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ON CRITICAL EXPONENTS FOR A SYSTEM OF HEAT EQUATIONS COUPLED IN THE BOUNDARY CONDITIONS

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ABSTRACT. In this paper, we consider the system

$$\begin{aligned} u_t &= \Delta u, & v_t &= \Delta v & x &\in \mathbb{R}^N_+, & t &> 0,\\ -\frac{\partial u}{\partial x_1} &= v^p, & -\frac{\partial v}{\partial x_1} &= u^q & x_1 &= 0, & t &> 0,\\ u(x,0) &= u_0(x), & v(x,0) &= v_0(x) & x &\in \mathbb{R}^N_+, \end{aligned}$$

where $\mathbb{R}^N_+ = \{(x_1, x') \mid x' \in \mathbb{R}^{N-1}, x_1 > 0\}$, $p, q > 0$, and u_0, v_0 nonnegative. We prove that if $pq \le 1$ every nonnegative solution is global. When $pq > 1$ we let $\alpha = \frac{1}{2}\frac{p+1}{pq-1}$, $\beta = \frac{1}{2}\frac{q+1}{pq-1}$. We show that if $\max(\alpha,\beta) \ge \frac{N}{2}$, all nontrivial nonnegative solutions are nonglobal; whereas if $\max(\alpha,\beta) < \frac{N}{2}$ there exist both global and nonglobal nonnegative solutions. When $N = 1$, we establish some results for the blow up rate for the nonglobal solutions and some results for the decay rate for the global solutions (in the supercritical case). We also construct a nontrivial solution with vanishing initial values when $pq < 1$.

1. INTRODUCTION

In this paper we study the large time behavior of nonnegative solutions of a system as follows:

$$\begin{aligned} u_t &= \Delta u, & v_t &= \Delta v & x &\in \mathbb{R}^N_+, & t &> 0,\\ -\frac{\partial u}{\partial x_1} &= v^p, & -\frac{\partial v}{\partial x_1} &= u^q & x_1 &= 0, & t &> 0,\\ u(x,0) &= u_0(x), & v(x,0) &= v_0(x) & x &\in \mathbb{R}^N_+, \end{aligned} \tag{1.1}$$

where $\mathbb{R}^N_+ = \{(x_1, x') \mid x' \in \mathbb{R}^{N-1}, x_1 > 0\}$ $(N \ge 1)$, $p, q > 0$, and both $u_0(x)$ and $v_0(x)$ are nonnegative bounded functions satisfying the compatibility condition

$$-\frac{\partial u_0}{\partial x_1} = v_0^p \quad \text{and} \quad -\frac{\partial v_0}{\partial x_1} = u_0^q \quad \text{at} \quad x_1 = 0. \tag{1.2}$$

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In order to motivate some of our results for the above system, we recall an old result of Fujita [**F**] for the initial problem

$$(1.3)\qquad \begin{aligned} u_t &= \Delta u + u^p \qquad && x \in \mathbb{R}^N, \quad t > 0, \\ u(x,0) &= u_0(x) && x \in \mathbb{R}^N, \end{aligned}$$

with nonnegative initial data u_0. He showed that (i) if $1 < p < 1+2/N$, then (1.3) possesses no global nonnegative solutions while (ii) if $p > 1+2/N$, both global and nonglobal nonnegative solutions exist. The number $1 + 2/N$ is called the critical exponent which turns out to belong to case (i). See [**We**] for an elegant proof by Weissler as well as references to earlier proofs of this result. This result can be reformulated in a second way. The number $\frac{1}{p-1}$ is the (algebraic) blow up rate for solutions of the initial value problem for the ordinary differential equation $y' = y^p$ for $p > 1$. On the other hand $\frac{N}{2}$ is the decay rate for solutions of $w_t = \Delta w$. Thus, Fujita's result says that there are no global, nontrivial solutions of (1.3) whenever the blow up rate for $y(t)$ is not smaller than the decay rate for $w(x,t)$ while there are both global, nontrivial solutions and nonglobal solutions of (1.3) if the blow up rate is positive and smaller than the decay rate. If the blow up rate is negative, all solutions of (1.3) are global.

Over the past a few years there have been a number of extensions of Fujita's result in various directions. These include similar results for other geometries and nonlinear equations of different types. For further details, we refer the reader to the survey paper by Levine [**L1**].

Recently, Escobedo and Herrero [**EH1**] investigated the initial value problem for a weakly coupled system

$$(1.4)\qquad \begin{aligned} u_t &= \Delta u + v^p, \qquad & v_t &= \Delta v + u^q \qquad & x &\in \mathbb{R}^N, \quad t > 0, \\ u(x,0) &= u_0(x) \geq 0, & v(x,0) &= v_0(x) \geq 0 & x &\in \mathbb{R}^N. \end{aligned}$$

Set, when $pq \neq 1$,

$$\alpha_1 = \frac{p+1}{pq-1}, \qquad \beta_1 = \frac{q+1}{pq-1}.$$

Then α_1, β_1 are the blow up rates for each component of the system of ordinary differential equations $y' = z^p$, $z' = y^q$. The decay rate for the linear "system" $w_{it} = \Delta w_i$ is still $\frac{N}{2}$. The results of [**EH1**] for (1.4) take the same form as for the single equation with $\frac{1}{p-1}$ replaced by $\max(\alpha_1, \beta_1)$. When this maximum is negative or not defined, all solutions with L^∞ initial values are global.

It is possible to extend this result, in the Lipschitz case, to the system (1.4) in a cone or in the exterior of a bounded domain. See [**L2**]. The decay rate for the linear system will, in general, be different in other geometries and the method of proof employed in the case of the initial problem does not carry over in every case to the initial boundary value problem for (1.4) in unbounded domains.

Galaktionov and Levine [**GL**] considered the boundary-value problem:

$$
\begin{aligned}
u_t &= u_{xx} && x > 0, \quad t > 0, \\
-u_x &= u^p && x = 0, \quad t > 0, \\
u(x,0) &= u_0(x) \geq 0 && x > 0; \\
-u_0'(0) &= u_0^p(0).
\end{aligned}
\tag{1.5}
$$

They showed that if $1 < p \leq 2$, then $u(x,t)$ blows up in a finite time for all nontrivial u_0; whereas if $p > 2$, then $u(x,t)$ becomes unbounded for large u_0 and $u(x,t)$ exists globally for small initial data. Their result extends to the half space problem

$$
\begin{aligned}
u_t &= \Delta u && x \in \mathbb{R}^N_+, && t > 0, \\
-\frac{\partial u}{\partial x_1} &= u^p && x_1 = 0, && t > 0, \\
u(x,0) &= u_0(x) \geq 0 && x \in \mathbb{R}^N_+, \\
-\frac{\partial u_0}{\partial x_1} &= u_0^p && x_1 = 0.
\end{aligned}
\tag{1.1*}
$$

where, with $\frac{1}{p-1}$ replaced by $\frac{1}{2(p-1)}$, it takes exactly the same form as the result for (1.3). Here the ordinary differential equation that replaces $y' = y^p$ is the equation $y' = y^{(2p-1)}$. This latter equation can be loosely interpreted as a differential equation in time for the trace of the solution of (1.5) on $x = 0$ (or $x_1 = 0$ in the case of (1.1*)).

The purpose of this paper is threefold. First, we extend the result of [**GL**] to the system (1.1). Secondly, we obtain some precise information concerning the nature of the blow up and decay of the solutions in the special case that $N = 1$. These results are intended to parallel those obtained for (1.1*) in [**FQ**] in so far as is possible. There has been a flurry of activity concerning the nature of single point blow up for (1.3) in the last few years. However, almost nothing is known for (1.4). Finally, we obtain a nonuniqueness result for (1.1) in one space dimension if $pq < 1$. While uniqueness probably does hold for $pq > 1$ for (1.1), this question remains open. However, for (1.4) this was recently established in [**EH2**]. It should be possible to adapt their arguments to the present situation.

In Section 2 we establish the Fujita type global existence — global nonexistence theorem while in Section 3 we discuss the blow up and decay rate results when $N = 1$.

Throughout the remainder of this paper we let

$$\alpha = \frac{\alpha_1}{2}, \qquad \beta = \frac{\beta_1}{2}.$$

2. The Fujita Type Blow Up Theorem

Theorem 2.1. *If* $pq \leq 1$ *all nonnegative solutions of* (1.1) *are global. If* $pq > 1$ *then there are no nontrivial global nonnegative solutions of* (1.1) *if* $\max(\alpha, \beta) \geq \frac{N}{2}$ *while both global nontrivial and nonglobal solutions exist if* $\max(\alpha, \beta) < \frac{N}{2}$.

Remark 2.1. Notice that, although the blow up rates (α, β) for (1.1) are not the same as for (1.4) (α_1, β_1) , the statement of this theorem is precisely the same as the corresponding result for (1.4).

The proof proceeds by a series of lemmas.

Lemma 2.2. *Assume* $0 < pq \leq 1$. *Every solution of* (1.1) *is global, that is, for any* $T > 0$, $\|u(\cdot, t)\|_\infty + \|v(\cdot, t)\|_\infty \leq C$ *for some constant* $C = C(T)$.

Proof. The proof follows by comparison. Suppose, without loss, that $p \leq q$. Let $\frac{M}{2} = \max(1, \|u_0\|_\infty, \|v_0\|_\infty)$ and define $h(x) = M + M^{p-q}(e^{-\sigma x_1} - 1)$ and $k(x) = M + e^{-\sigma x_1} - 1$ with $\sigma = M^q$. By introducing $\bar{u}(x, t) = h(x)e^{\sigma^2 t}$ and $\bar{v}(x, t) = k(x)e^{\frac{\sigma^2}{p}t}$, it is not hard to see that solutions of (1.1) are bounded above by $(\bar{u}, \bar{v})$. □

Of much greater difficulty are the next two lemmas.

Lemma 2.3. *Suppose that* $\max(\alpha, \beta) \geq \frac{N}{2}$. *Then all nontrivial nonnegative solutions of* (1.1) *are nonglobal.*

Lemma 2.4. *Suppose that* $\max(\alpha, \beta) < \frac{N}{2}$. *Then there exist both global and nonglobal nonnegative solutions of* (1.1).

When $p = q \geq 1$, we have uniqueness of solutions of (1.1). Then (1.1) reduces to the scalar problem (1.1*) if $u_0 = v_0$. Even when $p = q < 1$ and $u_0 = v_0$, it may happen that $u \equiv v$. In such cases, we recover the result of Galaktionov and Levine and our results read

Corollary 2.5. *If* $p \leq 1$, *all solutions of* (1.1*) *are global. If* $1 < p \leq 1 + 1/N$, *all nontrivial nonnegative solutions of* (1.1*) *are nonglobal; while if* $p > 1 + 1/N$, *there exist both global and nonglobal nonnegative solutions.*

The plan of this section is as follows: In Section 2.1 we establish the claim of Lemma 2.3, and then demonstrate the proof of Lemma 2.4 in Section 2.2. To show the global existence of solutions to (1.1), we adopt the supersolution argument. For the blow up case, the situation becomes more complicated, and we shall employ a quite different approach, namely, the iteration method, which was initially applied to problem (1.3) in [**AW**] and then successfully modified for (1.4). However, because the representation formula for solutions of (1.1) is distinct from that for (1.4), several notable differences appear at the technical level, and hence the relevant arguments will be presented in detail. For definiteness, we may always assume $p \leq q$ throughout the paper.

2.1 The Case $\max(\alpha, \beta) \geq \frac{N}{2}$

In this section we establish the global nonexistence claim of Theorem 2.1. Because our arguments parallel those of [**EH1**], we shall mainly focus on the salient differences. Without loss, we may assume $\beta \geq \alpha$.

Recall that the Green's function $G(x, y; t)$ for the heat equation in $\mathbb{R}^N_+$ satisfying $\frac{\partial G}{\partial y_1} = 0$ at $y_1 = 0$ is given by

$$G(x,y;t) = (4\pi t)^{-\frac{N}{2}} \exp\left(-\frac{|x'-y'|^2}{4t}\right) \times \left(\exp\left(-\frac{(x_1-y_1)^2}{4t}\right) + \exp\left(-\frac{(x_1+y_1)^2}{4t}\right)\right).$$

For any function $w(x_1, x') \in L^1_{loc}(\mathbb{R}^N_+)$, we then define

$$S(t)w(\cdot, x') = \int_{\mathbb{R}^{N-1}} (4\pi t)^{-\frac{N-1}{2}} \exp\left(-\frac{|x'-y'|^2}{4t}\right) w(\cdot, y')\, dy' \tag{2.1a}$$

and

$$S_1(t)w(x_1, \cdot) = \int_0^\infty (4\pi t)^{-\frac{1}{2}} \left(\exp\left(-\frac{(x_1-y_1)^2}{4t}\right) + \exp\left(-\frac{(x_1+y_1)^2}{4t}\right)\right) w(y_1, \cdot)\, dy_1. \tag{2.1b}$$

We have the representation formulae for the solution of (1.1),

$$u(x_1, x', t) = S(t)S_1(t)u_0(x_1, x') + \int_0^t (\pi(t-\eta))^{-\frac{1}{2}} \exp\left(-\frac{x_1^2}{4(t-\eta)}\right) S(t-\eta)v^p(0, x', \eta)\, d\eta \tag{2.2a}$$

and

$$v(x_1, x', t) = S(t)S_1(t)v_0(x_1, x') + \int_0^t (\pi(t-\eta))^{-\frac{1}{2}} \exp\left(-\frac{x_1^2}{4(t-\eta)}\right) S(t-\eta)u^q(0, x', \eta)\, d\eta. \tag{2.2b}$$

These are the so called "variation of constants formulae" (cf. [**LSU**], for example).

Remark 2.2. As in [**EH1**] it is possible to prove local (in time) existence of solutions for given L^∞ initial values using the variation of constants formulae (2.2) and the contraction mapping principle. The details are rather standard and we therefore omit them.

Lemma 2.1.1. *Suppose that $(u(x,t),v(x,t))$ is a nontrivial solution of* (1.1). *Then there exist $\tau=\tau(u_0,v_0)>0$ and constants $m>0$, $\sigma>0$ such that*

$$v(x,\tau)\geq m\exp(-\sigma|x|^2). \tag{2.3}$$

Proof. Since $v(x,t)\not\equiv 0$, we may assume, by shifting the time axis if necessary, that $v_0(x)\not\equiv 0$ and let $\delta=\inf\{v_0(y)|y\in\overline{\Omega}\subset\mathbb{R}^N_+\}>0$. By (2.2b) we then find

$$\begin{aligned} v(x,t)&\geq S(t)S_1(t)v_0(x)\\ &\geq\delta\exp\left(-\frac{|x|^2}{2t}\right)(4\pi t)^{-\frac{N}{2}}\int_{y\in\Omega}\exp\left(-\frac{|y|^2}{2t}\right)dy. \end{aligned}$$

For any fixed τ, letting $t=\tau$, $\sigma=\frac{1}{2\tau}$, and $m=\delta(4\pi\tau)^{-\frac{N}{2}}\int_{y\in\Omega}\exp\left(-\frac{|y|^2}{2\tau}\right)dy$, we obtain (2.3). □

We next establish several estimates for solutions of (1.1).

Lemma 2.1.2. *Suppose that $p\geq 1$ and $\frac{\partial v_0}{\partial x_1}\leq 0$. Then for any t in the existence interval,*

$$t^\beta\|S(t)S_1(t)v_0(0,x')\|_\infty\leq C, \tag{2.4}$$

where $C=C(p,q)$ is a constant.

Proof. By (2.2) one can see that

$$u(0,x',t)\geq\int_0^t(\pi(t-\eta))^{-\frac{1}{2}}S(t-\eta)v^p(0,x',\eta)\,d\eta \tag{2.5}$$

and

$$v(0,x',t)\geq S(t)S_1(t)v_0(0,x').$$

Applying Jensen's inequality yields

$$\begin{aligned} u(0,x',t)&\geq\int_0^t(\pi(t-\eta))^{-\frac{1}{2}}S(t-\eta)(S(\eta)S_1(\eta)v_0(0,x'))^p d\eta\\ &\geq\int_0^t(\pi(t-\eta))^{-\frac{1}{2}}(S(t-\eta)S(\eta)S_1(\eta)v_0(0,x'))^p d\eta\\ &=\int_0^t(\pi(t-\eta))^{-\frac{1}{2}}(S(t)S_1(\eta)v_0(0,x'))^p d\eta. \end{aligned} \tag{2.6}$$

Since $\frac{\partial v_0}{\partial x_1} \le 0$, for $0 \le \eta \le t$,

$$\begin{aligned} S_1(\eta)v_0(0,\cdot) &= \int_0^\infty v_0(\sqrt{4\eta}\xi_1,\cdot)e^{-\xi_1^2}\,d\xi_1 \\ &\ge \int_0^\infty v_0(\sqrt{4t}\xi_1,\cdot)e^{-\xi_1^2}\,d\xi_1 \\ &= S_1(t)v_0(0,\cdot), \end{aligned}$$

then

$$\begin{aligned} u(0,x',t) &\ge \int_0^t (\pi(t-\eta))^{-\frac{1}{2}}(S(t)S_1(t)v_0(0,x'))^p d\eta \\ &= 2\pi^{-\frac{1}{2}}(S(t)S_1(t)v_0(0,x'))^p t^{\frac{1}{2}}. \end{aligned} \tag{2.7}$$

Hence

$$\begin{aligned} v(0,x',t) &\ge 2^q\pi^{-\frac{q}{2}}\int_0^t (\pi(t-\eta))^{-\frac{1}{2}}S(t-\eta)(S(\eta)S_1(\eta)v_0(0,x'))^{pq}\eta^{\frac{q}{2}}d\eta \\ &\ge 2^q\pi^{-(q+1)/2}\int_0^t (t-\eta)^{-\frac{1}{2}}\eta^{\frac{q}{2}}(S(t)S_1(t)v_0(0,x'))^{pq}d\eta \\ &= 2^q\pi^{-(q+1)/2}B\left(1/2,q/2+1\right)(S(t)S_1(t)v_0(0,x'))^{pq}t^{(q+1)/2}, \end{aligned} \tag{2.8}$$

where $B(a,b)$ is the Beta function.

Substituting (2.8) into (2.5) leads to

$$\begin{aligned} u(0,x',t) &\ge 2^{pq}\pi^{-((q+1)p+1)/2}B^p(1/2,q/2+1) \\ &\qquad \times \int_0^t (t-\eta)^{-\frac{1}{2}}\eta^{(q+1)p/2}(S(t)S_1(t)v_0(0,x'))^{p^2q}d\eta \\ &= 2^{pq}\pi^{-((q+1)p+1)/2}B^p(1/2,q/2+1)B(1/2,(q+1)p/2+1) \\ &\qquad \times (S(t)S_1(t)v_0(0,x'))^{p^2q}t^{((q+1)p+1)/2}, \end{aligned}$$

and consequently,

$$\begin{aligned} v(0,x',t) &\ge 2^{pq^2}\pi^{-(q+1)(1+pq)/2}B^{pq}(1/2,q/2+1)B^q(1/2,(q+1)p/2+1) \\ &\qquad \times B(1/2,((q+1)p+1)q/2+1)(S(t)S_1(t)v_0(0,x'))^{(pq)^2}t^{(q+1)(1+pq)/2}. \end{aligned}$$

Thus by induction for any integer k

$$\begin{aligned} v(0,x',t) &\ge 2^{(1/p)(pq)^k}\pi^{-(q+1)(1+pq+\cdots+(pq)^{k-1})/2}C_k \\ &\qquad \times (S(t)S_1(t)v_0(0,x'))^{(pq)^k}t^{(q+1)(1+pq+\cdots+(pq)^{k-1})/2}, \end{aligned} \tag{2.9}$$

where

$$
\begin{aligned}
C_k = {} & B^{(pq)^{k-1}}(1/2, q/2+1)B^{(1/p)(pq)^{k-1}}(1/2,(q+1)p/2+1) \\
& \times B^{(pq)^{k-2}}(1/2,((q+1)p+1)q/2+1) \\
& \times B^{(1/p)(pq)^{k-2}}(1/2,(q+1)(1+pq)p/2+1) \\
& \cdots B^{pq}(1/2,((q+1)(1+pq+\cdots+(pq)^{k-3})p+1)q/2+1) \\
& \times B^{q}(1/2,(q+1)(1+pq+\cdots+(pq)^{k-2})p/2+1) \\
& \times B(1/2,((q+1)(1+pq+\cdots+(pq)^{k-2})p+1)q/2+1).
\end{aligned}
\tag{2.10}
$$

Recalling the formula $B(a,b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$ with $\Gamma(z)$ the Gamma function satisfying

$$\Gamma(1/2) = \pi^{\frac{1}{2}}, \Gamma(z+1) = z\Gamma(z), \quad \text{and} \quad \Gamma'(z) > 0 \quad \text{for} \quad z > 3/2,$$

we then find

$$C_k \geq (4\pi)^{((1+1/p)((pq)^{k-1}+\cdots+pq)+1)/2} A_k B_k,$$

where A_k and B_k are similar to those given in (4.7a) and (4.7b) of [**EH1**], respectively. Then arguing as in the proof of Lemma 4.1 of [**EH1**], we obtain the bound in (2.4). □

We also present the counterpart of Lemma 2.1.2.

Lemma 2.1.3. *Suppose that* $0 < p < 1$ *and* $\frac{\partial v_0}{\partial x_1} \leq 0$. *Then for any* t *in the existence interval,*

$$t^{p\beta}\|S(t)(S_1(t)v_0(0,x'))^p\|_\infty \leq C. \tag{2.11}$$

Proof. By Jensen's inequality, we find

$$
\begin{aligned}
u(0,x',t) &\geq \int_0^t (\pi(t-\eta))^{-\frac{1}{2}} S(t-\eta)v^p(0,x',\eta)d\eta \\
&\geq \int_0^t (\pi(t-\eta))^{-\frac{1}{2}} S(t-\eta)(S(\eta)S_1(\eta)v_0(0,x'))^p d\eta \\
&\geq \int_0^t (\pi(t-\eta))^{-\frac{1}{2}} S(t-\eta)S(\eta)(S_1(t)v_0(0,x'))^p d\eta \\
&= 2\pi^{-\frac{1}{2}} S(t)(S_1(t)v_0(0,x'))^p t^{\frac{1}{2}},
\end{aligned}
$$

and it follows that

$$
\begin{aligned}
v(0,x',t) \geq {} & \pi^{-(1+1/p)/2-(q-1/p)((pq)^k-1)/2(pq-1)} A_k B_k \\
& \times (S(t)(S_1(t)v_0(0,x'))^p)^{(1/p)(pq)^k} t^{(q+1)((pq)^k-1)/2(pq-1)}
\end{aligned}
$$

with A_k and B_k as those in the proof of Lemma 2.1.2.

Thus proceeding as before we obtain the estimate in (2.11). □

As a consequence, we have the following:

Lemma 2.1.4. *Suppose that* $\frac{\partial v_0}{\partial x_1} \leq 0$ *and a solution of* (1.1) *exists for all* $t > 0$. *Then if* $p \geq 1$

$$t^{\beta}\|S(t)S_1(t)v(0,x',t)\|_\infty \leq C \tag{2.12a}$$

while

$$t^{p\beta}\|S(t)(S_1(t)v(0,x',t))^p\|_\infty \leq C \tag{2.12b}$$

for $0 < p < 1$.

Proof. Since $\frac{\partial v_0}{\partial x_1} \leq 0$, by the maximum principle, it follows that $\frac{\partial v}{\partial x_1} \leq 0$ for all $t > 0$. Then making use of the autonomous nature of v, we draw the conclusion from Lemmas 2.1.2 and 2.1.3. □

We are now ready to prove Lemma 2.3. Noticing Lemma 2.1.1 and the fact that the system (1.1) is autonomous, without loss of generality, we may assume that $v_0(x) \geq m\exp(-\sigma|x|^2)$ and $\frac{\partial v_0}{\partial x_1} \leq 0$.

Recall

$$S(t)\exp(-\sigma|x'|^2) = (1+4\sigma t)^{-\frac{N-1}{2}}\exp\left(-\frac{\sigma|x'|^2}{1+4\sigma t}\right), \tag{2.13}$$

and we find

$$v(0,x',t) \geq S(t)S_1(t)v_0(0,x') \geq m(1+4\sigma t)^{-\frac{N}{2}}\exp\left(-\frac{\sigma|x'|^2}{1+4\sigma t}\right). \tag{2.14}$$

We first consider the case $0 < p < 1$. Combining (2.14) and (2.5), we have

$$\begin{aligned} u(0,x',t) &\geq m^p\int_0^t (\pi(t-\eta))^{-\frac{1}{2}}(1+4\sigma\eta)^{-\frac{pN}{2}}S(t-\eta)\exp\left(-\frac{p\sigma|x'|^2}{1+4\sigma\eta}\right)d\eta \\ &\geq 2m^p\left(\frac{\sigma}{\pi}\right)^{\frac{1}{2}}(1+4\sigma t)^{-\frac{N}{2}}\exp\left(-\frac{p\sigma|x'|^2}{1+4p\sigma t}\right) \\ &\quad\times\int_0^t (1+4\sigma\eta)^{\frac{N(1-p)-1}{2}}d\eta \\ &\geq \frac{m^p(\sigma\pi)^{-\frac{1}{2}}}{N(1-p)+1}(1+4\sigma t)^{-\frac{N}{2}}(4\sigma t)^{\frac{N(1-p)+1}{2}}\exp\left(-\frac{p\sigma|x'|^2}{1+4p\sigma t}\right). \end{aligned} \tag{2.15}$$

Thus by (2.2b)

$$\begin{aligned} v(x_1,x',t) &\geq \int_0^t (\pi(t-\eta))^{-\frac{1}{2}}\exp\left(-\frac{x_1^2}{4(t-\eta)}\right)S(t-\eta)u^q(0,x',\eta)\,d\eta \\ &\geq c(1+4pq\sigma t)^{-\frac{N-1}{2}}\exp\left(-\frac{pq\sigma|x'|^2}{1+4p\sigma t}\right) \\ &\quad\times\int_0^t (1+4\sigma\eta)^{-\frac{qN}{2}}(1+4p\sigma\eta)^{\frac{N-1}{2}} \\ &\quad\times(4\sigma\eta)^{(N(1-p)+1)q/2}(t-\eta)^{-\frac{1}{2}}\exp\left(-\frac{x_1^2}{4(t-\eta)}\right)d\eta, \end{aligned} \tag{2.16}$$

with $c = m^{pq}\sigma^{-q/2}\pi^{-(q+1)/2}(N(1-p)+1)^{-q}$, and we find

$$
\begin{aligned}
S_1(t)v(0,x',t) \geq {}& c(1+4pq\sigma t)^{-\frac{N-1}{2}} \exp\left(-\frac{pq\sigma|x'|^2}{1+4p\sigma t}\right) \\
& \times \int_0^t (1+4\sigma\eta)^{-\frac{qN}{2}} (1+4p\sigma\eta)^{\frac{N-1}{2}} (4\sigma\eta)^{(N(1-p)+1)q/2} \\
& \times \int_0^\infty (\pi t(t-\eta))^{-\frac{1}{2}} \exp\left(-\frac{y_1^2}{4t} - \frac{y_1^2}{4(t-\eta)}\right) dy_1\, d\eta \\
= {}& \frac{c}{2}(1+4pq\sigma t)^{-\frac{N-1}{2}} \exp\left(-\frac{pq\sigma|x'|^2}{1+4p\sigma t}\right) \int_0^t (2t-\eta)^{-\frac{1}{2}} \\
& \times (1+4\sigma\eta)^{-\frac{qN}{2}} (1+4p\sigma\eta)^{\frac{N-1}{2}} (4\sigma\eta)^{(N(1-p)+1)q/2}\, d\eta.
\end{aligned} \tag{2.17}
$$

Note that $1+4p\sigma\eta > p(1+4\sigma\eta)$ and $4\sigma\eta > (1+4\sigma\eta)/2$ for $\eta > 1/(4\sigma)$ and that $(1+4pq\sigma t)^{-\frac{N-1}{2}} \geq (pq)^{-\frac{N-1}{2}}(1+4\sigma t)^{-\frac{N-1}{2}}$ and $(2t-\eta)^{-\frac{1}{2}} \geq (2\sigma)^{\frac{1}{2}}(1+4\sigma t)^{-\frac{1}{2}}$, we then have

$$
S_1(t)v(0,x',t) \geq c_1(1+4\sigma t)^{-\frac{N}{2}} \exp\left(-\frac{pq\sigma|x'|^2}{1+4p\sigma t}\right) \int_{\frac{1}{4\sigma}}^t (1+4\sigma\eta)^{\lambda}\, d\eta,
$$

where

$$
\lambda = -\frac{qN}{2} + \frac{N-1}{2} + \frac{(N(1-p)+1)q}{2} = \frac{1}{2}((q+1)-(pq-1)N) - 1 \geq -1,
$$

since $pq \leq 1 + (\max(p,q)+1)/N$. Hence

$$
\begin{aligned}
S_1(t)v(0,x',t) &\geq c_1(1+4\sigma t)^{-\frac{N}{2}} \exp\left(-\frac{pq\sigma|x'|^2}{1+4p\sigma t}\right) \int_{\frac{1}{4\sigma}}^t (1+4\sigma\eta)^{-1}\, d\eta \\
&= \frac{c_1}{4\sigma}(1+4\sigma t)^{-\frac{N}{2}} \exp\left(-\frac{pq\sigma|x'|^2}{1+4p\sigma t}\right) \log\left(\frac{1+4\sigma t}{2}\right)
\end{aligned}
$$

whenever $t > 1/(4\sigma)$. In view of (2.13), we observe

$$
\begin{aligned}
S(t)(S_1(t)v(0,x',t))^p \geq {}& c_2(1+4\sigma t)^{-\frac{pN}{2}} \log^p\left(\frac{1+4\sigma t}{2}\right) \left(1+\frac{4p^2q\sigma t}{1+4p\sigma t}\right)^{-\frac{N-1}{2}} \\
& \times \exp\left(-\frac{p^2q\sigma|x'|^2}{1+4p\sigma(1+pq)t}\right).
\end{aligned}
$$

In particular, setting $x'=0$, we find

$$
(1+4\sigma t)^{\frac{pN}{2}} S(t)(S_1(t)v(0,t))^p \geq c_2(1+pq)^{-\frac{N-1}{2}} \log^p\left(\frac{1+4\sigma t}{2}\right), \tag{2.18}
$$

which would contradict (2.12b) if the solution of (1.1) is global.

Next for the case $p \geq 1$, after conducting a similar discussion, we finally reach

$$S_1(t)v(0,x',t) \geq c_3(1+4\sigma t)^{-\frac{N}{2}} \exp\left(-\frac{pq\sigma|x'|^2}{1+4\sigma t}\right) \log\left(\frac{1+4\sigma t}{2}\right).$$

Hence

$$S(t)S_1(t)v(0,x',t) \geq c_3(1+4\sigma t)^{-\frac{N}{2}} \log\left(\frac{1+4\sigma t}{2}\right)\left(1+\frac{4pq\sigma t}{1+4\sigma t}\right)^{-\frac{N-1}{2}} \times \exp\left(-\frac{pq\sigma|x'|^2}{1+4\sigma(1+pq)t}\right),$$

and consequently,

$$(1+4\sigma t)^{\frac{N}{2}} S(t)S_1(t)v(0,t) \geq c_3(1+pq)^{-\frac{N-1}{2}} \log\left(\frac{1+4\sigma t}{2}\right), \tag{2.19}$$

which, taking (2.12a) into account, does not permit the global existence of solutions of (1.1). The proof is then completed.

2.2 The Case $\max(\alpha,\beta) < \frac{N}{2}$

We begin this section by showing the global existence of solutions to (1.1) with small initial data. We shall use a modification of an argument in [**GL**]. To this end, we look for a supersolution of the self-similar type:

$$\bar{u}(x,t) = (t_0+t)^{-\alpha} f(\zeta), \quad \bar{v}(x,t) = (t_0+t)^{-\beta} g(\zeta) \tag{2.20}$$

where

$$\zeta = (\zeta_1, \zeta') \quad \text{with} \quad \zeta' = \frac{x'}{(t_0+t)^{1/2}}, \quad \zeta_1 = \frac{x_1}{(t_0+t)^{1/2}}, \tag{2.21}$$

here $t_0 > 0$ is a constant. As a supersolution, (f,g) must satisfy

$$\Delta f + \frac{1}{2}\zeta \cdot \nabla f + \alpha f \leq 0, \quad \Delta g + \frac{1}{2}\zeta \cdot \nabla g + \beta g \leq 0 \tag{2.22}$$

and

$$-\frac{\partial f}{\partial \zeta_1} \geq g^p, \quad -\frac{\partial g}{\partial \zeta_1} \geq f^q \quad \text{at} \quad \zeta_1 = 0. \tag{2.23}$$

Consider two cases.

Case 1: $p \geq 1$. Let

$$f(\zeta) = Ae^{-\sigma(|\zeta'|^2+(\zeta_1+\delta)^2)} \quad \text{and} \quad g(\zeta) = Be^{-\sigma(|\zeta'|^2+(\zeta_1+\delta)^2)},$$

where A, B, σ, and δ are positive constants. Then (2.22) is satisfied if

$$(\beta - 2N\sigma + 4\sigma^2\delta^2) + \sigma\delta(8\sigma - 1)\zeta_1 + \sigma(4\sigma - 1)|\zeta|^2 \leq 0 \tag{2.24}$$

for $\zeta \in \mathbb{R}^N_+$, and (2.23) is equivalent to

$$2\sigma\delta Ae^{\sigma(p-1)(|\zeta'|^2+\delta^2)} \geq B^p \quad \text{and} \quad 2\sigma\delta Be^{\sigma(q-1)(|\zeta'|^2+\delta^2)} \geq A^q. \tag{2.25}$$

One can see that (2.24) is valid for sufficiently small δ if

$$\beta - 2N\sigma < 0 \quad \text{and} \quad 4\sigma - 1 < 0,$$

i.e., if $\beta < N/2$.

Then letting $B = (2\sigma\delta A)^{1/p}e^{(p-1)\sigma\delta^2}$, we find that (2.25) holds if A is small enough to assure $(2\sigma\delta)^{p+1}e^{(pq-1)\sigma\delta^2} \geq A^{pq-1}$.

Case 2: $0 < p < 1$. Set

$$f(\zeta) = Ae^{-\sigma_1(|\zeta'|^2+(\zeta_1+\delta)^2)} \quad \text{and} \quad g(\zeta) = Be^{-\sigma_2(|\zeta'|^2+(\zeta_1+\delta)^2)}$$

where σ_1 and σ_2 are positive constants with $\sigma_1 = p\sigma_2$. Then (2.22) becomes

$$(\alpha - 2N\sigma_1 + 4\sigma_1^2\delta^2) + \sigma_1\delta(8\sigma_1 - 1)\zeta_1 + \sigma_1(4\sigma_1 - 1)|\zeta|^2 \leq 0 \tag{2.26a}$$

and

$$(\beta - 2N\sigma_2 + 4\sigma_2^2\delta^2) + \sigma_2\delta(8\sigma_2 - 1)\zeta_1 + \sigma_2(4\sigma_2 - 1)|\zeta|^2 \leq 0 \tag{2.26b}$$

for $\zeta \in \mathbb{R}^N_+$, and (2.23) is valid if

$$2\sigma_1\delta Ae^{-\sigma_1(|\zeta'|^2+\delta^2)} \geq B^pe^{-p\sigma_2(|\zeta'|^2+\delta^2)} \tag{2.27a}$$

and

$$2\sigma_2\delta Be^{-\sigma_2(\delta^2+|\zeta'|^2)} \geq A^qe^{-q\sigma_1(\delta^2+|\zeta'|^2)}. \tag{2.27b}$$

It is easy to check that (2.26b) holds for any $0 < \delta \ll 1$ if

$$\beta - 2N\sigma_2 < 0 \quad \text{and} \quad 4\sigma_2 - 1 < 0,$$

which again implies that $\beta < N/2$. Then (2.26a) is also true, since $(p+1)/2(pq-1) < (p+pq)/2(pq-1) < 2Np\sigma_2 = 2N\sigma_1$.

To ensure the validity of equations (2.27), we choose $B = (2\sigma_1\delta A)^{1/p}$ and then make A sufficiently small such that $p^{-p}(2\sigma_1\delta)^{p+1}e^{(pq-1)\sigma_1\delta^2} \geq A^{pq-1}$.

We now turn our attention to the blow up of solutions of (1.1) for large initial data. We shall discuss it in a similar manner as that in [**EH1**]. First consider the case $0 < p < 1$.

Using (2.2) and Jensen's inequality, we find

$$
\begin{aligned}
u(0,x',t) &\geq S(t)S_1(t)u_0(0,x') + \pi^{-\frac{p+1}{2}} \int_0^t (t-\eta)^{-\frac{1}{2}} S(t-\eta) \\
&\qquad \times \left(\int_0^\eta (\eta-\tau)^{-\frac{1}{2}} S(\eta-\tau) u^q(0,x',\tau) d\tau \right)^p d\eta \\
&\geq S(t)S_1(t)u_0(0,x') + 2^{p-1}\pi^{-\frac{p+1}{2}} \int_0^t (t-\eta)^{-\frac{1}{2}} \eta^{\frac{p-1}{2}} \\
&\qquad \times \int_0^\eta (\eta-\tau)^{-\frac{1}{2}} S(t-\tau) u^{pq}(0,x',\tau)\, d\tau\, d\eta \\
&\geq S(t)S_1(t)u_0(0,x') + 2^{p-1}\pi^{-\frac{p+1}{2}} t^{\frac{p-1}{2}} \int_0^t \int_\tau^t ((t-\eta)(\eta-\tau))^{-\frac{1}{2}} \\
&\qquad \times S(t-\tau) u^{pq}(0,x',\tau)\, d\eta\, d\tau \\
&= S(t)S_1(t)u_0(0,x') + 2^{p-1}\pi^{\frac{1-p}{2}} t^{\frac{p-1}{2}} \int_0^t S(t-\tau) u^{pq}(0,x',\tau)\, d\tau.
\end{aligned}
\tag{2.28}
$$

Suppose that $u_0(x) \geq Ce^{-\sigma|x|^2}$ with arbitrary $\sigma > 0$ and undetermined $C \geq 1$. Then by (2.13) we have

$$
\begin{aligned}
u(0,x',t) &\geq S(t)S_1(t)u_0(0,x') \\
&\geq C(1+4\sigma t)^{-\frac{N}{2}} \exp\left(-\frac{\sigma|x'|^2}{1+4\sigma t} \right) \equiv I_0(x',t).
\end{aligned}
\tag{2.29}
$$

Define

$$
I_1(x',t) = \frac{t^{\frac{p-1}{2}}}{2} \int_0^t S(t-\tau) I_0^\mu(x',\tau)\, d\tau,
$$

where $\mu = pq$.

From (2.28) and (2.29), we then obtain

$$
\begin{aligned}
u(0,x',t) &\geq I_0(x',t) + \frac{C^\mu}{2} t^{\frac{p-1}{2}} \int_0^t S(t-\tau)(I_0+I_1)^\mu(x',\tau)\, d\tau \\
&\geq I_0(x',t) + I_1(x',t) + \frac{t^{\frac{p-1}{2}}}{2} \int_0^t S(t-\tau) I_1^\mu(x',\tau)\, d\tau.
\end{aligned}
$$

Thus setting

$$
I_{k+1}(x',t) = \frac{t^{\frac{p-1}{2}}}{2} \int_0^t S(t-\tau) I_k^\mu(x',\tau)\, d\tau
\tag{2.30}
$$

for $k = 0, 1, 2, \ldots$, by induction we find

$$u(0, x', t) \geq \sum_{k=0}^{m} I_k(x', t) \tag{2.31}$$

for any integer $m > 0$.

Furthermore, for $k = 1, 2, \ldots$, $I_k(x', t)$ satisfies

$$\begin{aligned} I_k(x', t) \geq {} & C^{\mu^k} 2^{-\mu^{k-1}} \mu^{-(N-1)(\mu^{k-1}+2\mu^{k-2}+\cdots+(k-1)\mu+k)/2} \\ & \times t^{(p+1)(\mu^{k-1}+\mu^{k-2}+\cdots+\mu+1)/2} (1+4\sigma t)^{-(N/2)\mu^k} \exp\left(-\frac{\mu^k \sigma |x'|^2}{1+4\sigma t}\right) D_k, \end{aligned}$$

where

$$\begin{aligned} D_k = {} & \left(\frac{1}{(p+1)\mu+2}\right)^{\mu^{k-2}} \left(\frac{1}{(p+1)(\mu+1)\mu+2}\right)^{\mu^{k-3}} \\ & \cdots \left(\frac{1}{(p+1)(\mu^{k-2}+\mu^{k-3}+\cdots+1)\mu+2}\right). \end{aligned}$$

Then proceeding as in the proof of Theorem 4 of [**EH1**], we conclude that if C is large enough, then there exists a $T > 0$ such that $I_k(0, T) > C(1+4\sigma T)^{-N/2}$ for $k = 1, 2, \cdots$, and consequently $u(0, t) \to \infty$ as $t \to T^-$.

Next for the case $p \geq 1$, we have

$$\begin{aligned} u(0, x', t) \geq {} & S(t) S_1(t) u_0(0, x') + \pi^{-\frac{p+1}{2}} \int_0^t (t-\eta)^{-\frac{1}{2}} \\ & \times \left(\int_0^\eta (\eta-\tau)^{-\frac{1}{2}} S(t-\tau) u^q(0, x', \tau) d\tau\right)^p d\eta \\ \geq {} & S(t) S_1(t) u_0(0, x') + \pi^{-\frac{p+1}{2}} (2t)^{\frac{1-p}{2}} \\ & \times \left(\int_0^t \int_\tau^t ((t-\eta)(\eta-\tau))^{-\frac{1}{2}} S(t-\tau) u^q(0, x', \tau)\, d\eta\, d\tau\right)^p \\ \geq {} & S(t) S_1(t) u_0(0, x') + \left(\frac{\pi}{2}\right)^{\frac{p-1}{2}} t^{(3-p-2q)/2} \left(\int_0^t S(t-\tau) u(0, x', \tau)\, d\tau\right)^{pq}. \end{aligned}$$

Then by estimating

$$I_{k+1}(x', t) = t^{(3-p-2q)/2} \left(\int_0^t S(t-\tau) I_k(x', \tau) d\tau\right)^{pq}$$

for $k = 0, 1, 2, \ldots$, with $I_0(x', t)$ as that in (2.29), we can show that the solution of (1.1) must blow up in a finite time.

3. Blow Up and Decay Rates in One Space Dimension

In this section we study the one dimensional problem:

$$u_t = u_{xx}, \qquad v_t = v_{xx} \qquad x > 0, t > 0, \tag{3.1}$$

$$-u_x(0,t) = v^p(0,t), \qquad -v_x(0,t) = u^q(0,t) \quad t > 0, \tag{3.2}$$

$$u(\cdot,0) = u_0 \geq 0, \qquad v(\cdot,0) = v_0 \geq 0. \tag{3.3}$$

We first discuss the decay rate in x for global solutions when $\max(\alpha,\beta) < \frac{1}{2}$. We then establish the blow up rate for a suitable class of solutions which blow up in a finite time.

We consider some explicit self-similar solutions of (3.1) which blow up in finite time T. They are of the form

$$u(x,t) = (T-t)^{-\alpha} f_-(\xi), \qquad v(x,t) = (T-t)^{-\beta} g_-(\xi)$$

where we take

$$\xi = \frac{x}{\sqrt{T-t}}.$$

The functions f_-, g_- satisfy

$$\begin{aligned} &f_-''(\xi) - \frac{\xi}{2} f_-'(\xi) - \alpha f_-(\xi) = 0, \\ &g_-''(\xi) - \frac{\xi}{2} g_-'(\xi) - \beta g_-(\xi) = 0 \ \text{ for } \ \xi > 0, \end{aligned} \tag{3.4}$$

$$\begin{aligned} -f_-'(0) &= g_-^p(0), \\ -g_-'(0) &= f_-^q(0). \end{aligned} \tag{3.5}$$

Lemma 3.1. *Assume $pq > 1$. Then for any $T > 0$, there is a unique self-similar solution of* (3.1) *which blows up at time T and stays bounded as $x \to +\infty$ for $t \in [0,T)$. This solution has the following properties*

(i) $u_t > 0, v_t > 0$ *in* $[0,\infty) \times (0,T)$.

(ii) $u(x,T) = k_1 x^{-2\alpha}$, $v(x,T) = k_2 x^{-2\beta}$ *where*

$$k_1 = \pi^{-\frac{1}{2}} \left(\frac{\beta \Gamma(\beta + \frac{1}{2})}{\Gamma(\beta+1)} \right)^{\frac{p}{pq-1}} \left(\frac{\alpha \Gamma^{pq}(\alpha + \frac{1}{2})}{\Gamma(\alpha+1)} \right)^{\frac{1}{pq-1}}$$

and k_2 is obtained from k_1 by the interchange of α, β and p, q.

(iii) $x^{2\alpha} u(x,t) \to k_1$ *and* $x^{2\beta} v(x,t) \to k_2$ *as* $x \to \infty$ *for* $0 \leq t < T$.

Proof. As in [**FQ**, Lemma 3.1] one can show that the system (3.4), (3.5) has a unique bounded solution given by

$$f_-(\xi) = k_1 U(\alpha, \frac{1}{2}, \frac{\xi^2}{4}),$$
$$g_-(\xi) = k_2 U(\beta, \frac{1}{2}, \frac{\xi^2}{4})$$

where

$$U(a, b, r) = \frac{1}{\Gamma(a)} \int_0^\infty e^{-rt} t^{a-1} (1+t)^{b-a-1}\, dt.$$

To prove (i), we see that

$$\begin{aligned} u_t &= (T-t)^{-\alpha-1}(\alpha f_- + \frac{1}{2}\xi f_-') \\ &= k_1 (T-t)^{-\alpha-1}(\alpha U + \frac{1}{4}\xi^2 U_3). \end{aligned}$$

From this and the identity (cf. [**AS**])

$$a(1+a-b)U(a+1, b, r) = aU(a, b, r) + rU_r(a, b, r)$$

we easily see that $u_t > 0$. The remaining assertions (ii) and (iii) follow from the fact that

$$U(a, b, r) = r^{-a}[1 + O(r^{-1})] \quad \text{as} \quad r \to \infty. \tag{3.6}$$

□

Next we demonstrate the existence of positive global self-similar solutions of (3.1), (3.2) when $\max(\alpha, \beta) < \frac{1}{2}$. These take the form

$$u(x, t) = (T_0 + t)^{-\alpha} f_+(\zeta), \quad v(x, t) = (T_0 + t)^{-\beta} g_+(\zeta)$$

where we take

$$\zeta = \frac{x}{\sqrt{T_0 + t}}.$$

The functions f_+, g_+ satisfy

$$\begin{aligned} & f_+''(\zeta) + \frac{\zeta}{2} f_+'(\zeta) + \alpha f_+(\zeta) = 0, \\ & g_+''(\zeta) + \frac{\zeta}{2} g_+'(\zeta) + \beta g_+(\zeta) = 0 \quad \text{for } \zeta > 0, \end{aligned} \tag{3.7}$$

$$\begin{aligned} -f_+'(0) &= g_+^p(0), \\ -g_+'(0) &= f_+^q(0). \end{aligned} \tag{3.8}$$

Lemma 3.2. *Assume $pq > 1$ and $\max(\alpha, \beta) < \frac{1}{2}$. Let $T_0 > 0$ be fixed. Then*

(i) *There is a unique positive global self-similar solution of (3.1), (3.2), both of whose components decay in x like Gaussians for large x.*

(ii) *There is a one parameter family of positive global self-similar solutions of (3.1), (3.2) such that u decays like $x^{-2\alpha}$ and v decays like a Gaussian for large x.*

(iii) *There is a one parameter family of positive global self-similar solutions of (3.1), (3.2) such that v decays like $x^{-2\beta}$ and u decays like a Gaussian for large x.*

(iv) *There is a two parameter family of positive global self-similar solutions of (3.1), (3.2) such that u decays like $x^{-2\alpha}$ and v decays like $x^{-2\beta}$ for large x.*

Proof. The general solution of (3.7) is given by

$$f_+(\zeta) = e^{-\frac{\zeta^2}{4}}\left(c_1 U(\frac{1}{2}-\alpha, \frac{1}{2}, \frac{\zeta^2}{4}) + c_2 M(\frac{1}{2}-\alpha, \frac{1}{2}, \frac{\zeta^2}{4})\right),$$

$$g_+(\zeta) = e^{-\frac{\zeta^2}{4}}\left(d_1 U(\frac{1}{2}-\beta, \frac{1}{2}, \frac{\zeta^2}{4}) + d_2 M(\frac{1}{2}-\beta, \frac{1}{2}, \frac{\zeta^2}{4})\right)$$

where

$$M(a,b,r) = 1 + \frac{ar}{b} + \cdots + \frac{a(a+1)...(a+n-1)}{b(b+1)...(b+n-1)}r^n + ...$$

and

$$M(a,b,r) = \frac{\Gamma(a)}{\Gamma(b)}e^r r^{a-b}[1+O(r^{-1})] \quad \text{as} \quad r \to +\infty \tag{3.9}$$

In order to satisfy (3.8) we must have $c_1 > 0,\ d_1 > 0$ since

$$f_+'(0) = -c_1(\frac{1}{2}-\alpha)\frac{\sqrt{\pi}}{\Gamma(\frac{3}{2}-\alpha)},$$

$$g_+'(0) = -d_1(\frac{1}{2}-\beta)\frac{\sqrt{\pi}}{\Gamma(\frac{3}{2}-\beta)}.$$

From the definitions of U, M and equations (3.6), (3.9) we observe that f_+, g_+ will remain positive if and only if $c_2 \geq 0$, $d_2 \geq 0$.

If $c_2 = d_2 = 0$, then there is a unique pair (c_1, d_1) such that (3.8) is satisfied and this corresponds to the unique rapidly decaying solution claimed in (i).

If $d_2 = 0$, then there is a $c^* > 0$ such that for $c_2 = c^*$, there is a unique pair (c_1, d_1) such that (3.8) is satisfied while if $c_2 < c^*$, there are two such pairs. This establishes (ii). Claim (iii) follows by a similar argument if $c_2 = 0$.

In order to establish the existence of the slowly decaying solutions claimed in (iv) we observe that for any $c_2 > 0$, $d_2 > 0$, sufficiently small, there are c_1, d_1 such that (3.8) holds. □

From these lemmas we obtain

Theorem 3.3. *Assume* $pq > 1$ *and* $\max(\alpha, \beta) < \frac{1}{2}$.

(i) *If* u_0, $v_0 > 0$ *and*

$$\liminf_{x\to\infty} x^{2\alpha} u_0(x) \geq k_1 \quad \text{and} \quad \liminf_{x\to\infty} x^{2\beta} v_0(x) \geq k_2$$

where the k_i *are as in Lemma* (3.1)*, then the solution* (u, v) *of* (3.1)–(3.3) *blows up in finite time.*

(ii) *There are global solutions* (u, v) *such that*

$$\lim_{x\to\infty} x^{2\alpha} u(x,t) \quad \text{and} \quad \lim_{x\to\infty} x^{2\beta} v(x,t)$$

exist and are positive for all $t > 0$.

Proof. The first statement follows from Lemma 3.1 by comparison with the self-similar solution for sufficiently large T. The second statement follows from Lemma 3.2(iv). □

Remark 3.1. If we consider the scalar problem

$$\begin{aligned} u_t &= u_{xx} && x > 0, \quad t > 0, \\ -u_x(0,t) &= u^p(0,t) && t > 0, \\ u(x,0) &= u_0(x) \geq 0 && x \geq 0, \end{aligned}$$

then self-similar solutions give a precise characterization of the domain of attraction of zero if $p > 2$. Namely, u is global and decays to zero as $t \to \infty$ only if

$$u_0(x) = O(x^{-\frac{1}{p-1}}) \quad \text{as} \quad x \to \infty.$$

On the other hand, there are global solutions such that

$$\lim_{x\to\infty} x^{\frac{1}{p-1}} u(x,t)$$

exists and is positive for all $t > 0$.

For the system (3.1)–(3.3) we do not know whether or not a solution can be global if for example $u_0(x)$ behaves like $x^{-2\alpha-\varepsilon}$ while $v_0(x)$ behaves like $x^{-2\beta+\varepsilon}$.

Next we show that there is a class of solutions that blow up at the same rate in t as the self-similar solutions of Lemma 3.1.

Theorem 3.4. *Assume* $pq > 1$ *and* $\min(p, q) \geq 1$. *Assume that* u_0, $v_0 \in C^3$ *and that*

$$-u_0'(0) = v_0^p(0), \qquad -v_0'(0) = u_0^q(0), \tag{3.10}$$

$$-u_0'''(0) = pv^{p-1}(0)v_0''(0), \qquad -v_0'''(0) = qu^{q-1}(0)u_0''(0), \tag{3.11}$$

$$(-1)^i u_0^{(i)} \geq 0, \qquad (-1)^i v_0^{(i)} \geq 0, \quad i = 0, 1, 2, 3, \ x > 0, \tag{3.12}$$

$$\lim_{x\to\infty} u_0(x) = 0, \qquad \lim_{x\to\infty} v_0(x) = 0, \tag{3.13}$$

$$u_0' \leq -v_0^p, \qquad v_0' \leq -u_0^q, \quad x > 0. \tag{3.14}$$

Then the solution blows up in finite time $T > 0$ and for $t \in (0,T)$, we have

$$c_1 \le (T-t)^\alpha u(0,t) \le c_2, \tag{3.15}$$

$$c_3 \le (T-t)^\beta v(0,t) \le c_4 \tag{3.16}$$

where

$$c_1 = \left[\alpha p(2p)^{\frac{p-1}{p+1}}\right]^{-\alpha}, \qquad c_2 = \left(\frac{\alpha}{2}\right)^{-\alpha}(2q)^{\frac{p}{pq-1}},$$
$$c_3 = \left[\beta q(2p)^{\frac{q-1}{q+1}}\right]^{-\beta}, \qquad c_4 = \left(\frac{\beta}{2}\right)^{-\beta}(2p)^{\frac{q}{pq-1}}.$$

Proof. From (3.13) it follows that

$$\lim_{x\to\infty} u(x,t) = \lim_{x\to\infty} v(x,t) = 0. \tag{3.17}$$

By the maximum principle, it follows from (3.10)–(3.12) that

$$u,\ u_t,\ v,\ v_t \ge 0 \ \text{ and } \quad u_x,\ u_{xt},\ v_x,\ v_{xt} \le 0$$

for $t \in (0,T)$, T being the time of existence.

We now exploit an idea from [**FQ**]:

$$\begin{aligned}\frac{1}{2}v^{2p}(0,t) = \frac{1}{2}u_x^2(0,t) &= -\int_0^\infty u_{xx}(x,t)u_x(x,t)\,dx\\ &= -\int_0^\infty u_t(x,t)u_x(x,t)\,dx\\ &= -\lim_{x\to\infty} u_t(x,t)u(x,t) + u_t(0,t)u(0,t) + \int_0^\infty u_{xt}(x,t)u(x,t)\,dx.\end{aligned}$$

Hence on $(0,T)$

$$\frac{1}{2}v^{2p}(0,t) \le u_t(0,t)u(0,t) \tag{3.18}$$

and analogously

$$\frac{1}{2}u^{2q}(0,t) \le v_t(0,t)v(0,t). \tag{3.19}$$

Next we derive upper bounds for $u_t(0,t)$, $v_t(0,t)$. As in [**B**] for example, we define on $[0,\infty)\times[0,T)$

$$J(x,t) = u_x(x,t) + v^p(x,t), \qquad K(x,t) = v_x(x,t) + u^q(x,t).$$

The assumption (3.14) insures that

$$J(x,0) \leq 0, \qquad K(x,0) \leq 0.$$

A routine calculation yields

$$\begin{aligned} J_t - J_{xx} &= -p(p-1)v^{p-2}v_x^2 \leq 0, \\ K_t - K_{xx} &= -q(q-1)u^{q-2}u_x^2 \leq 0. \end{aligned}$$

Here we used the assumption that $\min(p,q) \geq 1$. From (3.17) and the additional observation that $u_x(x,t)$, $v_x(x,t) \to 0$ as $x \to \infty$, we have

$$\lim_{x\to\infty} J(x,t) = \lim_{x\to\infty} K(x,t) = 0\,.$$

Clearly

$$J(0,t) = K(0,t) = 0\,.$$

Thus, by the maximum principle $J \leq 0$, $K \leq 0$ in $[0,\infty) \times [0,T)$. Therefore, for every $t \in (0,T)$, the functions $J(\cdot,t)$, $K(\cdot,t)$ attain their maximum at $x = 0$. Consequently $J_x(0,t)$, $K_x(0,t) \leq 0$. Writing out these two inequalities, we see that on $(0,T)$

$$u_t(0,t) \leq pv^{p-1}(0,t)u^q(0,t), \tag{3.20}$$

$$v_t(0,t) \leq qu^{q-1}(0,t)v^p(0,t). \tag{3.21}$$

Combining (3.18) with (3.20) we obtain $\frac{1}{2}v^{2p}u^{-1} \leq pv^{p-1}u^q$ or

$$u \geq (2p)^{-\frac{1}{q+1}} v^{\frac{p+1}{q+1}} \tag{3.22}$$

on $(0,T)$. (Here and in the rest of the proof the argument $(0,t)$ is omitted). Using (3.22) in (3.19) we have

$$v_t \geq \frac{1}{2}(2p)^{-\frac{2q}{2q+1}} v^{1+\frac{1}{\beta}} \tag{3.23}$$

which implies that T is finite. Writing (3.23) in the form

$$-\beta(v^{-\frac{1}{\beta}})_t \geq \frac{1}{2}(2p)^{-\frac{2q}{2q+1}}$$

and integrating over $[t,T)$ we obtain the second inequality in (3.16).

Combining (3.19) and (3.21) we obtain

$$v \geq (2q)^{-\frac{1}{p+1}} u^{\frac{q+1}{p+1}}. \tag{3.24}$$

From the inequalities (3.22) and (3.24), we see that u, v blow up in the same finite time T. Using (3.24) in (3.18) and integrating, we obtain the second inequality in (3.15).

Combining (3.20) and (3.22) we have

$$u_t \le p(2p)^{\frac{p-1}{p+1}} u^{1+\frac{1}{\alpha}}$$

or

$$-\alpha(u^{-\frac{1}{\alpha}})_t \le p(2p)^{\frac{p-1}{p+1}}.$$

Integrating this over $[t, T)$ gives the lower bound in (3.15). The lower bound in (3.16) can be derived analogously using (3.21) and (3.24). □

Remark 3.2. The assumptions of Theorem 3.4 are satisfied if

$$u_0(x) = \rho^{-1}\sigma^{-p}a^{p+1+\rho(1-pq)}(x+a)^{-\rho},$$
$$v_0(x) = \sigma^{-1}\rho^{-q}a^{q+1+\sigma(1-pq)}(x+a)^{-\sigma}$$

where $a > 0$ is arbitrary and where ρ, σ solve the system

$$(\rho+1)(\rho+2) = p\sigma(\sigma+1), \tag{3.25}$$
$$(\sigma+1)(\sigma+2) = q\rho(\rho+1) \tag{3.26}$$

and satisfy the inequalities

$$\rho > 2\alpha, \quad \sigma > 2\beta.$$

We show that such ρ, σ exist if p, q are as in Theorem 3.4. We consider the cases (a) $1 = p < q$, (b) $1 < p = q$ and (c) $1 < p < q$.

(a) We see that (3.25) holds if $\rho = \sigma - 1$. But then (3.26) is a quadratic in σ with larger root

$$\sigma \ge \frac{q+3}{q-1} > 2\beta$$

and

$$\rho = \sigma - 1 \ge \frac{4}{q-1} > 2\alpha.$$

(b) If $p = q$, then $\phi(\rho) = \phi(\sigma)$ where $\phi(\rho) = \rho(\rho+1)^2(\rho+2)$. Therefore $\rho = \sigma$ if both are positive. The system then reduces to a single equation and we have

$$\rho = \sigma = \frac{2}{p-1} > 2\alpha = 2\beta = \frac{1}{p-1}.$$

(c) We will be done if we construct families of mappings F_τ and domains $\Omega_\tau \subset \mathbb{R}^2$, $1 \le \tau \le \tau_0$, such that

$$d(F_1, \Omega_1, 0) = -1 \quad \text{(Here } d \text{ is the Brouwer degree.)}, \tag{3.27}$$
$$F_\tau(\rho,\sigma) \ne 0 \quad \text{if } (\rho,\sigma) \in \partial\Omega_\tau, \tag{3.28}$$
$$F_{\tau_0}(\rho,\sigma) = 0 \quad \text{is equivalent to (3.25), (3.26)}, \tag{3.29}$$
$$\Omega_{\tau_0} \subset (2\alpha,\infty)\times(2\beta,\infty). \tag{3.30}$$

In order to do this set $r = \sqrt{pq}$, $\tau_0 = \sqrt{\frac{q}{p}}$,

$$F_\tau(\rho,\sigma) = \begin{pmatrix} f_\tau(\rho,\sigma) \\ g_\tau(\rho,\sigma) \end{pmatrix}$$

for $1 \le \tau \le \tau_0$,

$$\begin{aligned} f_\tau(\rho,\sigma) &= (\rho+1)(\rho+2) - \frac{r}{\tau}\sigma(\sigma+1), \\ g_\tau(\rho,\sigma) &= (\sigma+1)(\sigma+2) - r\tau\rho(\rho+1), \end{aligned}$$

$$\rho_0(\tau) = \frac{\frac{r}{\tau}+1}{r^2-1}, \quad \sigma_0(\tau) = \frac{r\tau+1}{r^2-1}, \quad R = \frac{2(r^2+1)}{r^2-1},$$

and

$$\Omega_\tau = (\rho_0(\tau), R) \times (\sigma_0(\tau), R).$$

With these definitions, (3.29), (3.30) hold.

By the argument in (b), $(\rho_1,\sigma_1) = (\frac{2}{r-1}, \frac{2}{r-1})$ is the unique root (in Ω_1) of $F_1(\rho,\sigma) = 0$. If we calculate the Jacobian $J_{F_1}(\rho_1,\sigma_1)$ we find that

$$J_{F_1}(\rho_1,\sigma_1) = -\frac{r^3+7r^2+7r+1}{r-1} < 0$$

and (3.27) follows.

In order to establish (3.28) we first show that $F_\tau(\rho,\sigma) \neq 0$ if $\rho = \rho_0(\tau)$. If $g_\tau(\rho,\sigma) = 0$, we find that

$$f_\tau(\rho,\sigma) = (1-r^2)\rho(\rho+1) + 2(\rho+1) + 2\frac{r}{\tau}(\sigma+1)$$

and hence

$$f_\tau(\rho_0(\tau),\sigma) = \frac{(\frac{r}{\tau}+r^2)(1-\frac{r}{\tau}) + 2\frac{r}{\tau}(r\tau+r^2)}{r^2-1} > 0$$

if $\sigma \ge \sigma_0(\tau)$. Analogously, if $f_\tau(\rho,\sigma_0(\tau)) = 0$, then $g_\tau(\rho,\sigma_0(\tau)) > 0$ for $\rho \ge \rho_0(\tau)$.

On the other parts of the boundary, we have:

$$\begin{aligned} f_\tau(R,\sigma) &< -2r(r-\tau^{-1})(R+1) < 0 \quad \text{if } \sigma \le R \quad \text{and} \quad g_\tau(R,\sigma) = 0, \\ g_\tau(\rho,R) &< -2r(r-\tau^{-1})(R+1) < 0 \quad \text{if } \rho \le R \quad \text{and} \quad f_\tau(\rho,R) = 0. \end{aligned}$$

Remark 3.3. The inequalities (3.15), (3.16) give upper and lower bounds for T in terms of $u_0(0)$, $v_0(0)$.

Finally we construct a nontrivial solution with zero initial data if $pq < 1$.

Theorem 3.5. *If $pq < 1$ then problem* (3.1)–(3.3) *with $u_0 \equiv v_0 \equiv 0$ has a nontrivial, nonnegative solution.*

Proof. If $pq < 1$ then $\max(\alpha, \beta) < 0$. We construct a self-similar solution of the form

$$u(x,t) = t^{-\alpha} f_+(\zeta), \quad v(x,t) = t^{-\beta} g_+(\zeta) \quad \text{for} \quad \zeta = \frac{x}{\sqrt{t}}$$

where (f_+, g_+) is a positive solution of (3.7), (3.8) that decays to $(0,0)$ as $\zeta \to \infty$. We take

$$f_+(\zeta) = c_1 e^{-\frac{\zeta^2}{4}} U(\frac{1}{2} - \alpha, \frac{1}{2}, \frac{\zeta^2}{4}),$$
$$g_+(\zeta) = d_1 e^{-\frac{\zeta^2}{4}} U(\frac{1}{2} - \beta, \frac{1}{2}, \frac{\zeta^2}{4})$$

where

$$c_1 = \pi^{-\frac{1}{2}} \left(\frac{(\frac{1}{2} - \beta)\Gamma(1-\beta)}{\Gamma(\frac{3}{2} - \beta)} \right)^{\frac{p}{pq-1}} \left(\frac{(\frac{1}{2} - \alpha)\Gamma^{pq}(1-\alpha)}{\Gamma(\frac{3}{2} - \alpha)} \right)^{\frac{1}{pq-1}}$$

and where d_1 is obtained from c_1 by the interchange of α, β and p, q. From (3.6) we see that (f_+, g_+) decays like $(e^{-\frac{\zeta^2}{4}} \zeta^{2\alpha-1}, e^{-\frac{\zeta^2}{4}} \zeta^{2\beta-1})$ as $\zeta \to \infty$. From the identity

$$U_r(a,b,r) = -aU(a+1, b+1, r)$$

we see that f_+, g_+ are decreasing. Therefore the solution (u, v) converges to $(0,0)$ as $t \to 0^+$ uniformly and in any $[L^r(R^+)]^2$. □

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