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OSCILLATION OF VOLTERRA INTEGRAL EQUATIONS AND FORCED FUNCTIONAL DIFFERENTIAL EQUATIONS

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1. Introduction

We provide some sufficient conditions under which oscillation phenomenon occurs for the linear Volterra integral equation of convolution type with delays

$$x(t) = f(t) + \int_0^t \sum_{i=1}^n a_i(t-s)x(s-r_i)\,ds, \quad t \geq 0, \tag{1.1}$$

and the difference-integral equation

$$x(t) - \sum_{j=1}^m p_j x(t-\sigma_j) = f(t) + \int_0^t \sum_{i=1}^n a_i(t-s)x(s-r_i)\,ds, \quad t \geq 0, \tag{1.2}$$

where $f \in C(\mathbb{R}^+, \mathbb{R})$, $a_i(\cdot) \in L^1_{\text{loc}}(\mathbb{R}^+)$, $r_i \in \mathbb{R}$, for $i = 1, 2, 3, \dots, n$, $p_j \in \mathbb{R}$, $\sigma_j \in \mathbb{R}$, for $j = 1, 2, 3, \dots, m$.

We also study oscillation of the following neutral differential equation with the forcing term f,

$$\frac{d}{dt}\left[x(t) + \sum_{j=1}^m p_j x(t-\sigma_j)\right] + \sum_{i=1}^n q_i x(t-r_i) = f(t), \quad \text{for } t \geq 0, \tag{1.3}$$

where $f \in C(\mathbb{R}^+, \mathbb{R})$, $p_j \in \mathbb{R}$, $\sigma_j \in \mathbb{R}$, for $j = 1, 2, 3, \dots, m$, $q_i \in \mathbb{R}$, $r_i \in \mathbb{R}$, for $i = 1, 2, 3, \dots, n$.

Our approach bases on the method of Laplace transform which has been used to study oscillation of delay differential equations [**1, 16**], oscillations of neutral differential equations without forcing terms [**24–26**] and oscillations of linear integro-differential equations [**10, 11**].

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To facilitate our discussions, let us first introduce the following notations and conventions.

Let $r := \max\{r_1, r_2, \dots, r_n\}$ and $T := \max\{\sigma_1, \sigma_2, \dots, \sigma_m, r\}$.

We say that x is a solution of (1.1) with initial function $\varphi \in C([-r,0],\mathbb{R})$ provided that x is continuous on $[0,\infty)$, satisfies (1.1) and $f(0) = \varphi(0)$, $x(s) = \varphi(s)$, for $s \in [-r,0]$.

The results concerning existence, uniqueness and continuous dependence of (1.1) can be found in [**2, 3, 6**] and the asymptotic behavior of the solutions has been studied elsewhere, see, e.g., [**30**] and the references cited there.

The function x is said to be a solution of (1.2) with initial function $\varphi \in C([-T,0],\mathbb{R})$ if x is continuous on $[0,+\infty]$, satisfies (1.2) and $\varphi(0)-\sum_{j=1}^{m} p_j\varphi(-\sigma_j) = f(0)$, $x(s) = \varphi(s)$ for $s \in [-T,0]$.

As we can see from Chapter 12 in [**17,** pp. 273–274], neutral differential equations defined there include difference equations. Thus, some basic properties of (1.2) can be found in [**17**].

Let $C^1([-r,0],\mathbb{R})$ denote the set of all continuously differentiable functions mapping $[-r,0]$ into $\mathbb{R}$.

The function x is said to be a solution of (1.3) with initial function $\varphi \in C^1([-T,0],\mathbb{R})$ if $x(t) + \sum_{j=1}^{m} p_j x(t-\sigma_j)$ is continuously differentiable for $t \geq 0$ and x satisfies (1.3) for $t \geq 0$, and $x(s) = \varphi(s)$, for $s \in [-T,0]$.

The fundamental theory of (1.3) is studied in [**4, 17**].

The following definitions of oscillation are used in this paper [**19, 20**].

A function x is said to be oscillatory if for any $t_1 \geq 0$, we have

$$\inf_{[t_1,+\infty)} x(t) < 0 < \sup_{[t_1,+\infty)} x(t)\,.$$

A function x is said to be strongly oscillatory if we have

$$\liminf_{t\to+\infty} x(t) < 0 < \limsup_{t\to+\infty} x(t)\,.$$

2. Preliminaries

In this section, we establish some results needed in the proofs of our main theorems.

To guarantee the existence of Laplace transforms of solutions of (1.1), (1.2) and (1.3), we assume that for the function f, there exist two real numbers $M \in \mathbb{R}^+$ and $b \in \mathbb{R}$ such that

$$|f(t)| \leq Me^{bt}, \qquad \text{for } t \geq 0.$$

To begin with, let us start from (1.3). By Theorem 7.3 in [**17,** p. 26], we see that every solution x of (1.3) satisfies

$$|x(t)| \leq M_1 e^{b_1 t}, \qquad \text{for } t \geq 0,$$

where $M_1 \in \mathbb{R}^+$, $b_1 \in \mathbb{R}$, which implies the existence of the Laplace transform of every solution x of (1.3).

For equation (1.2), let $H(s)$ denote the Heaviside function, defined as,

$$H(s) := \begin{cases} 1, & s \geq 0, \\ 0, & s < 0, \end{cases} \qquad \text{and} \qquad \overline{H}(s) := 1 - H(s).$$

Let $U := \{j \in \{1, 2, \dots, m\};\ \sigma_j = 0\}$ and $V := \{1, 2, \dots, m\} \setminus U$. Then we have

$$\sum_{j=1}^{m} p_j x(t - \sigma_j) = -\sum_{j \in U} p_j \int_0^t x(t-s)\, d\overline{H}(s) + \sum_{j \in V} p_j \int_0^t x(t-s)\, d\overline{H}(s - \sigma_j) \quad \text{for } t > 0.$$

As we know, the Laplace transform of the derivate of $H(s)$ exists. So we can write (1.2) in the form (1.1). Therefore, we only need to prove the existence of the Laplace transforms of solutions of (1.1). When $r = 0$, the existence of the Laplace transform of every solution of (1.1) has been proved in [**7, 13**]. So here we assume that $r \neq 0$.

The following conditions are used throughout this chapter.

There exist real numbers $b \in \mathbb{R}$ and $M \in \mathbb{R}^+$ such that

$$|f(t)| \leq M e^{bt}, \qquad \text{for } t \geq 0 \tag{2.1}$$

$$|a_i(t)| \leq M e^{bt}, \qquad \text{for } t \geq 0,\ i = 1, 2, \dots, n. \tag{2.2}$$

Lemma 2.1. *Assume that* (2.1) *and* (2.2) *hold. Then every solution of* (1.1) *has Laplace transform.*

Proof. Take a solution x of (1.1) with initial function $\varphi \in C([-r, 0], \mathbb{R})$; by (1.1) and (2.1), we have

$$|x(t)| \leq M e^{bt} + \sum_{i=1}^{n} \int_{-r_i}^{0} |a_i(t - s - r_i)|\, |\varphi(s)|\, ds + \sum_{i=1}^{n} \int_0^t |a_i(t - s - r_i)|\, |x(s)|\, ds,\ \ t \geq 0\,.$$

Multiplying both sides of this inequality by e^{-bt}, and taking into account (2.1) and (2.2), we obtain

$$e^{-bt}|x(t)| \leq M + M \sum_{i=1}^{n} e^{-br_i} \int_{-r_i}^{0} e^{-bs} |\varphi(s)|\, ds + M \sum_{i=1}^{n} e^{-br_i} \int_0^t e^{-bs} |x(s)|\, ds, \quad \text{for } t \geq 0.$$

By Gronwall's inequality, it follows

$$|x(t)| \leq \zeta_1 e^{(b+\zeta_2)t}, \quad t > 0,$$

where

$$\zeta_1 := M\left(1 + \sum_{i=1}^{n} e^{-br_i} \int_{-r_i}^{0} e^{-bs}|\varphi(s)|\, ds\right),$$
$$\zeta_2 := M\sum_{i=1}^{n} e^{-br_i},$$

which is a sufficient condition for the existence of the Laplace transform of x. $\square$

As we will see in the procedure of the proofs of our main theorems, the central role is played by an abscissa of convergence of Laplace transform, which is defined by

$$b := \inf\{\sigma \in \mathbb{R};\ X(\sigma) \text{ exists}\},$$

where $X(\lambda)$ is the Laplace transform of $x(t)$.

If b is the abscissa of convergence of $X(\lambda)$, then $X(\lambda)$ is analytic on $\operatorname{Re}\lambda > b$ in the complex plane $\mathbb{C}$. This can be found in [**32,** p. 347]. If $\dot{x}(t)$ exists, and b_1 is the abscissa of convergence of the Laplace transform of $\dot{x}(t)$, then

$$L\left[\dot{x}(t)\right] = -x(0) + \lambda L\left[x(t)\right],$$

and $b_1 \leq b$.

Now we will present some results which are needed below.

Lemma 2.2. *Assume that b is the abscissa of convergence of the Laplace transform $X(\lambda)$. Then for any $\varepsilon > 0$, $X(\lambda)$ has singular points in the region D_ε of the complex plane $\mathbb{C}$ defined by*

$$D_\varepsilon := \{\lambda = \lambda_1 + i\lambda_2;\ \lambda_1 \in (b-\varepsilon, b],\ \lambda_2 \in \mathbb{R}\}.$$

The proof of the lemma is implied by the definition of abscissa of convergence. In fact, if the lemma is not true, then there exists an $\varepsilon > 0$, such that $X(\lambda)$ is analytic in D_ε. Thus $X(\lambda)$ exists in $\operatorname{Re}\lambda > b-\varepsilon$. This is a contradiction with the definition of abscissa of convergence.

Lemma 2.3. *If $X(\lambda)$ is the Laplace transform of a nonnegative function x and has the abscissa of convergence $b > -\infty$, then $X(\lambda)$ has a singularity at the point $\lambda = b$ on the complex plane $\mathbb{C}$* [**31,** p. 58].

3. Oscillation of a Volterra Integral Equation

In this section, we shall present the main results for oscillation of the Volterra integral equation (1.1) via the method of Laplace transform.

Let $x_c(t)$ denote $x(t+c)$, where $c \in \mathbb{R}$. Then the Laplace transform $X_c(\lambda)$ of $x_c(t)$ exists and has the same abscissa of convergence as $X(\lambda)$ by noting the following formula

$$X_c(\lambda) = e^{\lambda c}\left[X(\lambda) - \int_0^c e^{-\lambda t} x(t)\, dt\right].$$

The last integral defines an entire function of the complex variable $\lambda \in \mathbb{C}$. It is clear that $X(\lambda)$ and $X_c(\lambda)$ have their singularities at the same points on the complex plane.

On the other hand, the translation of (1.1) along a solution x by $c \in \mathbb{R}$ is the following equation

$$x(t+c) = f(t+c) + \int_0^{t+c} \sum_{i=1}^{n} a_i(t+c-s)x(s-r_i)\, ds, \quad t \geq 0.$$

Multiplying the factor $e^{-\lambda t}$ in both sides of this equation, and integrating it from 0 to $+\infty$, we obtain

$$X_c(\lambda) = F_c(\lambda) + \int_0^{+\infty} e^{-\lambda t} \int_0^{t+c} \sum_{i=1}^{n} a_i(t+c-s)x(s-r_i)\, ds\, dt,$$

where $F_c(\lambda)$ denotes the Laplace transform of $f(t+c)$. Then, we find

$$\begin{aligned}
&\int_0^{+\infty} \int_0^{t+c} e^{-\lambda t} a_i(t+c-s)x(s-r_i)\, ds\, dt \\
&\quad = \int_0^c x(s-r_i) \int_0^{+\infty} e^{-\lambda t} a_i(t+c-s)\, ds\, dt \\
&\qquad + \int_0^{+\infty} \int_0^t e^{-\lambda t} a_i(t-s)x_c(s-r_i)\, ds\, dt \\
&\quad := I_1 + I_2 .
\end{aligned}$$

It is easy to see that

$$I_1 = \zeta_i(\lambda) + m_i(\lambda)A_i(\lambda), \qquad I_2 = A_i(\lambda)[\mu_i(\lambda) + e^{-\lambda r_i} X_c(\lambda)]$$

where

$$\begin{aligned}
\zeta_i(\lambda) &:= \int_0^c x(s-r_i)e^{-\lambda(s-c)} \int_{c-s}^0 e^{-\lambda t} a_i(t)\, dt\, ds, \\
m_i(\lambda) &:= \int_0^c x(s-r_i)e^{-\lambda(s-c)}\, ds, \\
\mu_i(\lambda) &:= \int_{-r_i}^0 e^{-\lambda(s+r_i)} \varphi_c(s)\, ds
\end{aligned}$$

and $A_i(\lambda)$ is the Laplace transform of $a_i(t)$. The functions $\zeta_i(\cdot)$, $m_i(\cdot)$ and $\mu_i(\cdot)$ are entire functions of the complex variable $\lambda \in \mathbb{C}$. Therefore we have

$$X_c(\lambda) = F_c(\lambda) + \sum_{i=1}^{n} \zeta_i(\lambda) + \sum_{i=1}^{n} (m_i(\lambda) + \mu_i(\lambda)) A_i(\lambda) + \sum_{i=1}^{n} e^{-\lambda r_i} A_i(\lambda) X_c(\lambda)\,.$$

Define $H(\lambda) := 1 - \sum_{i=1}^{n} e^{-\lambda r_i} A_i(\lambda)$. If $H(\lambda) = 0$ has no real roots, then we have

$$X_c(\lambda) = \frac{F_c(\lambda) + \sum_{i=1}^{n} \zeta_i(\lambda) + \sum_{i=1}^{n} (m_i(\lambda) + \mu_i(\lambda)) A_i(\lambda)}{H(\lambda)}\,. \tag{3.1}$$

As mentioned previously, the authors [**16**] have used the method of Laplace transform to study oscillation of delay differential equations. Here we will apply their method to study oscillation of (1.1).

Theorem 3.1. *Assume that the following conditions are satisfied:*

$$\begin{cases} a, a_1, a_2, \ldots, a_n \text{ are abscissas of convergence of } F(\lambda),\ A_1(\lambda), \\ A_2(\lambda),\ \ldots,\ A_n(\lambda), \text{ respectively, and } a > \max\{a_1,\ a_2,\ \ldots,\ a_n\}. \\ F(\lambda) \text{ has a singularity on } \operatorname{Re}\lambda = a, \text{ but is analytic at } \lambda = a. \end{cases} \tag{3.2}$$

(3.3) $H(\lambda)$ *has no real roots on* $[a, +\infty)$.

Then every solution of (1.1) *is oscillatory.*

Proof. Take a solution x of (1.1); for the sake of contradiction, we assume that x is not oscillatory. Then there exists a sufficiently large $T > 0$ such that either $x(t) \geq 0$ or $x(t) \leq 0$ for $t > T$.

Consider the case $x(t) \geq 0$ for $t > T$. (The case $x(t) \leq 0$ for $t > T$ can be treated in a similar way). Let us take a number $c > T$ such that $x_c(t) \geq 0$ for $t > 0$, namely, the function $x_c(t)$ is a nonnegative function. Assume that b is the abscissa of convergence of $X(\lambda)$, so $X_c(\lambda)$ is analytic on the half-plane $\operatorname{Re}\lambda > b$. By Lemma 2.3, $X_c(\lambda)$ can not be analytically continued to the point $\lambda = b$ from the right side, namely, there is no complex neighborhood of b on which we can find an analytic function which agrees with $X_c(\lambda)$ for $\operatorname{Re}\lambda > b$. By assumptions (3.2) and (3.3), we see that the function on the right side of (3.1) is analytic for $\operatorname{Re}\lambda > \max(a, b)$.

If $a > b$, in view of (3.2), $F(\lambda)$ has a singularity on $\operatorname{Re}\lambda = a$, and $A_i(\lambda)$, $i = 1, 2, 3, \ldots, n$, are analytic in $\operatorname{Re}\lambda \geq a$. Taking (3.3) into account, we see that $X_c(\lambda)$ has a singularity $\operatorname{Re}\lambda = a$, which contradicts that $X_c(\lambda)$ is analytic in $\operatorname{Re}\lambda > b$.

If $a < b$, by (3.2) and (3.3), the function on the right side of (3.1) is analytic in the region $\operatorname{Re}\lambda > a$ and at $\lambda = a$. This implies that $X_c(\lambda)$ is analytic even in the strip $a < \operatorname{Re}\lambda \leq b$. This is a contradiction.

If $a = b$, by the assumptions (3.2) and (3.3), we see that the function on the right side of (3.1) is analytic in $\operatorname{Re}\lambda = a$, but $X_c(\lambda)$ has a singularity at $\operatorname{Re}\lambda = b = a$, which is a contradiction.

The proof is complete. □

Theorem 3.2. *Assume that the following conditions are satisfied:*

$$(3.4)\qquad \begin{cases} a, a_1, a_2, \dots, a_n \text{ are the abscissas of convergence of } F(\lambda),\ A_1(\lambda), \\ A_2(\lambda),\ \dots,\ A_n(\lambda), \text{ respectively. There is an } i \in \{1,2,\dots,n\} \text{ such} \\ \text{that} \\ \qquad\qquad a_i > \max\{a, a_1, a_{i-1}, a_{i+1}, \dots, a_n\}. \\ A_i(\lambda) \text{ has a singularity on } \operatorname{Re}\lambda = a_i, \text{ but is analytic at } \lambda = a_i. \end{cases}$$

(3.5) $H(\lambda)$ *has no real roots on* $[a_i, +\infty)$.

Then every solution of (1.1) *is oscillatory.*

The proof of it is similar to the one of Theorem 3.1.

Note that in (1.1), if $a_i(t) = c_i w(t)$, $i = 1, 2, \dots, n$, c_i are real numbers, then $a_i(t)$, $(i = 1, 2, \dots, n)$, have the same abscissa d. If $d > a$, where a is the abscissa of convergence of $F(\lambda)$, then we can not apply Theorems 3.1 and 3.2. To cover the latter case, we have the following.

Theorem 3.3. *Assume that the following conditions are satisfied:*

$$(3.6)\qquad \begin{cases} a \text{ and } d \text{ are the abscissas of convergence of } F(\lambda) \text{ and } D(\lambda), \text{ and} \\ d > a \text{ where } D(\lambda) \text{ is the Laplace transform of } w(t).\ D(\lambda) \text{ has a} \\ \text{singularity on } \operatorname{Re}\lambda = d, \text{ but is analytic at } \lambda = d. \end{cases}$$

(3.7) $H(\lambda)$ *has no real roots on* $[d, +\infty)$.

Then every solution of the Volterra integral equation

$$(3.8)\qquad x(t) = f(t) + \int_0^t w(t-s) \sum_{i=1}^n c_i x(s - r_i)\, ds, \qquad t \ge 0,$$

is oscillatory.

Proof. Since $a_i(t) = c_i w(t)$, we see that (3.1) has the form

$$X_c(\lambda) = \frac{F_c(\lambda) + \sum_{i=1}^n \zeta_i(\lambda) + D(\lambda) \sum_{i=1}^n c_i (m_i(\lambda) + \mu_i(\lambda))}{H(\lambda)}.$$

where $H(\lambda) := 1 - D(\lambda) \sum_{i=1}^n c_i e^{-\lambda r_i}$.

The rest of the proof is similar to the one of Theorem 3.1. □

By the following example we show that for some Volterra integral equations, if (3.6) or (3.7), or both, are not true then not all solutions of the equation are oscillatory.

Example 3.1. Consider the Volterra integral equation

$$x(t) = 1 + \int_0^t 2x(s-1)\,ds, \qquad t \geq 0\,.$$

The abscissas of convergence of $F(\lambda)$ and $D(\lambda)$ are 0, namely, $a = b = 0$. Note that $D(\lambda) = 2/\lambda$ is singular at $\lambda = 0$. This means that (3.6) is not satisfied. Furthermore

$$H(\lambda) := 1 - \frac{2e^{-\lambda}}{\lambda} = \frac{\lambda - 2e^{-\lambda}}{\lambda}\,,$$

the function $L(\lambda) := \lambda - 2e^{-\lambda}$ has one real root $\overline{\lambda} \in [0, +\infty)$. So (3.7) does not hold.

On the other hand, if we only consider the solutions of the delay differential equation

$$\dot{x}(t) - 2x(t-1) = 0$$

with the initial functions $\varphi \in C([-1, 0]\,, \mathbb{R})$ and $\varphi(0) = x(0) = 1$, these solutions are also the solutions of the above Volterra integral equation. But it is clear that $x(t) = e^{\overline{\lambda} t}$ is a nonoscillatory solution of this delay differential equation. So the Volterra integral equation has a nonoscillatory solution.

The following example shows that the conditions in Theorem 3.1 are not necessary for (1.1) to have oscillatory solution.

Example 3.2. Consider the following Volterra integral equation

$$x(t) = \frac{\cos t - \sin t + e^{-t}}{2} + \int_0^t e^{-(t-s)} x(s - 2\pi)\,ds, \quad t \geq 0\,.$$

The Laplace transform of $f(t)$ and $a(t)$ are, respectively,

$$F(\lambda) = \frac{1}{2(1+\lambda)} + \frac{\lambda - 1}{2(1+\lambda^2)}, \qquad A(\lambda) = \frac{1}{1+\lambda}\,.$$

The abscissas of convergence of $F(\lambda)$ and $A(\lambda)$ are $a = 0$ and $b = -1$, respectively. Thus, condition (3.2) is satisfied. Moreover, the corresponding characteristic equation is

$$H(\lambda) = \frac{1}{1+\lambda}\left((1+\lambda) - e^{-2\pi\lambda}\right)$$

which has a real root $\lambda = 0 \in [0, +\infty)$. That is, condition (3.3) is not satisfied. However, $x(t) = \cos t$ is an oscillatory solution of the equation.

Example 3.3. Consider the Volterra integral equation

$$x(t) = \cos t - \frac{e^{-1}(\cos t + \sin t) - e^{-(1+t)}}{2} + \int_0^t e^{-(t-s+1)} x(s - 2\pi)\, ds, \quad t \geq 0\,.$$

It is easy to see that (3.2) and (3.3) are satisfied. By Theorem 3.1, we know that every solution of this Volterra integral equation is oscillatory, for example, $x(t) = \cos t$ is an oscillatory solution.

Generally speaking, it is not an easy task to check that the characteristic equation has no real roots in an interval. It is natural to ask: when does $H(\lambda) = 0$ have no real roots on $[\alpha, +\infty)$? where $\alpha \in \mathbb{R}$. In order to get an answer we restrict ourselves to the Volterra integral equation

$$x(t) = f(t) + \int_0^t a(t-s)x(s-r)\, ds, \quad t \geq 0\,. \tag{3.9}$$

The corresponding characteristic equation is

$$H(\lambda) := 1 - A(\lambda)e^{-\lambda r} = 0\,.$$

The following result is obvious.

Lemma 3.1. *Assume that there exist two real numbers ζ and $M > 0$ such that*

$$|a(t)| \leq Me^{\zeta t}, \quad \textit{for} \;\; t \geq 0, \tag{3.10}$$

$$\zeta < \alpha \quad \textit{and} \quad (\alpha - \zeta)e^{\alpha r} > M. \tag{3.11}$$

Then $H(\lambda)$ has no real roots on $[\alpha, +\infty)$.

The following result is a combination of Theorem 3.1 and Lemma 3.1, in which the more explicit conditions are provided.

Corollary 3.1. *Assume that the following conditions hold:*

(3.12) *a and b are the abscissas of convergence of $F(\lambda)$ and $A(\lambda)$, respectively, and $a > b$. $F(\lambda)$ has a singularity on* $\mathrm{Re}\,\lambda = a$, *but is analytic at $\lambda = a$.*

(3.13) *All conditions of Lemma* (3.1) *hold.*

Then every solution of (3.9) *is oscillatory.*

An illustrative example of Corollary 3.1 is the following.

Example 3.4. Consider the Volterra integral equation

$$x(t) = \sin t + \int_0^t e^{-2(t-s)} \cos(t-s) x(s-1)\, ds, \quad t \geq 0 .$$

It follows that

$$f(t) = \sin t, \qquad \text{and} \qquad a(t) = e^{-2t} \cos t .$$

We can see easily that all the conditions in Corollary 3.1 hold. Thus every solution of this Volterra integral equation is oscillatory.

Now we shall consider some specific cases of (1.1), especially some forced delay differential equations. To do this, assume that the function f in (3.8) is continuously differentiable and the function $w(t)$ in (3.8) has the form

$$w(t) = e^{bt} ,$$

where b is a real number. Then for $n = 1$, equation (3.8) has the following form

$$\dot{x}(t) - bx(t) + px(t-r) = \dot{f}(t) - bf(t), \quad t \geq 0, \tag{3.14}$$

where p is a real number.

Since $L[\dot{f}] = \lambda F(\lambda) + f(0)$, the abscissa of convergence of $L[\dot{f}]$ is less than or equal to the one of $F(\lambda)$. So the abscissa of convergence of the Laplace transform of $\dot{f}(t) - bf(t)$ is less than or equal to the one of $F(\lambda)$.

Corollary 3.2. *Consider equation* (3.14) *where assuming that the following conditions are satisfied.*

(i) *a is the abscissa of convergence of $F(\lambda)$. $F(\lambda)$ has a singularity on* $\operatorname{Re} \lambda = a$*, but $F(\lambda)$ is analytic at $\lambda = a$. And $a > b$.*

(ii) $(a-b)e^{ar} > p$.

Then every solution of (3.14) *with* $x(0) = f(0)$ *is oscillatory.*

Proof. Clearly the condition (3.2) is satisfied (for $n = 1$). So we only need to prove that the characteristic equation has no real roots in $[a, +\infty)$. Since

$$H(\lambda) = \frac{e^{-\lambda r}}{\lambda - b}((\lambda - b)e^{\lambda r} - p) ,$$

it is easy to prove that the function $L(\lambda) := (\lambda - b)e^{\lambda r} - p$ is an increasing function of λ ($\in \mathbb{R}$). Also by (ii), we obtain that $L(a) > 0$. So $L(\lambda) > 0$ for real $\lambda > a$. This means that

$$H(\lambda) = L(\lambda) \frac{e^{-\lambda r}}{\lambda - b}$$

has no real roots on $[a, +\infty)$. Thus by Theorem 3.1, every solution of the Volterra integral equation

$$x(t) = f(t) + p \int_0^t e^{b(t-s)} x(s-r)\, ds, \quad t \geq 0 ,$$

is oscillatory. This is equivalent to saying that every solution of (3.14) with the condition $x(0) = f(0)$ is oscillatory. $\square$

Example 3.5. Consider the delay differential equation

$$\dot{x}(t) + x(t) + \frac{1}{2}x(t-\pi) = 2\cos t + \sin t, \quad t \geq 0.$$

Corresponding to (3.14), we have that $b = -1$, $p = \frac{1}{2}$, $r = \pi$,

$$f(t) = \frac{1}{2}(\cos t + 3\sin t) - \frac{e^{-t}}{2},$$

and the Laplace transform $F(\lambda)$ of f has the abscissa of convergence $\lambda = 0$. Also $F(\lambda)$ has singularities on $\operatorname{Re}\lambda = 0$, but $F(\lambda)$ is analytic at $\lambda = 0$. By Corollary 3.2, we know that every solution of this delay differential equation with forcing term is oscillatory provided that $x(0) = f(0) = 0$. For example, $x(t) = 2\sin t$ is an oscillatory solution.

Having finished the study of the oscillation of (1.1), now let us turn to (1.2). In this case, (3.1) has the form

$$X(\lambda) = \frac{F(\lambda) + \sum_{j=1}^{m} p_j \int_{-\sigma_j}^{0} e^{-\lambda(s+\sigma_j)}\varphi(s)\,ds + \sum_{i=1}^{n} A_i(\lambda) \int_{-r_i}^{0} e^{-\lambda(s+r_i)}\varphi(s)\,ds}{H(\lambda)}$$

where $H(\lambda) = 1 - \sum_{j=1}^{m} p_j e^{-\lambda\sigma_j} - \sum_{i=1}^{n} A_i(\lambda)e^{-\lambda r_i}$.

Following the same way as we have done in Theorems 3.1, 3.2 and 3.3, we have the following results without further proving.

Theorem 3.4. *Assume that the following conditions hold.*

(i) $a, a_1, a_2, \ldots, a_n$ *are the abscissas of convergence of* $F(\lambda)$, $A_1(\lambda)$, $A_2(\lambda)$, $\ldots$, $A_n(\lambda)$, *respectively, and* $a > \max\{a_1, a_2, \ldots, a_n\}$. $F(\lambda)$ *has a singularity on* $\operatorname{Re}\lambda = a$, *but is analytic at* $\lambda = a$.

(ii) $H(\lambda)$ *has no real roots on* $[a, +\infty)$.

Then every solution of (1.2) *is oscillatory.*

Theorem 3.5. *Assume that the following conditions hold.*

(i) $a, a_1, a_2, \ldots, a_n$ *are the abscissas of convergence of* $F(\lambda)$, $A_1(\lambda)$, $A_2(\lambda)$, $\ldots$, $A_n(\lambda)$, *respectively. There is an* $i \in \{1, 2, \ldots, n\}$ *such that*

$$a_i > \max\{a, a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n\}.$$

$A_i(\lambda)$ *has a singularity on* $\operatorname{Re}\lambda = a_i$, *but is analytic at* $\lambda = a_i$.

(ii) $H(\lambda)$ *has no real roots on* $[a_i, +\infty)$.

Then every solution of (1.2) *is oscillatory.*

Theorem 3.6. *Assume that the following conditions hold.*

(i) *a and b are the abscissas of convergence of $F(\lambda)$ and $A(\lambda)$, respectively, where $b > a$. $A(\lambda)$ has a singularity on* $\operatorname{Re}\lambda = b$*, but $A(\lambda)$ is analytic at $\lambda = b$.*

(ii) *$H(\lambda)$ has no real roots on $[b, +\infty)$, where*

$$H(\lambda) := 1 - \sum_{j=1}^{m} p_j e^{-\lambda\sigma_j} - A(\lambda) \sum_{i=1}^{n} e^{-\lambda r_i}\,.$$

Then every solution of the difference-integral equation

$$x(t) - \sum_{j=1}^{m} p_j x(t - \sigma_j) = f(t) + \sum_{i=1}^{n} c_i \int_0^t a(t-s)x(s - r_i)\,ds, \quad t \geq 0. \tag{3.15}$$

is oscillatory.

As a special case of (3.15) *we consider the following equation*

$$x(t) - px(t-\sigma) = f(t) + \int_0^t a(t-s)x(s-r)\,ds, \quad t \geq 0, \tag{3.16}$$

where $p, \sigma, r \in \mathbb{R}^+$.

The characteristic equation of (3.16) *is*

$$H(\lambda) := 1 - pe^{-\lambda\sigma} - A(\lambda)e^{-\lambda r} = 0\,.$$

Corollary 3.3. *Assume that the following conditions hold.*

(i) *a and b are the abscissas of convergence of $F(\lambda)$ and $A(\lambda)$, respectively, where $a > b$. $F(\lambda)$ has a singularity on* $\operatorname{Re}\lambda = a$*, but is analytic at $\lambda = a$.*

(ii) *There are two real numbers $\zeta < a$ and $M > 0$ such that*

$$|a(t)| \leq Me^{\zeta t}, \qquad t > 0\,.$$

(iii) *$p \leq e^{a\sigma}$, and $(a - \zeta)(1 - pe^{-a\sigma)} > Me^{-ar}$.*

Then every solution of (3.16) *is oscillatory.*

Proof. In view of the conditions of Theorem 3.4 (when $n = 1$), we need only to prove that $H(\lambda) = 0$ has no real roots in $[a, +\infty)$. Since by (ii) we know that

$$H(\lambda) \geq \frac{1}{\lambda - \zeta}\left[(\lambda - \zeta)(1 - pe^{-\lambda\sigma}) - Me^{-\lambda r}\right], \quad \text{for } \lambda > \zeta\,.$$

Let $\Delta(\lambda) := (\lambda - \zeta)(1 - pe^{-\lambda\sigma}) - Me^{-\lambda r}$. It is easy to see from (ii) and (iii) that the function $\Delta(\lambda)$ is increasing and $\Delta(a) > 0$. Therefore, $H(\lambda) = 0$ has no real roots in $[a, +\infty)$. $\square$

As an application of Theorem 3.4, we discuss oscillation of the following neutral differential equation

$$\dot{x}(t) - p\dot{x}(t-\sigma) - bx(t) + bpx(t-\sigma) - cx(t-r) = \dot{f}(t) - pf(t)\,, \tag{3.17}$$

where $p, \sigma, r, c \in \mathbb{R}^+$ and f is continuously differentiable. As a matter of fact, setting $a(t) := ce^{bt}$ in (3.16), we can easily get (3.17).

Corollary 3.4. *Suppose that*

(i) *a is the abscissa of convergence of $F(\lambda)$. $F(\lambda)$ has a singularity on* $\operatorname{Re}\lambda = a$, *but $F(\lambda)$ is analytic at $\lambda = a$.*

(ii) *The inequalities $b < a$, $p < e^{a\sigma}$ and $(a-b)(1-pe^{-a\sigma}) > |c|e^{-ar}$ hold.*

Then every solution of (3.17) *with the condition $x(0) = f(0)$ is oscillatory.*

Proof. Since $a(t) = ce^{bt}$, we have $A(\lambda) = c/(\lambda - b)$. The rest of the proof can be fulfilled by following a similar way as for the proof of Corollary 3.3. □

Example 3.6. Consider the neutral differential equation

$$\dot{x}(t) - \dot{x}(t-\pi) - x(t) + x(t-\pi) = \sin t + \cos t\,.$$

It is clear that the condition (ii) in Corollary 3.4 is not satisfied. Moreover, it is easy to see that

$$x(t) = -\frac{1+\cos t}{2}$$

is a nonoscillatory solution with the property $x(0) = f(0) = -1$.

4. Oscillation of Nonhomogeneous Neutral Differential Equations

In this section we study oscillation of the neutral differential equation (1.3).

For the homogeneous neutral differential equations, necessary and sufficient conditions (in terms of their characteristic equations) for oscillation of all solutions have been obtained in [**8, 9, 12, 13, 15, 21, 27, 28**]. Also necessary and sufficient conditions for oscillation of homogeneous neutral differential equations of higher-order have been obtained in [**5, 23, 29**].

In [**16**] a method has been developed to study oscillation of nonhomogeneous delay differential equations. This method has been applied in [**24–26**] to get necessary and sufficient conditions of oscillation of some homogeneous neutral differential equations.

As we know, the characteristic equation of (1.3) is

$$H(\lambda) := \lambda + \lambda \sum_{j=1}^{m} p_j e^{-\lambda\sigma_j} + \sum_{i=1}^{n} q_i e^{-\lambda r_i} = 0 \tag{4.1}$$

and the Laplace transform $X(\lambda)$ of a solution of (1.3) is given by

$$X(\lambda) = \frac{F(\lambda) + P(\lambda)}{H(\lambda)}\,, \tag{4.2}$$

where $F(\lambda)$ is the Laplace transform of the function f,

$$\begin{aligned} P(\lambda) := {} & \sum_{j=1}^{m} p_j \varphi(-\sigma_j) - \lambda \sum_{j=1}^{m} p_j \int_{-\sigma_j}^{0} e^{-\lambda(s+\sigma_j)} \varphi(s)\, ds \\ & - \sum_{i=1}^{n} q_i \int_{-r_i}^{0} e^{-\lambda(s+r_i)} \varphi(s)\, ds, \end{aligned}$$

and φ is the initial function for the solution. It is obvious that $P(\lambda)$ is an entire function on the complex plane $\mathbb{C}$.

Theorem 4.1. *Assume that the following conditions hold.*

(i) *a is the abscissa of convergence of $F(\lambda)$. $F(\lambda)$ has a singularity on* $\operatorname{Re}\lambda = a$*, but is analytic at $\lambda = a$.*
(ii) *$H(\lambda)$ has no real roots on $[a, +\infty)$.*

Then every solution of (1.3) *is oscillatory.*

The proof of it follows the one for Theorem 3.1.

Example 4.1. It is easy to see that the nonhomogeneous neutral differential equation

$$\dot{x}(t) + 2\dot{x}(t-2\pi) + x(t-\pi) = 3(\cos t - \sin t)$$

has only oscillatory solutions. For example, $x(t) = \sin t$ is a solution. Indeed the conditions (i) and (ii) in Theorem 4.1 are satisfied, where the characteristic equation is as follows

$$H(\lambda) := \lambda + 2\lambda e^{-2\pi\lambda} + 3e^{-\pi\lambda} = 0\,,$$

which has obviously no real roots on $[0, +\infty)$. In fact $H(\lambda) > 0$ for $\lambda \geq 0$.

Note that if $p_j \geq 0$, $q_i \geq 0$ and $\sigma_j \geq$, $r_i \geq 0$, $(i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m)$, and $F(\lambda)$ has $\lambda = 0$ as the abscissa of convergence of the Laplace transform, has a singularity on $\operatorname{Re}\lambda = 0$ and is analytic at $\lambda = 0$, then the neutral differential equation (1.3) has only oscillatory solutions. Indeed its characteristic equation

$$H(\lambda) := \lambda + \lambda \sum_{j=1}^{m} p_j e^{-\lambda\sigma_j} + \sum_{i=1}^{n} q_i e^{-\lambda r_i} = 0$$

has no real roots on $[0, +\infty)$.

The previous results hold under the condition that the abscissa of convergence of $F(\lambda)$ is a real number. The case where $a = -\infty$ is covered by the following theorem.

We also assume that $\sigma := \min\{r_1, r_2, \ldots, r_n, \sigma_1, \sigma_2, \ldots, \sigma_m\} > 0$, and without loss of generality, we also assume that $r_1 := \max\{r_1, r_2, \ldots, r_n\}$.

Theorem 4.2. *Assume that the following conditions hold.*

$$\max\{r_1, r_2, \ldots, r_n\} > \max\{\sigma_1, \sigma_2, \ldots, \sigma_m\}. \tag{4.3}$$

$$\begin{cases} \textit{The abscissa of convergence of } F(\lambda) \textit{ is } -\infty, \textit{ and for some } \varepsilon > 0 \\ \textit{with the property that } \varepsilon < \min\{\sigma, r_1 - \max\{r_2,\ \ldots,\ r_n,\ \sigma_1,\ \sigma_2, \\ \ldots,\ \sigma_m\}\}, \textit{ it holds } F(\lambda) = o(e^{-\lambda(\sigma_j - \varepsilon)}) \textit{ and } F(\lambda) = o(e^{-\lambda(r_i - \varepsilon)}) \\ \textit{as } \lambda \to -\infty,\ i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m. \end{cases} \tag{4.4}$$

(4.5) *$H(\lambda)$ has no real roots on $\mathbb{R}$.*

Then every nontrivial solution of (1.3) *is oscillatory.*

Proof. For the contrary assume that (1.3) has a nonoscillatory solution x. That is, there exists a $T > 0$, such that $x(t) \geq 0$ (or $x(t) \leq 0$) and $x(t) \not\equiv 0$ for $t > T$. So we can always find a real number $c > T$ such that $x_c(t) \geq 0$ and $x_c(t) \not\equiv 0$ on $[-r, 0]$. The translation $x_c(t)$ of $x(t)$ is a nonpositive (or a nonnegative) solution of the equation

$$\frac{d}{dt}\left[x(t) + \sum_{j=1}^{m} p_j x(t-\sigma_j)\right] + \sum_{i=1}^{n} q_i x(t-r_i) = f(t+c), \quad \text{for } t \geq 0\,.$$

Fix such a $c > T$. Then for the Laplace transform $F_c(\lambda)$ of $f_c(t) = f(t+c)$, we have

$$F_c(\lambda) = e^{\lambda c}\left[F(\lambda) - \int_0^c e^{-\lambda t} f(t)\,dt\right].$$

Hence we have

$$\frac{F_c(\lambda)}{e^{-\lambda(\tau_i-\varepsilon)}} = e^{\lambda c}\frac{F(\lambda)}{e^{-\lambda(\tau_i-\varepsilon)}} - e^{\lambda c}\frac{\displaystyle\int_0^c e^{-\lambda t} f(t)\,dt}{e^{-\lambda(\tau_i-\varepsilon)}}\,, \tag{4.6}$$

$i = 1, 2, \ldots, n+m$, where $\tau_i = r_i$ if $i \in \{1, 2, \ldots, n\}$ and $\tau_i = \sigma_{i-n}$, if $i \in \{n+1, n+2, \ldots, n+m\}$. We can see that

$$\lim_{\lambda\to-\infty} e^{\lambda c} = 0\,, \tag{4.7}$$

and, by (4.4), we have

$$\lim_{\lambda\to-\infty} \frac{F(\lambda)}{e^{-\lambda(\tau_i-e)}} = 0\,. \tag{4.8}$$

Since $f \in C(\mathbb{R}^+, \mathbb{R})$, so there exists a positive number M such that

$$|f(t)| \leq M \qquad \text{on } [0, c].$$

Thus

$$\left|\int_0^c e^{-\lambda t} f(t)\,dt\right| \leq \int_0^c M e^{-\lambda t}\,dt = \frac{M}{\lambda}(1 - e^{-\lambda c})\,.$$

Therefore

$$\lim_{\lambda\to-\infty}\left|e^{\lambda c}\frac{\int_0^c e^{-\lambda t} f(t)\,dt}{e^{-\lambda(\tau_i-\varepsilon)}}\right| \leq \lim_{\lambda\to-\infty}\left|\frac{M(e^{\lambda c}-1)}{\lambda e^{-\lambda(\tau_i-\varepsilon)}}\right| = 0\,. \tag{4.9}$$

Now taking into account (4.6)–(4.9), we obtain the conclusion that $F_c(\lambda)$ also satisfies (4.4).

From the assumption and the above discussion, one can see that it is enough to assume that $x(t) \geq 0$ for $t > -r_1$. Furthermore, we know, from (4.2), (4.4) and (4.5), that $X(\lambda)$ has the abscissa of convergence $-\infty$ and $X(\lambda) > 0$, for $\lambda \in \mathbb{R}$.

By (4.1), we have

$$H(\lambda) = e^{-\lambda r_1}\left(\lambda e^{\lambda r_1} + \lambda \sum_{j=1}^{m} p_j e^{-\lambda(\sigma_j - r_1)} + q_1 + \sum_{i=2}^{n} q_i e^{-\lambda(r_i - r_1)}\right).$$

It is clear that

$$\lim_{\lambda \to -\infty} \lambda e^{\lambda r_1} = 0\,, \qquad \lim_{\lambda \to -\infty} \sum_{j=1}^{m} p_j e^{-\lambda(\sigma_j - r_1)} = 0\,,$$

and

$$\lim_{\lambda \to -\infty} \sum_{i=2}^{n} q_i e^{-\lambda(r_i - r_1)} = 0\,.$$

So the sign of $H(\lambda)$ depends eventually on the sign of q_1. Thus we have two cases: $q_1 > 0$ and $q_1 < 0$. In the following we shall discuss only the case $q_1 > 0$. The case $q_1 < 0$ can be treated in a similar way. Thus we have $H(\lambda) > 0$ eventually for $\lambda < 0$.

Let

$$\Phi_j(\lambda) := \lambda p_j \int_{-\sigma_j}^{0} e^{-\lambda(s+\sigma_j)} \varphi(s)\, ds \quad \text{and} \quad \Psi_i(\lambda) := q_i \int_{-r_i}^{0} e^{-\lambda(s+r_i)} \varphi(s)\, ds\,,$$

for $j = 1, 2, \ldots, m$ and $i = 1, 2, \ldots, n$. Then we observe that

$$\begin{aligned}\Psi_1(\lambda) := q_1 \int_{-r_1}^{0} e^{-\lambda(s+r_1)} \varphi(s)\, ds &= q_1 \varphi(\xi_0) e^{-\lambda r_1} \int_{-r_1}^{0} e^{-\lambda s}\, ds \\ &= q_1 \varphi(\xi_0) e^{-\lambda(r_1 - \varepsilon)} \frac{e^{\lambda(r_1-\varepsilon)} - e^{-\lambda\varepsilon}}{\lambda}\,,\end{aligned}$$

holds for some $\xi_0 \in [-r_1, 0]$, where ε satisfies the conditions in (4.4). Since $\varphi(t) \geq 0$ and $\varphi(t) \not\equiv 0$ on $[-r_1, 0]$, so $\Psi_1(\lambda) > 0$. This implies that $\varphi(\xi_0) \neq 0$, so we have that $\varphi(\xi_0) > 0$.

On the other hand, it is clear

$$\lim_{\lambda \to -\infty} \frac{e^{\lambda(r_1-\varepsilon)} - e^{-\lambda\varepsilon}}{\lambda} = +\infty\,.$$

Thus there exists a large number $N > 0$ such that

$$q_1\varphi(\xi_0)\frac{e^{\lambda(r_1-\varepsilon)} - e^{-\lambda\varepsilon}}{\lambda} > 1, \quad \text{for } \lambda < -N.$$

Therefore, we have

$$\Psi_1(\lambda) > e^{-\lambda(r_1-\varepsilon)}, \quad \text{for } \lambda < -N. \tag{4.10}$$

On the other hand, we also have

$$\Phi_j(\lambda) = \lambda p_j \int_{-\sigma_j}^{0} e^{-\lambda(s+\sigma_j)}\varphi(s)\, ds = \lambda p_j \sigma_j e^{-\lambda(-\xi_j+\sigma_j)}\varphi(-\xi_j)\,,$$

where $\xi_j \in (0, \sigma_j)$, $j = 1, 2, \ldots, m$, and

$$\Psi_i(\lambda) := q_i\varphi(-\eta_i)r_i e^{-\lambda(-\eta_i+r_i)},$$

where $\eta_i \in (0, r_i)$, $i = 2, 3, \ldots, n$.

So we have

$$\frac{|\Phi_j(\lambda)|}{\Psi_1(\lambda)} < \frac{|\lambda p_j|\, \varphi(-\xi_j)\sigma_j e^{-\lambda(\sigma_j-\xi_j)}}{e^{-\lambda(r_1-\varepsilon)}} = |\lambda p_j|\, \varphi(-\xi_j)\sigma_j e^{\lambda(r_1-\varepsilon-\sigma_j+\xi_j)},$$

for $\lambda < -N$, $j = 1, 2, \ldots, m$.
Since $r_1 - \varepsilon > \sigma_j > \sigma_j - \xi_j$, we find

$$\lim_{\lambda\to-\infty} e^{\lambda(r_1-\varepsilon-\sigma_j+\xi_j)} = 0, \quad \text{for } j = 1, 2, \ldots, m.$$

Similarly,

$$\frac{|\Psi_i(\lambda)|}{\Psi_1(\lambda)} < \frac{|q_i|\varphi(-\eta_i)r_i e^{\lambda(-\eta_j+r_i)}}{e^{-\lambda(r_1-\varepsilon)}} = |q_i|\, r_i\varphi(-\eta_i)\sigma_j e^{\lambda(r_1-\varepsilon-\eta_j+r_i)},$$

for $i = 2, 3, \ldots, n$.
From $r_1 - \varepsilon > r_i > r_i - \eta_i$, we have

$$\lim_{\lambda\to-\infty} e^{\lambda(r_1-\varepsilon-\eta_j+r_i)} = 0, \quad \text{for } i = 2, 3, \ldots, n.$$

Therefore we obtain

$$\lim_{\lambda\to-\infty}\frac{|\Phi_j(\lambda)|}{\Psi_1(\lambda)} = \lim_{\lambda\to-\infty}\frac{|\Psi_i(\lambda)|}{\Psi_1(\lambda)} = 0\,, \tag{4.11}$$

for $i = 2, 3, \ldots, n$ and $j = 1, 2, 3, \ldots, m$.

On the other hand,

$$F(\lambda) + P(\lambda) = F(\lambda) + \sum_{j=1}^{m} p_j\varphi(-\sigma_j) - \sum_{j=1}^{m} \Phi_j(\lambda) - \sum_{i=2}^{n} \Psi_i(\lambda) - \Psi_1(\lambda)$$

$$= \Psi_1(\lambda)\left[\frac{F(\lambda)}{\Psi_1(\lambda)} + \frac{\sum_{j=1}^{m} p_j\varphi(-\sigma_j)}{\Psi_1(\lambda)} - 1 - \frac{\sum_{j=1}^{m} \Phi_j(\lambda) + \sum_{i=2}^{n} \Psi_i(\lambda)}{\Psi_1(\lambda)}\right].$$

By (4.4) and (4.10), we know that

$$\lim_{\lambda\to-\infty} \frac{F(\lambda)}{\Psi_1(\lambda)} = 0\,.$$

Since $\sum_{j=1}^{m} p_j\varphi(-\sigma_j)$ is a real constant, so

$$\lim_{\lambda\to-\infty} \frac{\sum_{j=1}^{m} p_j\varphi(-\sigma_j)}{\Psi_1(\lambda)} = 0\,.$$

Then taking (4.11) into account, we obtain

$$\lim_{\lambda\to-\infty} (F(\lambda) + P(\lambda)) = -\infty\,.$$

Since $H(\lambda) > 0$, $X(\lambda) > 0$, so $H(\lambda)X(\lambda) > 0$, for $-\lambda$ ($\in \mathbb{R}^+$) large enough. Then from (4.2), we have,

$$0 < H(\lambda)X(\lambda) = F(\lambda) + P(\lambda) < 0\,,$$

for such a λ, a contradiction. So x is oscillatory. The proof is complete. □

Using Corollaries 1 and 2 in [**18**] we can obtain some explicit sufficient conditions under which the solutions of forced neutral differential equations of the form

$$\frac{d}{dt}\left[x(t) - px(t-\sigma)\right] + qx(t-r) = f(t), \quad t \geq 0 \tag{4.12}$$

are oscillatory.

Corollary 4.1. *Assume that the following conditions hold.*

(i) $r > \sigma$, $0 < p < 1$, $\sigma > 0$, $q > 0$.

(ii) *The abscissa of convergence of* $F(\lambda)$ *is* $-\infty$, *and, for some* $\varepsilon > 0$ *with* $\varepsilon < \min\{\sigma, r-\sigma\}$, *it holds* $F(\lambda) = o(e^{-\lambda(\sigma-\varepsilon)})$ *as* $\lambda \to -\infty$.

(iii) $\dfrac{q}{1-p}\left[r + \dfrac{p}{1-p}\sigma\right] > \dfrac{1}{e}\,.$

Then every nontrivial solution of the neutral differential equation (4.12) *is oscillatory.*

Corollary 4.2. *Assume that the conditions* (i) *and* (ii) *in Corollary* (4.1) *hold and* $p > 1$, $\sigma > 0$, $q > 0$ *and*

$$\frac{p^{1-k}q}{p-1}\left[\left(k+\frac{1}{p-1}\right)\sigma - r\right] > \frac{1}{e}\,,$$

where k *is the least positive integer such that* $k\sigma - r > 0$*. Then every nontrivial solution of the neutral differential equation* (4.12) *is oscillatory.*

As an application of Corollary 4.2, let us consider the neutral differential equation

$$\dot{x}(t) - 2\dot{x}(t-5\pi) + 3x(t-6\pi) = f(t) \tag{4.13}$$

where

$$f(t) := \begin{cases} \sin t, & t \in [2\pi, 3\pi]\ , \\ 0, & t \in [0, 2\pi) \cup (3\pi, -\infty)\,. \end{cases}$$

First, we see that the Laplace transform of f is

$$F(\lambda) = \frac{e^{-3\pi\lambda} + e^{-2\pi\lambda}}{1+\lambda^2}\,.$$

Note that at the complex points $\lambda = \pm i$, where $i = \sqrt{-1}$, the function F is bounded, and the derivate of F at the points $\lambda = \pm i$ exists. Therefore F is analytic on the complex plane $\mathbb{C}$. So the abscissa of convergence of F is equal to $-\infty$, namely, $b = -\infty$.

We observe that $p = 2$, $q = 3$, $\sigma = 5\pi$, $r = 6\pi$, so the number $k = 2$ is the least positive integer such that $5\pi k - 6\pi > 0$. Now take $\varepsilon \in (0, \pi)$, then we have

$$0 < \varepsilon < \min\{5\pi, 6\pi - 5\pi\} = \pi,$$

$$\lim_{\lambda\to-\infty} \frac{F(\lambda)}{e^{-\lambda(\sigma-\varepsilon)}} = \lim_{\lambda\to-\infty} \frac{e^{-3\pi\lambda} + e^{-2\pi\lambda}}{(1+\lambda^2)e^{-\lambda(5\pi-\varepsilon)}} = 0\,,$$

$$\lim_{\lambda\to-\infty} \frac{F(\lambda)}{e^{-\lambda(r-\varepsilon)}} = \lim_{\lambda\to-\infty} \frac{e^{-3\pi\lambda} + e^{-2\pi\lambda}}{(1+\lambda^2)e^{-\lambda(6\pi-\varepsilon)}} = 0\,,$$

and

$$\frac{p^{1-k}q}{p-1}\left[\left(k+\frac{1}{p-1}\right)\sigma - r\right] = \frac{3}{2}9\pi > \frac{1}{e}\,.$$

So all conditions of Corollary 4.2 hold.

On the other hand, $x \equiv 0$ is not a solution of (4.13). By Corollary 4.2, we know that every solution of (4.13) is oscillatory.

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