Acta Math. Univ. Comenianae
Vol. LXIII, 2(1994), pp. 315–321

ON THE NUMBER OF CYCLES IN k–CONNECTED GRAPHS

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Abstract. We give estimations for the minimum number of cycles in three special subclasses of the class of k-connected graphs.

1. Introduction

A collection of cycles (i.e., connected two-regular graphs) is said to be distinguishable if their vertex sets are pairwise distinct. As conjectured by Komlós, each graph with the minimum degree δ contains at least $2^{\delta+1} - \binom{\delta+1}{2} - \delta - 2$ distinguishable cycles, i.e., the worst case is given by the complete graph $K_{\delta+1}$. In [**4**] Tuza proved the following theorem:

Theorem A. *Each graph with the minimum degree $\delta \geq 3$ contains more than $2^{\delta/2}$ distinguishable cycles.*

Although the number of cycles is exponential in δ this gives no information on whether or not it is also exponential in the number n of vertices of the graph. Since the latter parameter is important when measuring input size in algorithmic complexity questions, it could be interesting to describe classes of graphs whose total number of cycles is bounded above by a polynomial in n.

Let G be a graph, $G = (V(G), E(G))$, such that $V(G) = \{a_i, b_i : 1 \leq i \leq \frac{n}{2}\}$ and $E(G) = \{a_ib_i, b_ia_{i+1}, a_ia_{i+1} : 1 \leq i \leq \frac{n}{2}\}$ (the addition is modulo $\frac{n}{2}$). Then G has at least $2^{n/2}$ cycles (each of them traverses all a_i and a prescribed subset of b_i). Thus, already in outerplanar graphs the number of cycles may be exponential in the number of vertices.

Let $c(G)$ denote the number of cycles in G and let Γ be a class of graphs. As shown above, the function $C_n(\Gamma) = \max\{c(G) : G \in \Gamma \text{ and } |V(G)| = n\}$ seems to be exponential in non-trivial cases. However, we show that $c_n(\Gamma) = \min\{c(G) : G \in \Gamma \text{ and } |V(G)| = n\}$ may be polynomial even if the graphs in Γ have large connectivity and large minimum degree.

We consider the following three classes of graphs: Γ_k is the class of k-connected graphs; $\Gamma_{k,\delta}$ denotes the class of k-connected graphs with minimum degree at

Received February 8, 1994.
1980 *Mathematics Subject Classification* (1991 *Revision*). Primary 05C38; Secondary 05C30, 05C40.

least δ; and by $\Gamma_{k,\delta,\Delta}$ we denote the class of k-connected graphs with minimum degree at least δ and maximum degree at most Δ. In this paper we give polynomial upper bounds on $c_n(\Gamma_k)$ and $c_n(\Gamma_{k,\delta})$, $k < \delta$, and lower bounds on $c_n(\Gamma_{k,\delta})$ and $c_n(\Gamma_{k,\delta,\Delta})$ for some values of k, δ, and Δ.

2. Preliminaries and Upper Bounds

All graphs considered in this paper are finite, without loops or multiple edges.

Let G be a graph. Then $V(G)$ denotes the vertex set of G and $E(G)$ the edge set of G. By n we always denote the number of vertices in G. The distance between two vertices u and v in G is denoted by $d_G(u,v)$; the complement of G is denoted by $\overline{G}$.

As usual, a graph G is said to be k-connected if and only if G has at least $k+1$ vertices and any two distinct vertices u, v are connected by at least k uv-paths that are pairwise disjoint, except for the vertices u and v. Using Menger's theorem (see e.g. [**1**, Section 9.2]) one can obtain the following statement that is often used throughout this paper: A graph G is k-connected if and only if G has at least $k+1$ vertices and each two sets $X, Y \subseteq V(G)$, $|X| \geq k$ and $|Y| \geq k$, are connected by at least k XY-paths that are pairwise disjoint. Note that the sets X and Y are not necessarily disjoint, and hence, some paths may have length 0.

We focus on the asymptotic behavior of $c_n(\Gamma)$. Let $f(x)$ and $g(x)$ be nonnegative functions. We write $f(x) = O(g(x))$ and $g(x) = \Omega(f(x))$ if and only if there are numbers c and x_0 such that $f(x) \leq c{\cdot}g(x)$ for every $x \geq x_0$. Moreover, if $f(x) = O(g(x))$ and $f(x) = \Omega(g(x))$, we write $f(x) = \Theta(g(x))$.

Definitions and notations not included here can be found in [**2**].

In what follows we give upper bounds on $c_n(\Gamma_k)$ and $c_n(\Gamma_{k,\delta})$ (by constructions).

Proposition 1. *$c_n(\Gamma_0) = c_n(\Gamma_1) = 0$; $c_n(\Gamma_2) = 1$; $c_n(\Gamma_3) = O(n^2)$; and $c_n(\Gamma_k) = O(n^k)$ if $k \geq 4$.*

Proof. Discrete graphs and trees give $c_n(\Gamma_k) = 0$ if $k \leq 1$, and single cycles give $c_n(\Gamma_2) \leq 1$. Since each 2-connected graph contains a cycle, we have $c_n(\Gamma_2) = 1$.

Joining a vertex w to all vertices of a cycle on $n-1$ vertices we obtain a wheel W_n. The W_n has just one cycle that does not pass through the vertex w, and $2\binom{n-1}{2}$ cycles that contain w. Since W_n is 3-connected, we have $c_n(\Gamma_3) = O(n^2)$.

The complete bipartite graph $K_{k,n-k}$ on n vertices belongs to Γ_k if $n \geq 2k$. Since in $K_{k,n-k}$ there are at most $\sum_{i=2}^{k}\binom{k}{i}\binom{n-k}{i}$ different vertex sets of cycles, the $K_{k,n-k}$ contains at most $\sum_{i=2}^{k}\binom{k}{i}\binom{n-k}{i}\frac{(2i-1)!}{2} = O(n^k)$ cycles if $k \geq 2$. Thus, $c_n(\Gamma_k) = O(n^k)$ if $k \geq 4$. □

The graphs in the proof of Proposition 1 have the smallest possible minimum degree. One can expect that in k-connected graphs with greater minimum degree,

$\Gamma_{k,\delta}$, there are much more cycles. In the next section we show that this is true if $k \leq 3$ (see Corollary 10). However, first we give an upper bound for $c_n(\Gamma_{k,\delta})$:

Proposition 2. *We have $c_n(\Gamma_{k,\delta}) = O(n)$ if $k \leq 1$ and $\delta \geq 3$; and $c_n(\Gamma_{k,\delta}) = O(n^k)$ if $k \geq 2$ and $\delta > k$.*

Proof. Let G_m^1 consist of $m > \delta$ copies of the complete graph $K_{\delta+1-k}$, and let G_m^2 consist of k isolated vertices. Let G_m consist of G_m^1, G_m^2, and the edges u_1u_2, $u_1 \in V(G_m^1)$ and $u_2 \in V(G_m^2)$. Clearly, $G_m \in \Gamma_{k,\delta}$.

Since $c(K_t) = \sum_{i=3}^{t} \binom{t}{i} \frac{i!}{2i} < 2t!$, there are at most $2m(\delta+1-k)!$ cycles in G_m that contain no vertex of G_m^2. Further, K_t contains exactly $\sum_{i=1}^{t} \binom{t}{i} i!$ labelled paths. Note that $\sum_{i=1}^{t} \binom{t}{i} i! = t! \sum_{i=1}^{t} \frac{1}{(t-i)!} < t! \sum_{i=0}^{\infty} \frac{1}{i!} = et!$. Hence, there are at most $\binom{m}{i} e(\delta+1-k)! \binom{k}{i} \frac{(2i)!}{2i}$ cycles in G_m that contain exactly i vertices of G_m^2. Since $m = \frac{n-k}{\delta+1-k}$, we have $c(G_m) < 2m(\delta+1-k)! + \sum_{i=1}^{k} \binom{m}{i} e(\delta+1-k)! \binom{k}{i} \frac{(2i)!}{2i} < 3(\delta+1-k)! (\frac{n-k}{\delta+1-k} + \sum_{i=1}^{k} \binom{\frac{n-k}{\delta+1-k}}{i} \binom{k}{i} \frac{(2i)!}{2i})$. Thus, $c_n(\Gamma_{0,\delta}) = O(n)$, and $c_n(\Gamma_{k,\delta}) = O(n^k)$ if $k \geq 1$. □

3. Lower Bounds

In this section we give some lower bounds on $c_n(\Gamma)$ using k-minimal subgraphs.

Since $\Gamma_{k,\delta} \supseteq \Gamma_{k,\delta+1}$ and $\Gamma_{k,\delta,\Delta} \supseteq \Gamma_{k,\delta+1,\Delta}$ if $k \leq \delta < \Delta$, we have $c_n(\Gamma_{k,\delta}) \leq c_n(\Gamma_{k,\delta+1})$ and $c_n(\Gamma_{k,\delta,\Delta}) \leq c_n(\Gamma_{k,\delta+1,\Delta})$. Hence, it is enough to give "good" lower bounds for "small" values of δ.

Proposition 3. *Let $k \leq 1$. Then $c_n(\Gamma_{k,3}) = \Omega(n)$.*

Proof. Let $G \in \Gamma_{k,3}$, $k \leq 1$, and let H be a subgraph of G with maximum number of edges containing no cycle. Clearly, $|E(H)| \leq n-1$.

Let $e \in E(G)-E(H)$. Then there is exactly one cycle in $H \cup e$ containing e. Since there are at least $\frac{3n}{2} - (n-1) > \frac{n}{2}$ edges in $E(G)-E(H)$, we have $c_n(\Gamma_{k,\delta}) = \Omega(n)$. □

In what follows we utilize the simple idea of the previous proof.

A graph H is called *k-minimal* if H is k-connected, but loses this property after the deletion of any edge (see [**3**]). We remark that each k-connected graph always contains a k-minimal spanning subgraph. In [**3**] Mader proved the following lemma:

Lemma 4. *Each k-minimal graph on n vertices contains at least $\frac{k-1}{2k-1}n$ vertices of degree k.*

Before an analogue of Proposition 3 will be given for higher connectivities, we need to prove two auxiliary results:

Lemma 5. *Let $H \in \Gamma_2$ and $e_1, e_2 \in E(\overline{H})$. Then $H \cup \{e_1, e_2\}$ contains a cycle passing through e_1 and e_2.*

Proof. Let $e_i = u_i v_i$, $1 \le i \le 2$. Since H is 2-connected, there are two disjoint paths (possibly of length 0) connecting the vertex sets $\{v_1, u_1\}$ and $\{v_2, u_2\}$ in H. Thus, there is a cycle in $H \cup \{e_1, e_2\}$ containing e_1 and e_2. □

Lemma 6. *Let $H \in \Gamma_3$, and let $e_1, e_2, e_3 \in E(\overline{H})$ be edges that do not form a claw (i.e., the complete bipartite graph $K_{1,3}$). Then $H \cup \{e_1, e_2, e_3\}$ contains a cycle passing through e_1, e_2, and e_3.*

Proof. Let $e_i = u_i v_i$, $1 \le i \le 3$, be edges that do not form a claw. We distinguish two cases:

1. Suppose that $u_1 = u_2$. Then $u_3, v_3 \in V(H) - u_1$, since e_1, e_2, and e_3 do not form a claw. Moreover, $H - u_1$ is 2-connected (by Menger's theorem). Hence, there are two disjoint paths (possibly of length 0) connecting the vertex sets $\{v_1, v_2\}$ and $\{u_3, v_3\}$ in $H - u_1$. Thus, there is a cycle in $H \cup \{e_1, e_2, e_3\}$ containing e_1, e_2, and e_3.

2. Suppose that the six nodes u_i, v_i, $1 \le i \le 3$, are distinct. Let $S_1 = \{u_1, v_1, u_3\}$ and $S_2 = \{u_2, v_2, v_3\}$. Since H is 3-connected, there are three pairwise disjoint paths, say P_1, P_2, and P_3, connecting S_1 with S_2, see Fig. 1. If $F = \cup_{i=1}^{3} (P_i \cup e_i)$ forms a cycle, we are done. Otherwise, F consists of two disjoint cycles $C_1 = P_1 \cup e_1 \cup P_2 \cup e_2$ and $C_2 = P_3 \cup e_3$. Let $S_1' = V(C_1)$ and $S_2' = V(C_2)$. Clearly, $|S_1'| \ge 3$ and $|S_2'| \ge 3$. Since H is 3-connected, there are at least three pairwise disjoint paths, say R_1, R_2, and R_3, connecting S_1' with S_2'. Obviously, at least two of them, say R_1 and R_2, connect P_3 with P_j for some j, $1 \le j \le 2$. Then $F \cup R_1 \cup R_2$ contains a cycle passing through the e_i, $1 \le i \le 3$, see Fig. 1. □

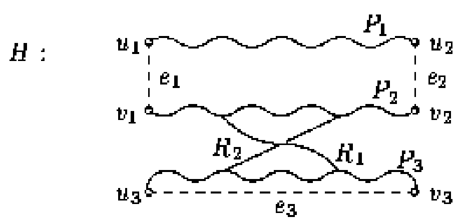

Figure 1.

Theorem 7. $c_n(\Gamma_{2,3}) = \Omega(n^2)$.

Proof. Let $G \in \Gamma_{2,3}$, and let H be a spanning subgraph of G that is 2-minimal. Then $|E(G) - E(H)| \ge \frac{1}{6} n$, since H contains at least $\frac{1}{3} n$ vertices of degree 2, by Lemma 4.

By Lemma 5 for each pair of edges $e_1, e_2 \in E(G) - E(H)$ there is a cycle in $H \cup \{e_1, e_2\}$ containing e_1 and e_2. Since different pairs of edges from $E(G) - E(H)$ give different cycles in G, we have $c(G) \geq \binom{n/6}{2}$. Thus $c_n(\Gamma_{2,3}) = \Omega(n^2)$. □

Since the minimum degree of each 3-connected graph is at least three, the following corollary is implied by Proposition 1 and Theorem 7:

Corollary 8. *For* Γ_3 *it is* $c_n(\Gamma_3) = \Theta(n^2)$.

Clearly $c_n(\Gamma_{3,4}) = \Omega(n^2)$, since $\Gamma_{3,4} \subseteq \Gamma_3$. However, for 3-connected graphs with minimum degree at least 5 we have the following theorem:

Theorem 9. $c_n(\Gamma_{3,5}) = \Omega(n^3)$.

Proof. Let $G \in \Gamma_{3,5}$, and let H be a spanning subgraph of G that is 3-minimal. Then $|E(G) - E(H)| \geq \frac{2}{5}n$, since H contains at least $\frac{2}{5}n$ vertices of degree 3, by Lemma 4.

Let e_1 and e_2 be two edges from $E(G) - E(H)$. Then there are at least $\frac{2n}{5} - 4$ vertices of degree 3 in H that are not endvertices of e_1 or e_2. Since minimum degree in G is at least 5, each one of the $\frac{2n}{5} - 4$ vertices belongs to an edge from $E(G) - E(H)$ that does not form a claw with e_1 and e_2. Since there are at least $\binom{2n/5}{2}$ pairs of edges in $E(G) - E(H)$, there are at least $\frac{1}{3}\binom{2n/5}{2}(\frac{2n}{5} - 4) = \Omega(n^3)$ triples of edges in $E(G) - E(H)$ that do not form a claw.

By Lemma 6 for each triple of edges $e_1, e_2, e_3 \in E(G) - E(H)$ that do not form a claw there is a cycle in $H \cup \{e_1, e_2, e_3\}$ containing e_1, e_2, and e_3, so that $c_n(\Gamma_{3,5}) = \Omega(n^3)$. □

The following corollary summarizes our results concerning $c_n(\Gamma_{k,\delta})$:

Corollary 10. $c_n(\Gamma_{k,\delta}) = \Theta(n)$ *if* $k \leq 1$ *and* $\delta \geq 3$; $c_n(\Gamma_{2,\delta}) = \Theta(n^2)$ *if* $\delta \geq 3$; *and* $c_n(\Gamma_{3,\delta}) = \Theta(n^3)$ *if* $\delta \geq 5$.

Although we are not able to give the expected lower bound for $c_n(\Gamma_{3,4})$, surprisingly, we have such bound for $c_n(\Gamma_{3,4,\Delta})$. (One can expect that if large degrees are allowed, then more cycles will appear. But in fact $c_n(\Gamma_{k,\delta}) \leq c_n(\Gamma_{k,\delta,\Delta})$, since $\Gamma_{k,\delta} \supseteq \Gamma_{k,\delta,\Delta}$ if $k \leq \delta \leq \Delta$.)

Theorem 11. *We have* $c_n(\Gamma_{3,4,\Delta}) = \Omega(n^3)$ *for any fixed* $\Delta \geq 4$.

Proof. Let $G \in \Gamma_{3,4,\Delta}$, and let H be a spanning subgraph of G that is 3-minimal. Then $|E(G) - E(H)| \geq \frac{1}{5}n$, by Lemma 4.

Since the maximum degree in G is at most Δ, there are at least $\frac{1}{6}\frac{1}{5}n(\frac{1}{5}n - 2\Delta)(\frac{1}{5}n - 4\Delta) = \Omega(n^3)$ triples of non-adjacent edges in $E(G) - E(H)$. Thus, $c_n(\Gamma_{3,4,\Delta}) = \Omega(n^3)$ by Lemma 6, as non-adjacent edges do not form a claw. □

We conclude this section with a theorem that generalizes Theorem 11 for higher connectivities:

Theorem 12. *Let l be the largest integer such that $l \le \sqrt{k-1}$, $k \ge 4$. Then $c_n(\Gamma_{k,k+1,\Delta}) = \Omega(n^{2l})$ for any fixed $\Delta \ge k+1$.*

Proof. Let $G \in \Gamma_{k,k+1,\Delta}$, and let H be a spanning subgraph of G that is k-minimal. By Lemma 4 $|E(G)-E(H)| \ge \frac{k-1}{2k-1}\frac{n}{2} \ge \frac{n}{6}$, since $k \ge 2$.

Let $e_i = u_iv_i$, $1 \le i \le 2l$, be edges from $E(G)-E(H)$, such that $d_G(e_i,e_j) \ge \frac{k-2}{2}$ for each $i \ne j$. (By $d_G(e_i,e_j)$ we mean $\min\{d_G(x,y) : x \in \{u_i,v_i\}$ and $y \in \{u_j,v_j\}\}$.) We show that there is a cycle in $H^* = H \cup \{e_i : 1 \le i \le 2l\}$ containing the edges e_i, $1 \le i \le 2l$.

Let $S_1 = \{u_i,v_i : 1 \le i \le l\}$ and $S_2 = \{u_i,v_i : l+1 \le i \le 2l\}$. Then $|S_1| = |S_2| = 2l$. Since H is k-connected and $k \ge 2l$, there are $2l$ pairwise disjoint paths, say $P_1,\dots,P_{2l}$, connecting the vertices from S_1 to the vertices in S_2. The paths P_i together with the edges e_i, $1 \le i \le 2l$, form a collection of m pairwise disjoint cycles, say $C_1,\dots,C_m$, in H^*. If $m = 1$, we are done. Suppose that $m \ge 2$. Moreover, suppose that there is no collection of $m-1$ pairwise disjoint cycles containing the edges e_i, $1 \le i \le 2l$, in H^*.

Let $S_1' = V(C_1)$ and $S_2' = V(C_2) \cup \dots \cup V(C_m)$. Since $d_G(e_i,e_j) \ge \frac{k-2}{2}$ if $i \ne j$, we have $|S_1'| \ge k$ and $|S_2'| \ge k$. Since H is k-connected, there are k pairwise disjoint paths, say $P_1',\dots,P_k'$, connecting the vertices from S_1' to the vertices in S_2'. Note that $k \ge l^2+1$ as $l \le \sqrt{k-1}$. Suppose that C_1 contains p paths among $P_1,\dots,P_{2l}$. Since $p(2l-p) \le l^2 < k$, there are at least two paths among $P_1',\dots,P_k'$, say P_1' and P_2', that connect two distinct vertices from P_{i_1} in S_1' to two distinct vertices in P_{i_2} in S_2', for some i_1 and i_2. Assume that P_{i_2} is contained in C_2. Then $C_1 \cup C_2 \cup P_1' \cup P_2'$ contains a cycle C', passing through all those edges e_i that have been in C_1 and C_2. Hence, there is a collection $C', C_3, \dots, C_m$ of $m-1$ pairwise disjoint cycles containing the edges e_i, $1 \le i \le 2l$, a contradiction.

Finally, we show that there is "many" $2l$-tuples of edges $e_1,\dots,e_{2l}$ in $E(G)-E(H)$, such that $d_G(e_i,e_j) \ge k-1$ whenever $i \ne j$. (Clearly, $k-1 \ge \frac{k-2}{2}$.)

Let $e \in E(G)-E(H)$. Then there are at most 2Δ edges in G at distance 0 from e; $2\Delta(\Delta-1)$ edges in G at distance 1 from e; etc. Thus, there are at most $2\Delta\sum_{j=0}^{k-1}(\Delta-1)^j = 2\Delta\frac{(\Delta-1)^k-1}{\Delta-2} \le 4\Delta^k$ edges in G at distance at most $k-1$ from e. Since $E(G)-E(H) \ge \frac{n}{6}$, there are at least $\frac{1}{(2l)!}\frac{n}{6}(\frac{n}{6} - 4\Delta^k)(\frac{n}{6} - 2{\cdot}4\Delta^k)\dots(\frac{n}{6} - (2l-1)4\Delta^k) = \Omega(n^{2l})$ required $2l$-tuples of edges in $E(G)-E(H)$. □

4. Concluding Remarks

We remark that Theorems 7, 9, 11, and 12 can be slightly improved. Namely, the class $\Gamma_{k,\delta}$ ($\Gamma_{k,\delta,\Delta}$) can be replaced by the class of k-connected graphs in which each graph contains at least $c \cdot n$ vertices with degree at least δ (and maximum degree at most Δ), $c > \frac{k}{2k-1}$. Analogously Proposition 3 can be improved.

Further, classes of graphs can be constructed such that $c_n(\Gamma_{k,\delta,\delta}) = O(n^k)$ if $k = 1,2$ and $\delta \ge 3$. However, no polynomial upper bound for $c_n(\Gamma_{k,\delta,\delta})$ seems to be known if $k \geq 3$. Moreover, we know no polynomial upper bound for $c_n(\Gamma_{k,\delta,\Delta})$ if $k \geq 3$ and $\Delta \geq k+1$. (Note that the graphs in the proof of Proposition 2 have not bounded maximum degree.) We conjecture that $c_n(\Gamma_{k,\delta,\Delta}) = \Theta(n^k)$ if $k > 2$ and $k < \delta < \Delta$.

Acknowledgement. The author is grateful to the anonymous referee for valuable suggestions.

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