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ON SURJECTIVE KERNELS OF PARTIAL ALGEBRAS

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Abstract. A partial algebra $\mathbf{A} = (A, F)$ is called surjective if each of its elements lies in the range of some of its operations. By a transfinite iteration construction over the class of all ordinals it is proved that in each partial algebra $\mathbf{A}$ there exists the largest surjective subalgebra $\operatorname{Skr}\mathbf{A}$, called the surjective kernel of $\mathbf{A}$. However, what might be found a bit surprising, for each ordinal α there is an algebra $\mathbf{A}$ with only finitary operations (even with a single unary operation), such that the described construction stops exactly in α steps. The result is compared with the classical ones on perfect kernels of first countable topological spaces.

We use standard set-theoretical notation and terminology; in particular Y^X denotes the set of all functions from the set X into the set Y, each ordinal α is represented as the set of all ordinals $\beta < \alpha$, the least ordinal of cardinality $\aleph_\gamma$ is denoted by ω_γ, and $\omega = \omega_0$.

Under the term "partial algebra" we will understand a pair $\mathbf{A} = (A, F)$, where A is an arbitrary set and F is a set of partial (finitary or infinitary) operations on A (we do not exclude any of the possibilities $A = \emptyset$ or $F = \emptyset$). For an operation $f \in F$ we denote by $\operatorname{ar}(f)$ the arity and by $\mathrm{D}(f)$ the domain of f. This is to say that to each $f \in F$ two sets $\operatorname{ar}(f)$ (in most cases $\operatorname{ar}(f)$ is assumed to be an ordinal) and $\mathrm{D}(f) \subseteq A^{\operatorname{ar}(f)}$ are assigned, such that $f\colon \mathrm{D}(f) \to A$. A partial algebra $\mathbf{A} = (A, F)$ will be called finitary if $\operatorname{ar}(f)$ is finite for each $f \in F$. $\mathbf{A}$ will be called a total algebra, or simply an algebra if all its operations are total, i.e., $\mathrm{D}(f) = A^{\operatorname{ar}(f)}$ for each $f \in F$.

Any subset $B \subseteq A$ closed with respect to all operations $f \in F$, i.e. $f(\mathbf{b}) \in B$ whenever $\mathbf{b} \in \mathrm{D}(f) \cap B^{\operatorname{ar}(f)}$, will be called a subalgebra of $\mathbf{A}$ and it will be identified with the corresponding partial algebra $\mathbf{B} = (B, F_B)$, where $F_B = \{f_B;\ f \in F\}$ and f_B denotes the restriction of f to B, i.e. $\operatorname{ar}(f_B) = \operatorname{ar}(f)$, $\mathrm{D}(f_B) = \mathrm{D}(f) \cap B^{\operatorname{ar}(f)}$ and $f_B(\mathbf{b}) = f(\mathbf{b})$ for $\mathbf{b} \in \mathrm{D}(f_B)$. Obviously, every subalgebra of a total algebra is total, as well.

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Let $\mathbf{A} = (A, F)$ be a partial algebra, $X \subseteq A$ and $H \subseteq F$. We put

$$\begin{aligned} H[X] &= \bigcup_{f \in H} f\bigl[X^{\mathrm{ar}(f)}\bigr] \\ &= \bigl\{f(\mathbf{a});\ f \in H\ \ \&\ \ \mathbf{a} \in \mathrm{D}(f) \cap X^{\mathrm{ar}(f)}\bigr\}. \end{aligned}$$

The partial algebra $\mathbf{A} = (A, F)$ will be called **surjective** if $A = F[A]$. The largest surjective subalgebra of $\mathbf{A}$ (we will prove that it always exists) will be called the **surjective kernel** of $\mathbf{A}$ and denoted by $\operatorname{Skr} \mathbf{A}$.

Concerning the asymptotic behaviour of a (partial) finitary algebra $\mathbf{A}$, it suffices to deal with its surjective kernel $\operatorname{Skr} \mathbf{A}$, as the remaining elements of A do not matter at all. It is not our aim to make fully precise the intuitive meaning of the previous sentence in this short note. We expect that the article [Z], devoted to this topic and developing some ideas from [M–Z], will be submitted in the nearest future.

For every partial algebra $\mathbf{A} = (A, F)$ and any subset $X \subseteq A$ one can construct a sequence of subsets $F^{(n)}[X] \subseteq A$ by recursion over the set ω of all natural numbers putting

$$\begin{aligned} F^{(0)}[X] &= X, \\ F^{(n+1)}[X] &= F\bigl[F^{(n)}[X]\bigr]. \end{aligned}$$

If B is a subalgebra of $\mathbf{A}$, then obviously, $F^{(n+1)}[B] \subseteq F^{(n)}[B]$ holds for each n, all the sets $F^{(n)}[B]$ are subalgebras of $\mathbf{A}$, and they are nonempty provided B is. On the other hand, taking for $\mathbf{A}$ the algebra with the underlying set ω and the successor operation, one can see that the intersection $\bigcap_{n<\omega} F^{(n)}[A]$ may well be empty.

But more surprising is (at least for the author was) the fact that the expected and offering “theorem,” asserting

$$\operatorname{Skr} \mathbf{A} = \bigcap_{n<\omega} F^{(n)}[A],$$

is not true even for finitary algebras, as it will be shown within short.

This leads us to prolong the above sequence $F^{(n)}[B]$ over the class Ω of all ordinals by transfinite recursion. For every partial algebra $\mathbf{A} = (A, F)$, any its subalgebra B, each ordinal α and each limit ordinal $\lambda > 0$ we put

$$\begin{aligned} F^{(0)}[B] &= B, \\ F^{(\alpha+1)}[B] &= F\bigl[F^{(\alpha)}[B]\bigr], \\ F^{(\lambda)}[B] &= \bigcap_{\beta<\lambda} F^{(\beta)}[B]. \end{aligned}$$

We write $f^{(\alpha)}[B]$ instead of $\{f\}^{(\alpha)}[B]$.

Again, each $F^{(\alpha)}[B]$ is a subalgebra of $\mathbf{A}$ and $F^{(\beta)}[B] \subseteq F^{(\alpha)}[B]$ for all ordinals $\alpha \le \beta$. Also, if C is another subalgebra of $\mathbf{A}$ and $B \subseteq C$, then $F_B^{(\alpha)}[B] \subseteq F^{(\alpha)}[C]$ holds for each $\alpha \in \Omega$.

Proposition. *For every partial algebra* $\mathbf{A} = (A, F)$ *there is an ordinal number* ϑ *such that*

$$\operatorname{Skr}\mathbf{A} = F^{(\vartheta)}[A] = \bigcap_{\alpha\in\Omega} F^{(\alpha)}[A].$$

Proof. As A is a set, the sequence $\{F^{(\alpha)}[A]\}_{\alpha\in\Omega}$ cannot be strictly decreasing. Let us denote ϑ the least ordinal such that $F^{(\vartheta)}[A] = F^{(\vartheta+1)}[A]$. Then obviously

$$F^{(\vartheta)}[A] = \bigcap_{\alpha\in\Omega} F^{(\alpha)}[A],$$

and it is a surjective subalgebra of $\mathbf{A}$. On the other hand, if B is any surjective subalgebra of $\mathbf{A}$, then for each ordinal α we have $B = F^{(\alpha)}[B]$. In particular,

$$B = F^{(\vartheta)}[B] \subseteq F^{(\vartheta)}[A].$$

Hence $F^{(\vartheta)}[A] = \operatorname{Skr}\mathbf{A}$ is the largest surjective subalgebra of $\mathbf{A}$. □

The least ordinal ϑ such that $F^{(\vartheta)}[A] = F^{(\vartheta+1)}[A]$ will be called the **depth** of $\mathbf{A}$ and denoted by $\vartheta_{\mathbf{A}}$. Thus

$$\operatorname{Skr}\mathbf{A} = F^{(\vartheta_{\mathbf{A}})}[A].$$

If A is finite, then obviously $\vartheta_{\mathbf{A}} \le \operatorname{card}(A) - 1$. If $\operatorname{card}(A) = \aleph_\gamma$, say, then, as it will be shown during the proof of the next Theorem, one cannot prove more than the obvious inequality $\vartheta_{\mathbf{A}} < \omega_{\gamma+1}$.

Given a partial algebra $\mathbf{A} = (A, F)$, we will introduce the surjectivity rank function on $\mathbf{A}$ putting

$$\operatorname{rank}_{\mathbf{A}}[x] = \begin{cases} \alpha & \text{if } x \in F^{(\alpha)}[A] \setminus F^{(\alpha+1)}[A], \\ \Omega & \text{if } x \in \operatorname{Skr}\mathbf{A} \end{cases}$$

for $x \in A$.

Theorem. *For each ordinal* α *there exists an algebra* $\mathbf{A} = (A, f)$ *with a single unary operation, such that* $\vartheta_{\mathbf{A}} = \alpha$.

Proof. We will construct a sequence of algebras $\mathbf{T}_\alpha = (T_\alpha, f_\alpha)$ with a single unary operation by transfinite recursion over Ω. Each $\mathbf{T}_\alpha$ in fact will be a tree with finite branches only, and f_α will be the tree-predecessor operation along the branches. Let t be any element, distinct from any finite sequence of ordinals (hence from all the nodes to be added during the construction); it will be the root of each of the trees $\mathbf{T}_\alpha$.

We start with the trivial tree, i.e.,

$$T_0 = \{t\} \qquad \text{and} \qquad f_0(t) = t.$$

Then for each $\alpha \in \Omega$ we put

$$T_{\alpha+1} = T_\alpha \cup \{\alpha\} \qquad \text{and} \qquad f_{\alpha+1}(x) = \begin{cases} f_\alpha(x), & \text{for } x \in T_\alpha,\ x \neq t \neq f_\alpha(x), \\ \alpha, & \text{for } x \in T_\alpha,\ x \neq t = f_\alpha(x), \\ t, & \text{for } x = \alpha \text{ or } x = t. \end{cases}$$

Thus $\mathbf{T}_{\alpha+1}$ is the tree obtained by inserting a new node α between the root t and the rest of the tree.

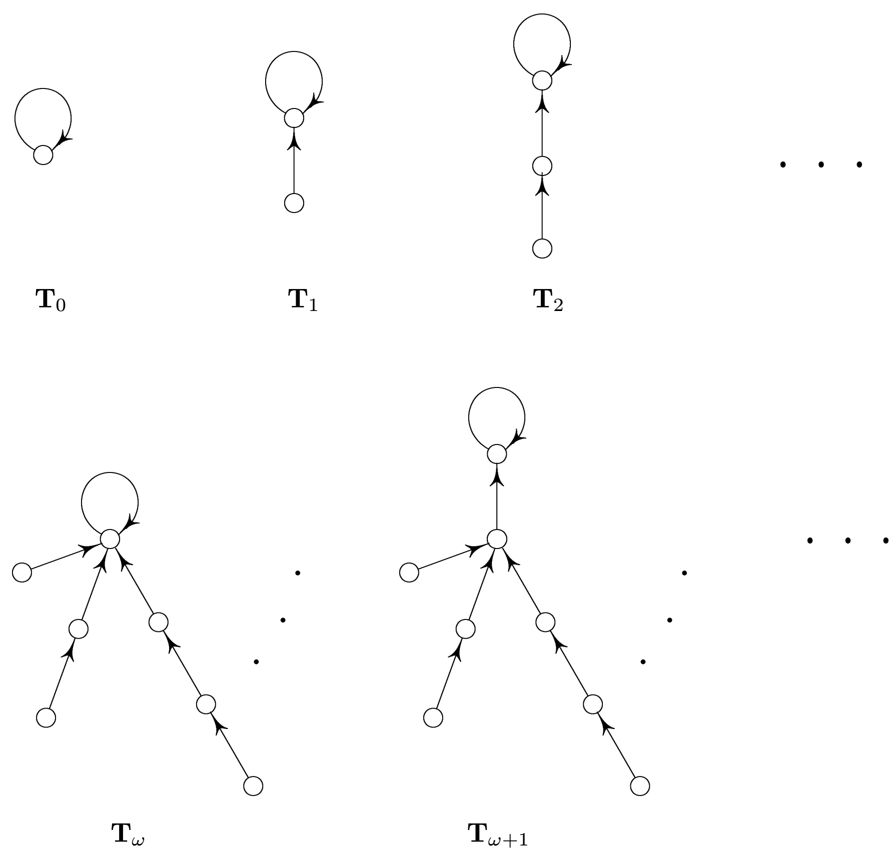
Further, for each limit ordinal $\lambda > 0$ we define

$$T_\lambda = \{t\} \cup \bigcup_{\alpha<\lambda} \big(T_\alpha \setminus \{t\}\big) \times \{\alpha\},$$

$$f_\lambda(t) = t \qquad \text{and} \qquad f_\lambda(x, \alpha) = \begin{cases} (f_\alpha(x), \alpha) & \text{if } f_\alpha(x) \neq t, \\ t & \text{if } f_\alpha(x) = t, \end{cases}$$

whenever $\alpha < \lambda$, $x \in T_\alpha$, $x \neq t$. In other words, $\mathbf{T}_\lambda$ is the tree obtained by identifying the roots (but no other nodes) of all the preceding trees $\mathbf{T}_\alpha$, $\alpha < \lambda$.

Some of the trees $\mathbf{T}_\alpha$ are in the following picture:



Now, following the described construction, one can easily prove by transfinite induction that every tree $\mathbf{T}_\alpha$ has indeed finite branches only, and that

$$f_\alpha^{(\alpha)}[T_\alpha] = \{t\} = f_\alpha^{(\alpha+1)}[T_\alpha] \qquad \text{and} \qquad \operatorname{rank}_{\mathbf{T}_\alpha}[x] < \alpha$$

for each $\alpha \in \Omega$ and each $x \in T_\alpha$, $x \neq t$. Consequently

$$\operatorname{Skr} \mathbf{T}_\alpha = \{t\} \qquad \text{and} \qquad \vartheta_{\mathbf{T}_\alpha} = \alpha$$

for each $\alpha \in \Omega$. □

Any first countable topological space $(M, \mathcal{O})$, in particular any metric space (M, d), raises to a partial algebra $\mathbf{M} = (M, \lim)$ where lim denotes the partial ω-ary operation of taking limits of convergent seqences $\mathbf{a} \in M^\omega$. It is clear that a set $C \subseteq M$ is a subalgebra of $\mathbf{M}$ if and only if it is closed in the corresponding topological space $(M, \mathcal{O})$. However, the above construction applied to the partial algebra $\mathbf{M}$, though it reminds of the Cantor's derivative, leads to trivial results, only. For each set $X \subseteq M$, $\lim[X]$ is namely the closure of X, and $\lim[C] = C$ for each closed set $C \subseteq M$. Nevertheles our algebraic construction can be modified to include the Cantor's derivative in the following way.

For every partial algebra $\mathbf{A} = (A, F)$ and all $X \subseteq A$, $H \subseteq F$ we put

$$H\langle X\rangle = \bigl\{ f(\mathbf{a});\ f \in H \ \ \&\ \ \mathbf{a} \in \mathrm{D}(f) \cap X^{\operatorname{ar}(f)} \ \ \&\ \ (\forall p \in \operatorname{ar}(f))(\mathbf{a}(p) \neq f(\mathbf{a})) \bigr\}.$$

Obviously, $\lim\langle X\rangle$ is the set of all accumulation points of the set $X \subseteq M$ in the first countable topological space $(M, \mathcal{O})$, or if you like, the Cantor's derivative of X.

Now, a partial algebra $\mathbf{A} = (A, F)$ can be be called perfect if $A = F\langle A\rangle$.

Similarly as in the previous case, iterating this new construction for a given subalgebra B of $\mathbf{A}$, one can produce a transfinite sequence $\bigl\{F^{(\alpha)}\langle B\rangle\bigr\}_{\alpha\in\Omega}$ of subalgebras of $\mathbf{A}$, isotone in B and antitone in α. Then one can show that for each partial algebra $\mathbf{A}$ the sequence $\bigl\{F^{(\alpha)}\langle A\rangle\bigr\}_{\alpha\in\Omega}$ stabilizes starting from some ordinal τ and the respective sequence item is the largest perfect subalgebra of $\mathbf{A}$ which can be called the perfect kernel of $\mathbf{A}$ and denoted by $\operatorname{Pkr} \mathbf{A}$. Also the correspondig characteristic $\tau_{\mathbf{A}} = \tau$, called the order of $\mathbf{A}$, and the rank function $\operatorname{rank}_{\mathbf{A}}\langle x\rangle$ on elements of A can be introduced in the obvious way.

It is a well known result that for each second countable topological space $(M, \mathcal{O})$ it holds that $\tau_{\mathbf{M}} < \omega_1$, and each ordinal $< \omega_1$ can occure. More generally, for each ordinal α there is a metric space (M, d) satisfying $\tau_{\mathbf{M}} = \alpha$. At a glance the fact that infinitely many iterations of the operation $\lim\langle X\rangle$ are needed, seems to be caused by the ω-arity of the partial operation lim. However, introducing a slight modification of the algebras $\mathbf{T}_\alpha$ we will show that this is not the reason.

Let s be an arbitrary element not belonging to any of the sets T_α. For every $\alpha \in \Omega$ we put

$$S_\alpha = T_\alpha \cup \{s\} \qquad \text{and} \qquad g_\alpha(x) = \begin{cases} f_\alpha(x) & \text{if } t \neq x \in T_\alpha, \\ s & \text{if } x = t, \\ t & \text{if } x = s. \end{cases}$$

Now, substituting the algebras $\mathbf{S}_\alpha = (S_\alpha, g_\alpha)$ in the places of the algebras $\mathbf{T}_\alpha$ and inspecting once more the proof of the Theorem, one can find that

$$g_\alpha^{(\beta)}\langle S_\alpha\rangle = g_\alpha^{(\beta)}[S_\alpha]$$

holds for all $\alpha, \beta \in \Omega$. With this fact in mind it can be easily seen that

$$g_\alpha^{(\alpha)}\langle S_\alpha\rangle = \{s,t\} = g_\alpha^{(\alpha+1)}\langle S_\alpha\rangle \qquad \text{and} \qquad \operatorname{rank}_{\mathbf{S}_\alpha}\langle x\rangle < \alpha$$

for each $\alpha \in \Omega$ and each $x \in S_\alpha \setminus \{s,t\}$. Hence

$$\operatorname{Pkr}\mathbf{S}_\alpha = \{s,t\} \qquad \text{and} \qquad \tau_{\mathbf{S}_\alpha} = \alpha$$

for each $\alpha \in \Omega$. Thus also in this case every ordinal can occur as the order of a finitary algebra.

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