar{\varphi}_i - \varphi_i \right|^2_\Gamma$$

$$\le \frac{\tau^2\varepsilon}{2}\text{IV}_{11} + C\tau^2 \sum_{k=1}^{m} \left| \sum_{i=1}^{k} \frac{L_\varphi}{\tau} \int_{I_i} |t - t_i| \, dt \right|^2_\Gamma \le \frac{\tau^2\varepsilon}{2}\text{IV}_{11} + C\tau .$$

At last we estimate the term on the right hand-side of (3.4):

$$
\begin{aligned}
|\mathrm{V}| &= \left| \sum_{k=1}^{m} \int_{I_k} \left(\sum_{i=1}^{k} \tau(\bar{f}^i - f(t_i,\beta(u_{i-1}))), e_\theta \right) dt \right| \\
&\le \sum_{k=1}^{m} \int_{I_k} \tau \left| \sum_{i=1}^{k} (\bar{f}^i - f(t_i,\beta(u_{i-1}))) \right|_\Omega |e_\theta|_\Omega \, dt \\
&\le \frac{\varepsilon}{2} \sum_{k=1}^{m} \int_{I_k} |e_\theta|_\Omega^2 \, dt + \frac{1}{2\varepsilon} \sum_{k=1}^{m} \int_{I_k} \left| \tau \sum_{i=1}^{k} (\bar{f}^i - f(t_i,\beta(u_{i-1}))) \right|_\Omega^2 dt \\
&= \frac{\varepsilon}{2} \mathrm{I}_1^m + \frac{1}{2\varepsilon} \sum_{k=1}^{m} \tau \left| \sum_{i=1}^{k} \tau(\bar{f}^i - f(t_i,\beta(u_{i-1}))) \right|_\Omega^2 = \frac{\varepsilon}{2} \mathrm{I}_1^m + \frac{1}{2\varepsilon} \mathrm{V}_1 \ .
\end{aligned}
$$

For estimating V_1 we use the following inequality:

$$
\begin{aligned}
|\bar{f}^i - f(t_i,\beta(u_{i-1}))| &\le \frac{1}{\tau} \int_{I_i} |f(t,\beta(u)) - f(t_i,\beta(u_{i-1}))| \, dt \\
&\le \frac{L_f}{\tau} \int_{I_i} |\beta(u) - \beta(u_{i-1})| \, dt + L_f \tau \\
&\le \frac{L_f}{\tau} \int_{I_i} |e_\theta| \, dt + L_f \tau + \frac{L_f}{\tau} \int_{I_i} \left| \frac{u_i - u_{i-1}}{\mu_i} \right| dt \\
&= \frac{L_f}{\tau} \int_{I_i} |e_\theta| \, dt + L_f \tau + L_f \left| \frac{u_i - u_{i-1}}{\mu_i} \right| \ ,
\end{aligned}
$$

where we used (2.2) and (1.9). Hence

$$
\begin{aligned}
|\bar{f}^i - f(t_i,\beta(u_{i-1}))|^2 &\le \frac{4L_f^2}{\tau^2} \left(\int_{Ii} |e_\theta| \, dt \right)^2 + 4L_f^2 \tau^2 + 2L_f^2 \frac{|u_i - u_{i-1}|^2}{\mu_i^2} \\
&\le \frac{4L_f^2}{\tau} \int_{Ii} e_\theta^2 \, dt + 4L_f^2 \tau^2 + 2L_f^2 \frac{|u_i - u_{i-1}|^2}{\mu_i^2} \ .
\end{aligned}
$$

For V_1 we then have:

$$
\begin{aligned}
\mathrm{V}_1 &= \sum_{k=1}^{m} \tau \left| \sum_{i=1}^{k} \tau(\bar{f}^i - f(t_i,\beta(u_{i-1}))) \right|_\Omega^2 \le T \sum_{k=1}^{m} \tau^2 \sum_{i=1}^{k} |\bar{f}^i - f(t_i,\beta(u_{i-1})|_\Omega^2 \\
&\le 4L_f^2 T \sum_{k=1}^{m} \tau \sum_{i=1}^{k} \left| \int_{I_i} e_\theta^2 \, dt \right|_\Omega^2 + 2L_f^2 T \sum_{k=1}^{m} \tau^2 \sum_{i=1}^{k} \left| \frac{u_i - u_{i-1}}{\mu_i} \right|_\Omega^2 + C\tau \\
&\le C\tau \sum_{k=1}^{m} I_1^k + C\tau \ ,
\end{aligned}
$$

because of (2.12) and the properties of μ_i. We have finally:

$$\mathrm{V} \le \frac{\varepsilon}{2} I_1^m + \frac{C\tau}{2\varepsilon}\sum_{k=1}^{m} \mathrm{I}_1^k + \frac{C}{2\varepsilon}\tau \ . \tag{3.12}$$

Collecting all the previous bounds (3.6), (3.7), (3.9), (3.10), (3.11), (3.12) yields:

$$\begin{aligned}
&\|e_\theta\|^2_{L_2(0,t_m,L_2(\Omega))} + l_\beta \|h(u) - h(u_{i-1})\|^2_{L_2(0,t_m,L_2(\Omega))} + \left|\nabla \int_0^{t_m} e_\theta(t)\, dt\right|^2_\Omega \\
&\quad + \tau^2 \sum_{k=1}^{m} |\nabla(\bar\beta^k - \theta_k)|^2_\Omega + C\tau^2 \sum_{k=1}^{m} |\bar\beta^k - \theta_k|^2_\Gamma + \frac{\gamma}{2}\left|\int_0^{t_m} e_\theta\, dt\right|^2_\Gamma \\
&\le C \sum_{k=1}^{m} \int_{I_k} \left(\left(1 - \frac{1}{H\mu_k}\right)(u_k - u_{k-1}), e_\theta\right) dt \\
&\quad + C \sum_{k=1}^{m} \int_{I_k} \left\langle h(u) - h(u_{k-1}), \frac{u_k - u_{k-1}}{\mu_k}\right\rangle dt \\
&\quad + \tau \|\partial_t u\ \|_{L_2(0,t_m,H^*(\Omega))} \|e_\theta\|_{L_2(0,t_m;H(\Omega))} + C\tau + C\tau \sum_{k=1}^{m} \mathrm{I}_1^k \ .
\end{aligned} \tag{3.13}$$

Further we consider two cases: if $\beta(s)$ is stictly increasing function, (i.e. $l_\beta > 0$) or $\beta(s)$ is nondecreasing function, (i.e. $l_\beta \ge 0$). We can express conclusions for these two cases in following two theorems.

Theorem 1. *Under the assumptions* (2.1)–(2.5) *for* $l_\beta > 0$ *in* (2.1) *we have*

$$\|e_\theta\|^2_{L_2(Q)} + ess \sup_{0\le t\le T} \left\|\nabla \int_0^t e_\theta\, ds\right\|^2_{L_2(\Omega)} \le C\tau \ , \tag{3.14}$$

where constant C depends only on initaial data, and some norms of u and $\beta(u)$.

Proof.

We estimate the second term on the right-hand side of (3.13)

$$\begin{aligned}
&\sum_{k=1}^{m} \int_{I_k} \left\langle h(u) - h(u_{k-1}), \frac{u_k - u_{k-1}}{\mu_k}\right\rangle dt \\
&\quad \le \frac{\eta}{2} \sum_{k=1}^{m} \int_{I_k} |h(u) - h(u_{k-1})|^2_\Omega\, dt + \frac{2}{\eta} \sum_{k=1}^{m} \int_{I_k} \left|\frac{u_k - u_{k-1}}{\mu_k}\right|^2_\Omega dt \\
&\quad \le \frac{\eta}{2} ||h(u) - h(u_{k-1})\|^2_{L_2(0,t_m,L_2(\Omega))} + C\tau \ ,
\end{aligned} \tag{3.15}$$

where (2.12) is applied. We chose η such that $\frac{C\eta}{2} \leq l_\beta$ and (3.15) together with (3.13) yields:

$$
\begin{aligned}
&\|e_\theta\|^2_{L_2(0.t_m,L_2(\Omega))} + \left|\nabla \int_0^{t_m} e_\theta(t)\,dt\right|^2_\Omega \\
&\quad \leq C\tau \sum_{k=1}^m I_1^k + C\tau + \sum_{k=1}^m \int_{I_k} \left(\left(1 - \frac{1}{H\mu_k}\right)(u_k - u_{k-1}), e_\theta\right) dt \\
&\qquad + \tau \|\partial_t u\|_{L_2(0,t_m;H^*(\Omega))} \|e_\theta\|_{L_2(0,t_m;H(\Omega))} \ .
\end{aligned}
$$

Last term on the right-hand side of this inequality is estimated by $C\tau$ because of properties of $\partial_t u$, $\beta(u)$ and the estimates (2.12). After estimating the third term on the right hand-side as follows:

$$
\begin{aligned}
&\sum_{k=1}^m \int_{I_k} \left(\left(1 - \frac{1}{H\mu_k}\right)(u_k - u_{k-1}), e_\theta\right) dt \leq C \sum_{k=1}^m \int_{I_k} |u_k - u_{k-1}|_\Omega |e_\theta|_\Omega \, dt \\
&\quad \leq \frac{C\tau}{2\varepsilon} \sum_{k=1}^m |u_k - u_{k-1}|^2_\Omega + \frac{C}{2}\varepsilon I_1^m \leq C\tau + \frac{C\varepsilon}{2} I_1^m \ ,
\end{aligned}
$$

and now after applying the discrete Gronwall inequality we have finally (3.14). □

Theorem 2. *Under the assumptions* (2.1)–(2.5) *for* $l_\beta = 0$ *we have*

$$
|e_\theta|^2_{L_2(Q)} + ess \sup_{0\leq t\leq T} \left\|\nabla \int_0^t e_\theta \, ds\right\|_{L_2(\Omega)} \leq C\tau^{\frac{1}{2}} \ , \tag{3.16}
$$

where the constant C *has the same properties as in Theorem* 1.

Proof. We must split the second term on the right hand-side of (3.13) as follows:

$$
\begin{aligned}
&C \sum_{k=1}^m \int_{I_k} \left\langle h(u) - h(u_{k-1}), \frac{u_k - u_{k-1}}{\mu_k} \right\rangle dt \\
&\quad = C \sum_{k=1}^m \int_{I_k} \left\langle u, \frac{u_k - u_{k-1}}{\mu_k} \right\rangle dt - C\tau \sum_{k=1}^m \left(u_{k-1}, \frac{u_k - u_{k-1}}{\mu_k}\right) \\
&\quad - \frac{C}{H} \sum_{k=1}^m \int_{I_k} \left(e_\theta, \frac{u_k - u_{k-1}}{\mu_k}\right) dt - C\tau \sum_{k=1}^m \left(\frac{u_k - u_{k-1}}{H\mu_k}, \frac{u_k - u_{k-1}}{\mu_k}\right) \ .
\end{aligned}
$$

This implies:

$$
\begin{aligned}
&\left| C \sum_{k=1}^{m} \int_{I_k} \left\langle h(u) - h(u_{k-1}), \frac{u_k - u_{k-1}}{\mu_k} \right\rangle dt \right| \\
&\le C\|u\|_{L_2(Q)} \left[\tau \sum_{k=1}^{m} |u_k - u_{k-1}|_\Omega^2 \right]^{\frac{1}{2}} + C\tau \left| \sum_{k=1}^{m} \left(u_{k-1}, \frac{u_k - u_{k-1}}{\mu_k} \right) \right| + c\varepsilon I_1^m + C\tau \\
&\le C\tau^{\frac{1}{2}} + C\tau \left| \sum_{k=1}^{m} \left(u_{k-1}, \frac{u_k - u_{k-1}}{\mu_k} \right) \right| + C\varepsilon I_1^m \\
&\le C\varepsilon I_1^m + C\tau^{\frac{1}{2}} + C \left| \sum_{k=1}^{m} \left(\tau^{\frac{3}{4}} u_{k-1}, \tau^{\frac{1}{4}} \frac{u_k - u_{k-1}}{\mu_k} \right) \right| \\
&\le C\varepsilon I_1^m + C\tau^{\frac{1}{2}} + C\tau^{\frac{3}{2}} \sum_{k=1}^{m} |u_{k-1}|_\Omega^2 + C\tau^{\frac{1}{2}} \sum_{k=1}^{m} \left| \frac{u_k - u_{k-1}}{\mu_k} \right|_\Omega^2 \\
&\le C\varepsilon I_1^m + C\tau^{\frac{1}{2}} \ ,
\end{aligned}
$$

because of (2.12), (2.13) and (2.1). Using this estimate in (3.13) and an analogous assertion as in Theorem 1 for other terms on the right hand-side we get after aplying the discrete Gronwall inequality the conclusion:

$$
\|e_\theta\|_{L_2(Q)}^2 + \operatorname{ess} \sup_{0 \le t \le T} \left\| \nabla \int_0^t e_\theta(s)\, ds \right\|_{L_2(\Omega)}^2 \le C\tau^{\frac{1}{2}} \ .
$$

□

Consequence. *If the estimate*

$$
\|e_\theta\|_{L_2(Q)}^2 + ess \sup_{0 \le t \le T} \left\| \nabla \int_0^t e_\theta \, ds \right\|_{L_2(\Omega)}^2 = O(\tau^\omega)
$$

holds, then for the unknown u we have the following error bound:

$$
\|e_u\|_{L_2(0,T,H^*(\Omega))}^2 = O(\tau^\omega) \ . \tag{3.17}
$$

Proof. This assertion can be proved in the same way as in [**5**] or [**7**]. □

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