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FRONT–DIVISORS OF TREES

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Abstract. Generalizing the concept of graph covering we introduce the notion of semicovering of graphs. We prove that the relationship between front-divisors of graph G and semicoverings defined on G has the same character as the well-known relationship between divisors and coverings. Main attention is paid on trees. We show that each front-divisor of a tree T may be obtained by factoring T under any appropriate subgroup of the automorphism group $\operatorname{Aut} T$.

1. Introduction and Preliminaries

Let G be a directed graph; we allow G to have multiple arcs (= directed edges) as well as loops. A partition of its vertex set $V(G) = \cup_{i=1}^{n} V_i$ is called an **equitable partition** if and only if there exists a square matrix $\mathbf{M} = (d_{ij})$ of order n such that for every i, $j \in \{1, 2, \dots, n\}$ and for every vertex $x \in V_i$ there are exactly d_{ij} arcs emanating from x and terminating at some vertices of V_j. The (directed) graph D determined by the adjacency matrix $\mathbf{M}$ is called a **front-divisor** of G. The fact that D is a front divisor of G will be symbolically denoted by $D|G$. Obviously, vertices of D represent the classes V_i of the equitable partition of G.

The most important property of a front-divisor D of a graph G is that the characteristic polynomial of D divides the characteristic polynomial of G [**1**; Theorem 4.5]. For more information the reader is referred to [**1**; Chapter 4], where basic properties (including references) of front divisors can be found.

The concept of an equitable partition was introduced in [**4**], where the equivalence classes are induced by an action of a subgroup of the automorphism group of G (such divisors are known as regular, and we deal with them in Section 3).

The motivation for our research comes from the study of spectra of some classes of trees. This led us to the investigation of front-divisors of trees, and consequently, to the study of front-divisors in general. It turns out that sometimes it is more convenient to consider the natural projection from a graph G onto its front-divisor D rather than the front-divisor itself. In Section 2, the relationship of the natural

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projection mapping a of graph G onto its front-divisor to the covering projections defined on G is discussed. In this context one needs to relax the conditions that are usually imposed on covering projections (as they are commonly known in algebraic topology). Surprisingly enough, this gives rise to a theory of semicoverings of (directed) graphs that is parallel to the existing theory of permutation voltage assignments (see [**2**; pp. 81–86]).

Special attention is paid to front-divisors of trees. Our main theorem in Section 3 shows that every front-divisor of a tree T may be obtained by factoring T by an appropriate subgroup of the automorphism group $\mathrm{Aut}\, T$. The main advantage of this fact is that it allows us to reduce the problem of determining the spectrum of T to the problem of determining the spectra of divisors and codivisors (see [**1**; Chapter 4]) of T. The effectiveness of this method is directly proportional to the number of symmetries of T.

Throughout, by a graph we always mean a directed graph; multiple directed edges and loops are allowed. Although sometimes also undirected graphs will appear, they will always be referred to as "undirected". By the **characteristic polynomial** of a graph G, denoted by $P_G(\lambda)$, we mean the characteristic polynomial of the adjacency matrix of G. If S is a set, then by $|S|$ we indicate the number of its elements. Let $G = (V, E)$ and $H = (V_1, E_1)$ be two graphs. A map $\varphi : G \to H$, mapping the vertex set V into the vertex set V_1 and the arc set E into the arc set E_1 is called a **morphism** if an arc e originating at u and terminating at v is mapped onto the arc $\varphi(e)$ originating at $\varphi(u)$ and terminating at $\varphi(v)$.

The notions such as front-divisor, equitable partition, and semicovering projection introduced for (directed) graphs will be used also for undirected graphs in the following sense: Let G be an undirected graph. Denote by $\vec{G}$ the directed graph which arises from G by replacing each edge of G by a couple of oppositely directed arcs. Note that both G and $\vec{G}$ have identical adjacency matrices. Thus, as regards properties which depend only on the adjacency matrix, we are free in interchanging G by $\vec{G}$ and vice versa.

2. Front-divisors and Semicovering Projections

It is well-known [**1**; p. 117], that divisors and covers of undirected graphs are closely related. The main goal of this section is to introduce the notion of a semicovering projection of a directed graph by generalizing the concept of combinatorial coverings. We show that the relationship between front-divisors and semicoverings has the same character as the above mentioned relationship between divisors and covers for undirected graphs.

Let G and H be two graphs. An epimorphism $\varphi : G \to H$ will be called a **semicovering projection** if for every vertex $u \in V(G)$, the restriction of φ onto the set of arcs originating at u is a bijection.

Gross and **Tucker** in [**2**; Chapter 2] introduced the concept of a covering projection for undirected graphs. Clearly, every covering projection $\Pi : G \to H$ (in the sense of [**2**]) is a semicovering projection. On the other hand, a semicovering projection $\Pi' : G \to H$, where G, H are undirected graphs, is a covering projection if and only if for every vertex $u \in V(G)$ the restriction of Π' onto the set of arcs terminating at u is a bijection. Thus the notion of a semicovering projection of graphs is a natural generalization of the notion of a covering projection of undirected graphs. The following two propositions explain the relationship between front-divisors of a graph G and semicovering projections defined on G.

Proposition 2.1. *Let G be a graph and D be its front-divisor determined by an equitable partition $\mathcal{T}$. Then the natural projection $\Pi\colon V(G) \to V(D)$, mapping a vertex x onto the class $[x]$ of $\mathcal{T}$, can be extended to a semicovering projection $G \to D$.*

Proof. Since $D|G$, the vertex set $V(G)$ of G is partitioned into classes $V_1 \cup V_2 \cup \cdots \cup V_n$ of $\mathcal{T}$. Denote by $M = (d_{ij})$ the adjacency matrix of D. Let $x \in V_i$ be an arbitrary vertex of G. Then there are exactly d_{ij} arcs $e_1, e_2, \ldots, e_{d_{ij}}$ joining x to vertices of V_j (for $j = 1, 2, \ldots, n$). Since $D|G$ we have exactly d_{ij} arcs $f_1, f_2, \ldots, f_{d_{ij}}$ joining V_i to V_j in D. Put $\Pi^*(e_i) = f_i$ for each $i \in \{1, 2, \ldots, d_{ij}\}$. Since each arc e of G joins a vertex $u \in V_i$ to a vertex $v \in V_j$, Π^* is defined on the arc set of G. Set $\Pi^*(u) = \Pi(u)$ for every vertex $u \in V(G)$. Then Π^* is a semicovering projection $G \to D$. $\square$

Proposition 2.2. *Let $\Pi\colon G \to H$ be a semicovering projection. Then H is a front-divisor of G.*

Proof. We show that the partition $V(G) = \cup_{u \in V(H)} \Pi^{-1}(u)$ is an equitable partition determining H. Let u, v be two arbitrary vertices in H, and let $d(u, v)$ be the number of arcs of H joining u to v. Since Π is a semicovering projection, there exist exactly $d(u, v)$ arcs joining a vertex $x \in \Pi^{-1}(u)$ to vertices from $\Pi^{-1}(v)$. Thus $\cup_{u \in V(H)} \Pi^{-1}(u)$ is an equitable partition of $V(G)$ corresponding to H. $\square$

Gross and **Tucker** [**2**] introduced the concept of permutation voltage graphs; they represent a nice combinatorial tool for description of covers of undirected graphs. The following definition of functional voltage graphs may be considered to be a generalization of this concept.

Let H be a graph. Let $\eta\colon V(H) \to N$ be a function assigning to each vertex $u \in V(H)$ a natural number $\eta(u)$. For every arc e joining a vertex u to a vertex v let $\alpha(e)\colon \{1, 2, \ldots, \eta(u)\} \to \{1, 2, \ldots, \eta(v)\}$ be a function called a **functional voltage assignment** of e. The triple (H, η, α) will be called a **functional voltage graph**. The graph H will be called the **base graph**. Each functional voltage graph (H, η, α) defines the **derived graph** G^α_η, whose vertex set is $V(G^\alpha_\eta) = \cup_{u \in V(H)} \{u\} \times \{1, 2, \ldots, \eta(u)\}$ and whose arc set is $E(G^\alpha_\eta) = \cup_{e=uv \in E(H)} \{e\} \times \{1, 2, \ldots, \eta(u)\}$. The arc e_i $i \in \{1, 2, \ldots, \eta(u)\}$ joins the vertex u_i to the vertex

v_j, $j \in \{1, 2, \ldots, \eta(v)\}$ in G^α_η if there is an arc e joining u to v in H assigned by $\Pi = \alpha(e)$ such that $\Pi(i) = j$. The next two theorems indicate that the set of graphs that can be obtained from a functional voltage assignment of a given graph H is precisely the set of graphs that semicover the graph H.

Theorem 2.3. *Let (H, η, α) be a functional voltage graph and $G^\alpha_\eta = G$ be the derived graph. Then the natural projection $\Pi\colon G \to H$ mapping each vertex u_i onto the vertex u, and each arc e_i (with the initial vertex u_i) onto e for $i \in \{1, 2, \ldots, \eta(u)\}$ is a semicovering projection.*

Proof. From the definition of the derived graph $G = G^\alpha_\eta$ it follows that Π is an epimorphism. Further, if e_i and f_i are two arcs with an initial vertex $u_i \in \Pi^{-1}(u)$ and $\Pi(e_i) = \Pi(f_i)$ then from the definition of G we have $e = f$. Hence, the restriction of Π onto a set of arcs with the same initial vertex is injective, and the statement follows. □

Theorem 2.4. *Let $\Pi\colon G \to H$ be a semicovering projection. Then there exists a function $\eta\colon V(H) \to N$ and a functional voltage assignment α on $E(H)$ such that G is isomorphic to the derived graph G^α_η associated with (H, η, α).*

Proof. Put $\eta(u) = |\Pi^{-1}(u)|$ for any vertex u in H. For every vertex u of H label the vertices of $\Pi^{-1}(u)$ by $u_1, u_2, \ldots, u_{\eta(u)}$. Let e be an arc of H joining a vertex u to a vertex v. Since Π is a semicovering projection, the preimage $\Pi^{-1}(e)$ consists of $\eta(u)$ arcs, originating one by one successively at the vertices $u_1, u_2, \ldots, u_{\eta(u)}$. Denote the arc of $\Pi^{-1}(e)$ that originates at u_i by e_i, for $i = 1, 2, \ldots, \eta(u)$. If we match $\Pi^{-1}(u)$ to $\Pi^{-1}(v)$, the arcs of $\Pi^{-1}(e)$ define a function $\alpha_e\colon \Pi^{-1}(u) \to \Pi^{-1}(v)$. That is, u_i is matched by e_i to v_j if and only if $\alpha_e(i) = j$. Now, $\alpha = \alpha_e$ is a functional voltage assignment on the arcs of H and the triple (H, η, α) is a functional voltage graph. It follows from the definitions of η and α that $G = G^\alpha_\eta$.□

Corollary 2.5. *Let (H, η, α) be a functional voltage graph and let $G = G^\alpha_\eta$ be the derived graph. Then the characteristic polynomial of the graph H divides the characteristic polynomial of the graph G.*

Proof. It is a consequence of Theorem 2.3 and Proposition 2.2 that H is a front-divisor of G. Now the result follows from [**1**; Theorem 4.7]. □

Let H be a fixed graph. It follows from Proposition 2.1 and Theorem 2.4 that every graph G such that $H|G$ can be obtained as a derived graph from H using an appropriate functional voltage assignment defined on H. This fact has an interesting consequence for characteristic polynomials of H and G. Namely, by the Sachs-Petersdorf Fundamental lemma [**1**; Theorem 4.7] $P_H(\lambda)$ divides $P_G(\lambda)$. Thus using the construction of derived graphs we are able to produce a large family of graphs G with the property that $P_H(\lambda)$ divides $P_G(\lambda)$.

Semicovering projections have a lot of nice properties which correspond to properties of covering projections. For instance, it can be easily observed that if W is a (directed) walk in a functional voltage graph (H, η, α) originating at a vertex u, then for each vertex u_i $(i = 1, 2, \dots, \eta(u))$ in the derived graph $G = G_\eta^\alpha$ there is a unique lift W_i of W that starts at u_i. Moreover, the functional voltage assignment α can be extended from arcs onto arbitrary walks as follows: If $W = e_1 e_2 \dots e_k$ is an $u - v$ walk in H, then set

$$\alpha_W(i) = \alpha_{e_1} \cdot \alpha_{e_2} \dots \alpha_{e_k}(i) = \alpha_{e_k}(\alpha_{e_{k-1}}(\dots(\alpha_{e_1}(i))\dots)),$$

for $i = 1, 2, \dots, \eta(u)$. Clearly, α_W is a function mapping $\{1, 2, \dots, \eta(u)\}$ onto $\{1, 2, \dots, \eta(v)\}$. Moreover, if W is an $u - v$ walk in H, $i \in \{1, 2, \dots, \eta(u)\}$, then the lifted walk W_i starting at the vertex u_i terminates at the vertex v_j if and only if $j = \alpha_W(i)$.

Sometimes it is convenient to know under what conditions on a voltage assignment on the arcs of a base graph the derived graph is connected. The following theorem gives us such a condition.

Theorem 2.6. *Let (H, η, α) be a functional voltage graph. Let H be strongly connected. Then the derived graph $G = G_\eta^\alpha$ is strongly connected if and only if for every vertex $u \in V(H)$ and $i, j \in \{1, 2, \dots, \eta(u)\}$ there exists an $u - u$ walk W in H such that $\alpha_W(i) = j$.*

Proof.
$(\Rightarrow)$ Since G is strongly connected there is an $u_i - u_j$ walk W in G. Let $L = \Pi(W)$, where Π is the natural projection. Obviously, $j = \alpha_L(i)$.

$(\Leftarrow)$ Let u_i, v_j, $i \in \{1, 2, \dots, \eta(u)\}$, $j \in \{1, 2, \dots, \eta(v)\}$ be two arbitrary vertices of G. Since H is strongly connected there is an $u - v$ walk W in H. Then the lift W_i of W originates at u_i and terminates at v_k, for some $k \in \{1, 2, \dots, \eta(v)\}$. By the assumption there exists a $v - v$ walk W' in H such that $\alpha_{W'}(k) = j$. Then the lift W'_k joins the vertex v_k to the vertex v_j. Clearly, the walk $\overline{W} = W_i W'_k$ originates at u_i and terminates at v_j. Since the vertices u_i and v_j were chosen arbitrarily, the graph G is strongly connected. □

3. Regular Divisors and Divisors of Trees

The relationship between the automorphism group of a graph G and divisors of G was observed by Petersdorf [**3**]. Let G be a graph and $\Gamma \subseteq \operatorname{Aut} G$ be a subgroup of the automorphism group of G. Let $\{V_1, V_2, \dots, V_s\}$ be the system of orbits into which the vertex set V of G is partitioned by the action of Γ on V. Clearly, the number of arcs emanating from any vertex of V_i and terminating in vertices of V_j depends only on i and j $(i, j \in \{1, 2, \dots, s\})$. Denote this number by d_{ij}. Then the square matrix (d_{ij}) of order s is the adjacency matrix of a front-divisor D of G (see [**1**; Chapter 4]). Such a front-divisor will be called a **regular front-divisor** of G. The natural semicovering projection $\Pi\colon G \to D = G/\Gamma$ will be called a

regular semicovering projection. Equivalently, any semicovering projection $\Pi\colon G \to H$ is regular if the fiber-preserving subgroup of Aut G acts transitively on each fiber $\Pi^{-1}(x)$ $(x \in V(H))$.

Let G be an undirected graph. Then a regular semicovering projection $\Pi\colon G \to G/\Gamma$, $(\Gamma \subseteq \operatorname{Aut} G)$ is a regular covering projection (in sense of [**2**]) if and only if Γ acts freely on G.

It is well-known that any covering projection defined on a tree T is trivial; in particular, it is regular. In the following theorem we show that this observation can be generalized to semicovering projections.

Theorem 3.1. *Every front-divisor of a tree is regular.*

Proof. We shall proceed by induction on the number of vertices of T. The result is trivial for the trees K_1 and K_2. There are exactly three semicovering projections defined on K_1 or K_2. (See Fig. 1.) Clearly, each of them is a regular semicovering projection.

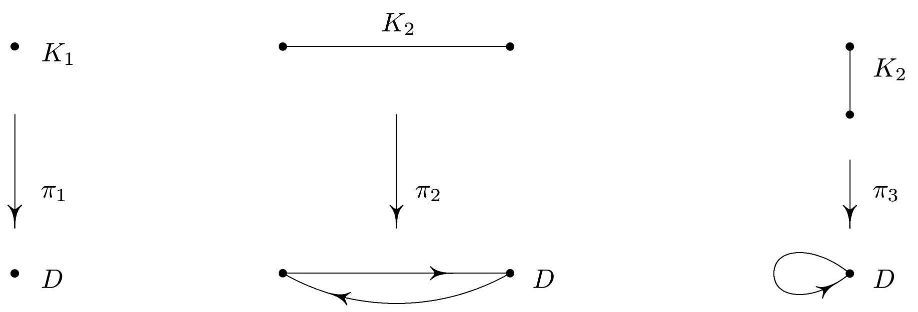


Figure 1.

Now, let the statement hold for every tree with fewer than n vertices $(n \geq 3)$. Suppose that T has n vertices and let $\Pi\colon T \to D$ be a semicovering projection. Then, $T' = T - \{v \in V(T); \deg v = 1\}$ is a non-empty tree with fewer than n vertices. Since Π is a semicovering projection, then $v \in V_1(D)$ if and only if $\Pi^{-1}(v) \subseteq V_1(T)$, where $V_1(D)$ and $V_1(T)$ are the sets of vertices with outdegree one in D and T, respectively. Therefore, $\Pi' = \Pi/T'$ is a semicovering projection which maps $T' = T - V_1(T)$ onto $D' = D - V_1(D)$. Because Π and Π' are semicovering projections, the following statement holds: If x, y are two vertices of the tree T which belong to the same fiber $\Pi^{-1}(u)$, then $\deg_o(x,T) - \deg_o(x,T') = \deg_o(y,T) - \deg_o(y,T')$. That is, x and y have the same number of neighbours of degree one in the tree T. By the induction hypothesis, Π' is a regular semicovering projection. Thus, there exists an automorphism $\varphi' \in \operatorname{Aut} T'$ such that $\varphi'(x) = y$. It follows that there exists $\varphi \in \operatorname{Aut} T$ with the property that $\varphi/T' = \varphi'$. To complete the proof it is sufficient to show that for every two vertices x, y of degree one in T with $\Pi(x) = \Pi(y)$ there exists an automorphism $\varphi \in \operatorname{Aut} T$ such that $\varphi(x) = y$. Let z be the vertex in T adjacent to the vertex x and let w be a vertex of T adjacent to the vertex y. If $z = w$, then the existence of such $\varphi \in \operatorname{Aut} T$ is obvious. Now, let $e = xz$, $f = yw$, and $z \neq w$. Since Π is a semicovering projection and $\Pi(x) = \Pi(y)$, we have $\Pi(e) = \Pi(f)$. Hence $\Pi(z) = \Pi(w)$. It follows from the regularity of Π' that there exists $\varphi' \in \operatorname{Aut} T'$ with $\varphi'(z) = w$. Then there is an automorphism $\varphi \in \operatorname{Aut} T$ which is an extension of φ' such that $\varphi(x) = y$. The proof is complete. $\square$

Corollary 3.2. *Let T be a tree. Then there exists a front-divisor $D^* = T/\operatorname{Aut} T$ such that $D^*|D$ for every front-divisor D of T.*

The front-divisor D^* from Corollary 3.2 will be called the **canonical divisor** of T.

Let G be a strongly connected graph, v be a vertex of G and let $Z(G)$ be the center of G. Denote by $\operatorname{rad}(v)$ the length of a shortest directed path joining the center $Z(G)$ of G with v. Let P be an $u - v$ path in G. A path P in G will be called a **radial path** if u is a central vertex and the length of P is equal to $\operatorname{rad}(v)$. A partition $V = \cup_i V_i$, $i \in \{0, 1, \dots, \operatorname{rad}(G)\}$, of the vertex-set of G will be called a **radial partition** if, for each vertex $v \in G$, $v \in V_i$ if and only if $i = \operatorname{rad}(v)$. Clearly, V_0 is the center of G.

Corollary 3.3. *Every equitable partition of a tree T is a refinement of the radial partition of T.*

Proof. Let $V = \cup_i V_i$ be a radial partition of the vertex set of a tree T, $i \in \{1, 2, \dots, \operatorname{rad}(T)\}$. Let $V = \cup_j U_j$ be an equitable partition of the vertex set of T. It is sufficient to show that for every j there exists i such that $U_j \subseteq V_i$. Let $x \in U_j$ be an arbitrary vertex. Since every $\varphi \in \operatorname{Aut} T$ maps the center $Z(T)$ onto $Z(T)$ and a radial path P of the length s onto a radial path P' of the length s, $\operatorname{rad}(x) = \operatorname{rad}(\varphi(x))$. Now, if we put $i = \operatorname{rad}(x)$, the statement follows from Theorem 3.1. $\square$

The following proposition shows that each front-divisor of a tree has a tree-like structure.

Proposition 3.4. *Let D be a front-divisor of a tree T and $\Pi\colon T \to D$ be the natural semicovering projection. Then the following statements hold:*

(i) *The image of the center $Z(T)$ of D is either $\vec{K}_2$, or K_1, or a directed $\Pi(u)$-based loop ($u \in Z(T)$).*
(ii) *If e is a loop in the front-divisor, then the $\Pi(Z(T))$ consists of one vertex u and e is u-based. In particular, either D has exactly one loop, or it has none.*
(iii) *Let u, v be two vertices in D such that there exist vertices $y \in \Pi^{-1}(v)$ and $x \in \Pi^{-1}(u)$ such that y is a successor of x in any radial path of T. Then $|\Pi^{-1}(u)| \leq |\Pi^{-1}(v)|$ and there exists exactly one $v - u$ path in D.*

(iv) *D has no cycle of length greater than two.*

(v) *Let $U = \{u \in D;\ |\Pi^{-1}(u)| = 1\}$. Then $Z(T) \subseteq U$ and the preimage of U, $\Pi^{-1}(U)$, is a connected subgraph of T.*

Proof.

(i): By Theorem 3.1 the front-divisor D of T is regular and $D = T/\Gamma$, where Γ is a subgroup of the group $\operatorname{Aut} T$. Since φ is an automorphism of T, we have $\varphi(Z(T)) = Z(T)$ for every $\varphi \in \Gamma$. We distinguish three cases:

1. $Z(T) = \{u\}$, $\varphi(u) = u$. Then $\Pi(Z(T)) = K_1$.

2. $Z(T) = \{u, v\}$, $\varphi(u) = u$, $\varphi(v) = v$ for every $\varphi \in \Gamma$. Then the vertices u, v belong to two distinct classes of the equitable partition and $\Pi(Z(T)) = \vec{K}_2$.

3. $Z(T) = \{u, v\}$, and there exists an automorphism $\varphi \in \Gamma$ such that $\varphi(u) = v$, $\varphi(v) = u$. Then u, v belong to the same orbit and $\Pi(Z(T))$ is a directed $\Pi(u)$-based loop.

(ii): Because of (i) it is sufficient to prove that every loop in the front-divisor D belongs to $\Pi(Z(T))$. Suppose this is not the case and e is a loop in D such that $e \notin \Pi(Z(T))$. Then T contains an undirected edge $\{xy\}$ joining two vertices in same class U_i of the equitable partition of T. By the Corollary 3.3 there exists a class V_j, $j \geq 1$, of a radial partition of T such that $\{xy\} \in \Pi^{-1}(e) \in V_j$. Let P_x, P_y be radial paths terminating in x, y, respectively. Then the subgraph of T formed by P_x, P_y, xy and $Z(T)$ contains a cycle, which is a contradiction.

(iii): By Corollary 3.3 there exist classes V_k, V_n of the radial partition $V_k \supseteq \Pi^{-1}(u) = A$, $V_r \supseteq \Pi^{-1}(v) = B$, $r > k$. It is sufficient to prove that the statement holds for $r = k + 1$. In order to derive a contradiction assume that $A = \{v_1, v_2, \dots, v_t\}$, $B = \{w_1, w_2, \dots, w_s\}$ and $s < t$. Since A, B are two classes of an equitable partition of T, for every $v_k \in A$, $k = 1, 2, \dots, t$, there are exactly d arcs joining v_k to vertices of B and for every $w_{k'} \in B$, $k' = 1, 2, \dots, s$, there are exactly d' arcs joining $w_{k'}$ to vertices of A. Since T is undirected, $d \cdot t = d' \cdot s$. It follows that $d' > d \geq 1$. Now, for $w_1 \in B$ there are at least two undirected edges e_1, e_2 joining w_1 to vertices of A. Let $e_1 = \{w_1 v_i\}$, $e_2 = \{w_1 v_j\}$, $i, j \in \{1, 2, \dots, t\}$, $i \neq j$. Let P_{v_i}, P_{v_j} be the corresponding radial paths. Then the subgraph of T that is formed by e_1, e_2, P_{v_i}, P_{v_j} and $Z(T)$ contains a cycle, a contradiction. Now, the existence of exactly one $v - u$ path in D is easily seen. Otherwise, we can always find a cycle in T.

(iv): Let C be a cycle of D of length n, $n > 2$. Any component of a lift of the cycle C is a cycle C' in T which has length $\geq n$. This is a contradiction.

(v): If the statement did not hold, we would obtain a contradiction with the statement (iii). $\square$

4. The Canonical Divisor of a Tree

The main aim of this section is to give a characterization of canonical divisors of trees. In order to do it we shall encode each front-divisor $D|T$ of a tree T using its underlying (undirected) spanning tree. By a **weighted rooted tree** we mean a couple (W, μ), where W is either a rooted tree or a rooted tree with a loop based at the root of W, and $\mu : E(W) \to N$ is the weight function from the set of edges of W into the set of natural numbers. If e is a loop in W then we put $\mu(e) = 1$.

Theorem 4.1. *Let $D|T$ be a front-divisor of a tree T and $\Pi\colon T \to D$ be the natural semicovering projection. Then there exists a weighted rooted tree $W(D)$ satisfying the following conditions:*
(1) *$W(D)$ has the same vertex-set as D.*
(2) *There is an edge $\{x, y\}$ in $W(D)$ if and only if there is an arc joining the vertex x to vertex y in D.*
(3) *Let y be a successor of a vertex x in $W(D)$. Then $\mu(\{x, y\})$ is the number of arcs joining x to y in D.*
(4) *The preimage of the root v of $W(D)$ is a subset of the center of T; i.e. $\Pi^{-1}(v) \subseteq Z(T)$.*
(5) *$W(D)$ has a loop at the root v if and only if D has such a loop.*

Proof. Consider the undirected graph W' defined by the conditions (1) and (2). Since T is strongly connected and $\Pi\colon T \to D$ is a semicovering projection, D is strongly connected, and consequently, W' is connected. By Proposition 3.4(iv) D contains no cycles of length greater than 2. Consequently, W' contains no cycles of length > 2, too. By the definition, W' cannot contain multiple edges. It follows from the above facts that W' is a tree, or a tree with loops at vertices. However, it follows from Proposition 3.4(ii) that W' may contain at most one loop based at a vertex v, and if this is the case, then $\Pi^{-1}(v) \subseteq Z(T)$. Thus we may choose the root v in W' in such way that the conditions (4) and (5) are satisfied. Now, set $W(D) = (W', \mu)$, where μ is defined by (3). We thus proved that $W(D)$ is a weighted rooted tree for which the conditions (1)–(5) hold. $\square$

The weighted rooted tree $W(D)$ defined by the conditions (1)–(5) will be called the **tree associated** to the front-divisor D. It follows from 3.4(iii) and the definition of $W(D)$ of a front-divisor $D|T$ of a tree T that both graphs D and T can be reconstructed from $W(D)$ (see Fig. 2). Thus the weighted rooted tree $W(D)$ may be considered as an encoding of the tree T and its front-divisor D. The encoding of T by $W(D)$ is most effective if we use the canonical divisor D^* of T. We say that a weighted rooted tree W **represents** a front-divisor $D|T$ of a tree T if $W = W(D)$.

Theorem 4.2. *A weighted rooted tree W represents a front-divisor of a tree T if and only if either W has a loop at the root, or the root of W belongs to the*

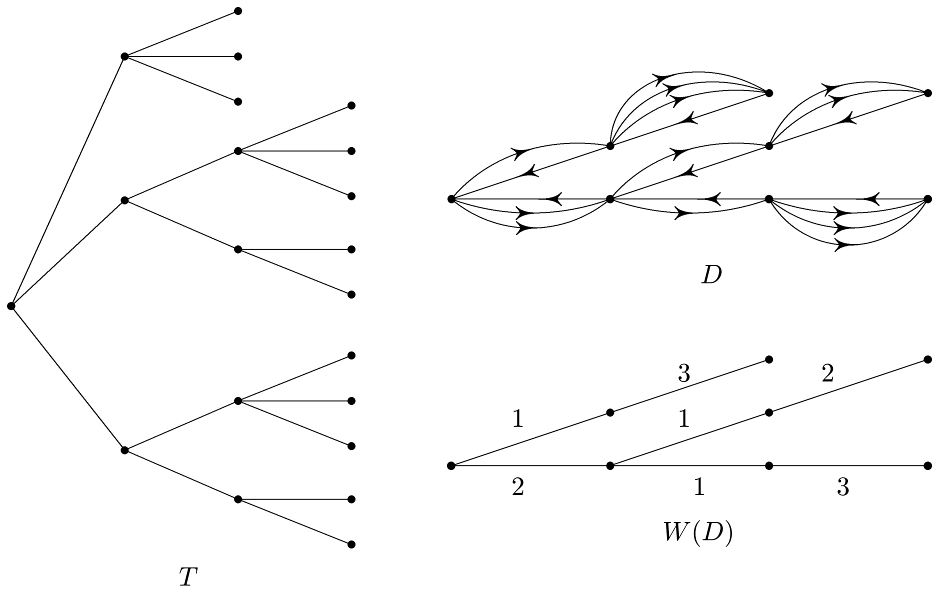


Figure 2.

center of W, or the first edge on the path joining the root of W with the center of W has weight greater than 1.

Proof.
($\Rightarrow$) Let W represent a tree T. In order to derive a contradiction, assume that the statement does not hold. That means that W is looples, the root v of W does not belong to its center and the first edge e on the path P joining the root v of W with the center of W has weight $\mu(e) = 1$. Recall that by the condition (4) in the definition of W, the preimage $\Pi^{-1}(v) \subseteq Z(T)$. First observe that if T has a central edge, then W has either a loop or a central edge incident with v. Thus T has no central edge, and consequently, $\Pi^{-1}(v) = Z(T)$ contains exactly one central vertex, say c. Since $\mu(e) = 1$, $\Pi^{-1}(e)$ contains exactly one edge e_c incident with the vertex c. Since $c \in Z(T)$, c is the central vertex of a path Q of length $2\,\mathrm{rad}(T)$ in T. Since Π is an epimorphism of graphs, we have

$$\mathrm{rad}(W) = \mathrm{rad}(D) \leq \mathrm{rad}(T). \tag{1}$$

Let Q_1, Q_2 be the two directed subpaths of Q of length $\mathrm{rad}(T)$ originating at c. Since Q_1, Q_2 are radial paths, they are mapped by Π onto directed paths Q_1' , Q_2' of length $\mathrm{rad}(T)$ and originating at $v = \Pi(c)$. Clearly, either Q_1 or Q_2 does not contain the edge e_c. Let it be Q_1. Then Q_1' does not contain e, and we have

$$|Q_1'| + |P| = \mathrm{rad}(T) + |P| \leq \mathrm{rad}(W).$$

Since $|P| > 1$, we have $\mathrm{rad}(W) > \mathrm{rad}(T)$, a contradiction with (1).

($\Leftarrow$) We shall construct a tree T such that $W = W(D)$ for some front-divisor D of T. Let v be a root of W. First suppose W is loopless. Let u be a vertex of W and let the $v - u$ path in W use the edges $e_1, e_2, \dots, e_k$ ($k \le \mathrm{rad}(W)$) in that order. Then set $\Pi^{-1}(u) = \{u_{x_1,\dots,x_k};\ 1 \le x_i \le \mu(e_i) \text{ for } 1 \le i \le k\}$. If $u = v$, put $\Pi^{-1}(v) = \{v_1\}$. The vertex-set $V(T)$ will be defined by $V(T) = \cup_{u \in W} \Pi^{-1}(u)$. Two vertices $u_{x_1,\dots,x_k}$ and $w_{y_1,\dots,y_s}$ will be adjacent in T if the vertex w succeeds u in W, $s = k+1$ and $y_i = x_i$ for $1 \le i \le k$. By the definition, $\cup_{u \in W} \Pi^{-1}(u)$ is an equitable partition of T defining a front-divisor D of T such that $W = W(D)$. Let c be the center of W and P be the path of W joining v to c. Clearly, there is a radial path Q of W of length $\mathrm{rad}(W)$ originating at c and not containing an edge of P. Since the weight of the first edge in P is > 1, the path $P \cup Q$ of W is lifted onto at least two disjoint paths of length $|P| + \mathrm{rad}(W)$ originating at v_1. Therefore $\Pi^{-1}(v) = v_1$ is the central vertex of T.

If W contains a loop e at the root v, then set $W' = W - e$. As before, from W' we form the tree T' such that $W' = W(D')$ for some front-divisor D' of T'. The required tree T is then constructed by joining the central vertices of two copies of T' with an edge. $\square$

Now we are ready to give a characterization of canonical front-divisors of trees.

Theorem 4.3. *A weighted rooted tree W represents the canonical divisor of a tree if and only if the following three conditions are satisfied:*
(a) *W represents a divisor of a tree,*
(b) *for every vertex u of W, $W - u$ does not contain two isomorphic components,*
(c) *if e is the central edge of W and the root of W is incident with e, then the two components of $W - e$ are non-isomorphic.*

Proof.
($\Rightarrow$) Let W represents the canonical divisor of a tree T. Then, clearly, the condition (a) holds. We show that (b), (c) are satisfied, too.

Suppose to the contrary that there is a vertex u such that $W - u$ contains two weighted subtrees W_1, W_2, $W_1 \cong W_2$. Let uw_1, uw_2 be the two edges joining u to W_1 and u to W_2, respectively. Form a new weighted rooted tree $W' = (W - W_2, \mu')$, where μ' is a weight function defined as follows:

$$\mu'(uw_1) = \mu(uw_1) + \mu(uw_2) \qquad \text{and} \qquad \mu'(e) = \mu(e) \quad \text{for} \quad e \ne uw_1.$$

One can easily check that W' "lifts" onto the tree T and $D'|D$, where $D \ne D'$ are the divisors of T such that $W = W(D)$ and $W' = W(D')$, a contradiction with the canonical character of D.

Finally, assume that $W - e$ consists of two isomorphic weighted rooted trees $W_1 \cong W_2$ and the root of W is incident with e. Then form a new weighted rooted tree W' from W by deleting the vertices of W_2 and attaching a loop to the root of W_1 (the root of W_1 is the vertex incident with e in W). As before, we are able to prove that W' represents a divisor of T which divides the divisor which is represented by W.

($\Leftarrow$) In order to derive a contradiction assume that for $W = W(D)$ the conditions (a), (b) and (c) hold and that there exists a weighted rooted tree $W' = W(D')$, $W' \neq W$ such that $D'|D$. Then there is a semicovering projection $p\colon D \to D'$. Since $D' \neq D$ there exists a vertex u in D' such that $p^{-1}(u)$ contains at least two vertices w_1, w_2. Let t_1, t_2 be two end-vertices of out-degree 1 of two radial paths of D originating at the root v of D passing through w_1, w_2 and terminating at t_1, t_2, respectively. Then t_1, t_2 are mapped by p onto the same vertex t of out-degree 1 in D'. Choose a vertex t between the vertices of out-degree 1 in D' such that $p^{-1}(t)$ contains two vertices t_1, t_2, whose predecessor u is at minimum distance from t_1 and t_2. First suppose that $u \neq v$, where v is the root of G. Let D_1, D_2 be the components of $D - u$ containing t_1 and t_2, respectively. The image $p(D_1)$ and $p(D_2)$ is isomorphic with D_1 and D_2, respectively, otherwise we would obtain a contradiction with the minimality of u. Hence $D_1 \cong D_2$, and $W - u$ contains two isomorphic components corresponding to D_1 and D_2, a contradiction with (b). If $u = v$ then we distinguish two cases. If the distance $\rho(t_1, v) = \rho(t_2, v)$ then using the same arguments as before we see that $D_1 \cong D_2$. Otherwise $\rho(t_1, v) \neq \rho(t_2, v)$. By Proposition 3.3 there is a central edge e incident with v in D. Then $W - e$ consists of two isomorphic components, a contradiction with (c). □

The following algorithm for constructing a canonical divisor of any tree T is based on the above theorem.

ALGORITHM:

Input data: Undirected tree T.

Variables:

$V(T) = V_1 \cup V_2 \cup V_3$;
V_1 – labeled vertices;
V_2 – vertices with exactly one unlabeled neighbor,
V_3 – other vertices;

Sets V_1, V_2 work as heaps (first in, first out).

1. [Initialization]:
   $V_1 :=$ vertices of degree one.
   $V_2 :=$ pendant vertices in $T - V_1$
2. If $V_2 = \emptyset$ then go to 3.
   Otherwise: Select a vertex $v \in V_2$. $V_2 := V_2 - \{v\}$. Let $u_1, u_2, \ldots, u_k$ be labeled vertices adjacent to v. Let $T_1, T_2, \ldots, T_k$ be the subtrees of

the tree T with roots $u_1, u_2, \dots, u_k$. Let $\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_r$, $1 \le r \le k$ be isomorphism classes of $\cup_{i=1}^{k} T_i$ and $|\mathcal{T}_i| = k_i$. Without loss of a generality we can assume that $T_1, T_2, \dots, T_r$ are representatives of $\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_r$, respectively. Then put $T := T - \cup_{j=r+1}^{k} V(T_j)$, $w(v, u_i) := k_i$, $V_1 := V_1 \cup \{v\}$. Let u be the only unlabeled neighbour of v. If u does not exist, then go to 2.

Otherwise: If every neighbour of u except one is labeled then put $V_2 := V_2 \cup u$, go to 2.

Otherwise: Go to 2.

3. If there exists an edge $e = vu$ without a weight then:

If subtrees T_v, T_u of T with roots v, u, respectively are isomorphic then put $T := T_v \cup e$, where e is a v-based loop and v is the root of T. Go to 4.

Otherwise: Put $T := T$, $w(e) := 1$, a root of T is v or u. Go to 4.

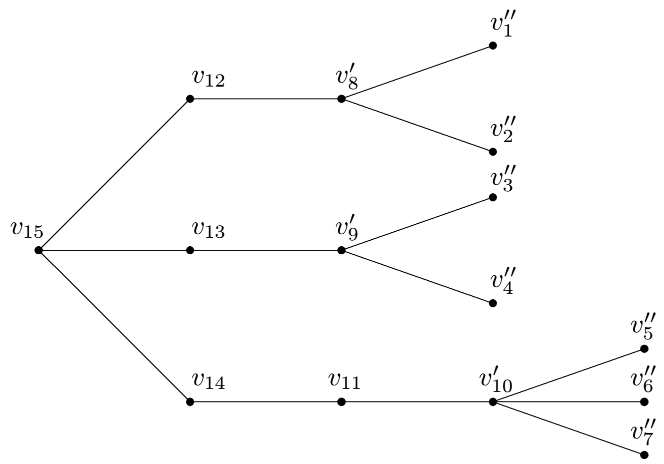
Otherwise: The root of T is v. Go to 4.

4. Stop.

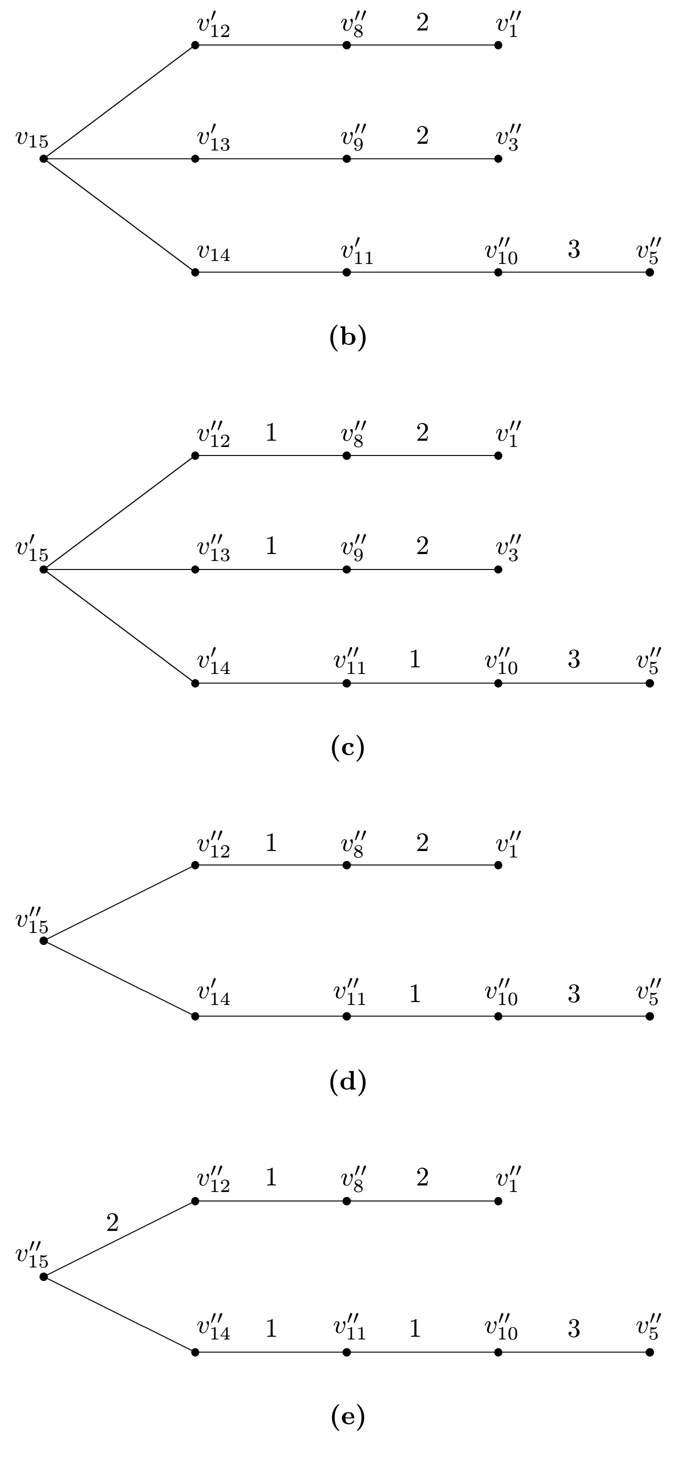
The steps of the algorithm are illustrated in Fig. 3., where elements of V_1 are primed and those of V_2 are double primed.

We conclude the paper by a list of canonical divisors for trees of diameter ≤ 6. In Fig. 4, $T_1, T_2, \dots, T_k$ are canonical divisors of trees of diameter 4 which are pairwise non-isomorphic.

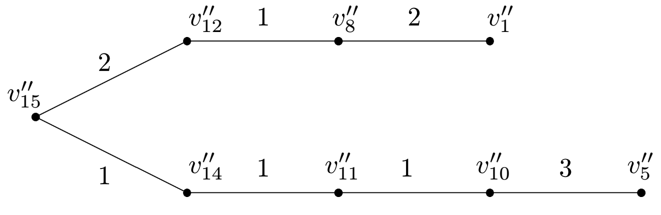
Figure 3.



(a)

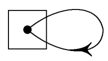

v'_{12} v''_8 2 v''_1
v_{15} v'_{13} v''_9 2 v''_3
v_{14} v'_{11} v''_{10} 3 v''_5
(b)
v''_{12} 1 v''_8 2 v''_1
v'_{15} v''_{13} 1 v''_9 2 v''_3
v'_{14} v''_{11} 1 v''_{10} 3 v''_5
(c)
v''_{12} 1 v''_8 2 v''_1
v''_{15}
v'_{14} v''_{11} 1 v''_{10} 3 v''_5
(d)
v''_{12} 1 v''_8 2 v''_1
2
v''_{15}
v''_{14} 1 v''_{11} 1 v''_{10} 3 v''_5

(e)

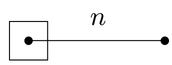


(f)

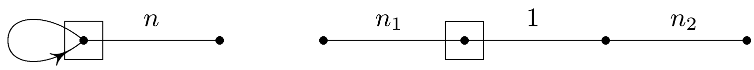
Figure 4.



D_1^*

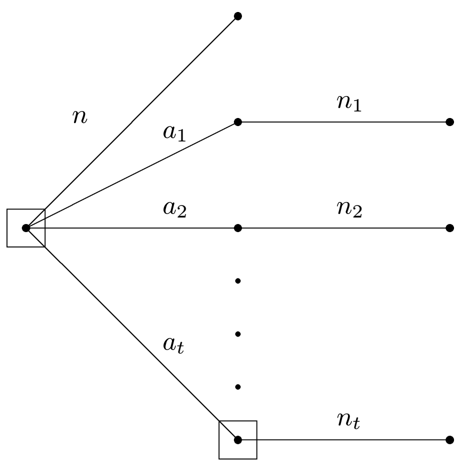


D_2^*



$n_1 \neq n_2$

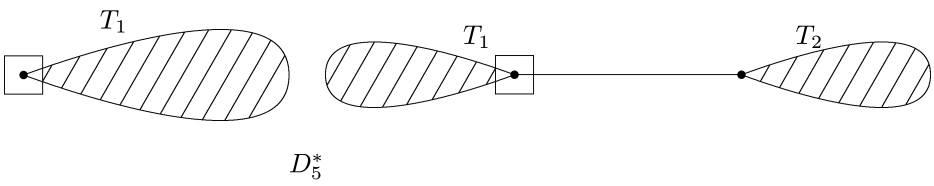
D_3^*



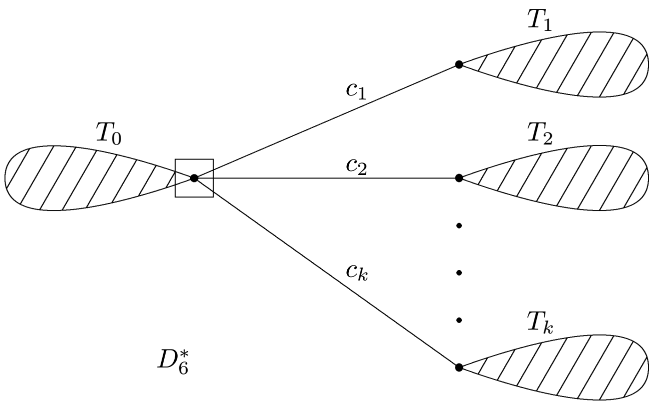
$n_i \neq n_j$, $i, j \in \{1, 2, \ldots, t\}$,

$\sum a_i \geq 2$

D_4^*



T_1, T_2 are the canonical divisors of trees of diameter 4, $T_1 \not\cong T_2$



T_i, $1 \leq i \leq k$ are the canonical divisors of trees of diameter 4, T_0 is the canonical divisor of a tree of diameter ≤ 4.

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