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MONOTONE SEMIFLOWS GENERATED BY NEUTRAL EQUATIONS WITH DIFFERENT DELAYS IN NEUTRAL AND RETARDED PARTS

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1. Introduction

Let $r \geq r_0 > 0$ be given constants, $C = C([-r, 0]; R)$ the Banach space of continuous functions from $[-r, 0]$ into R with the norm $||\phi|| = \sup_{-r \leq s \leq 0} |\phi(s)|$, $\phi \in C$. We consider the following scalar neutral functional differential equation

$$\frac{d}{dt} Dx_t = f(x_t) \tag{1.1}$$

where

(D) $Dx_t = x(t) - \int_{-r}^{-r_0} x(t+s)d\nu(s)$, $\nu : [-r, r_0] \to R$ is nondecreasing and $\nu(-r_0) - \nu(-r) < 1$;

(f) $f : C \to R$ is locally Lipschitz continuous;

and $x_t \in C$ is defined in the usual way by $x_t(s) = x(t+s)$ for $s \in [-r, 0]$.

In [**13**], an ordering $\geq_D$ of the space C is introduced as follows

$$\phi \geq_D 0 \ \text{ iff } \phi(s) \geq 0 \ \text{ on } [-r, 0] \ \text{ and } D(\phi) \geq 0,$$

so that under certain conditions, the solution semiflow defined by (1.1) is eventually strongly monotone in the sense of [**6**]. In particular, the eventual strong monotonicity is obtained for solutions of the following scalar neutral equation

$$\frac{d}{dt}[x(t) - cx(t-\tau)] = g(x(t), x(t-\omega)) \tag{1.2}$$

if $\tau = \omega$, $0 \leq c < 1$ and $\frac{\partial}{\partial y} g(x, y) > 0$ for all $(x, y) \in R^2$. In verifying the above strong monotonicity of the solution semiflow, it is crucial to assume that the delay τ in the neutral part and the delay ω in the retarded part are the same. However, in some applications, this assumption is unrealistic. For example, a special case is

$$\frac{d}{dt}[x(t) - cx(t-\tau)] = -h(x(t)) + h(x(t-\omega)), \tag{1.3}$$

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where $h\colon R \to R$ is continuously differentiable and $\frac{d}{dx}h(x) > 0$ for $x \in R$, was used to model the transmission dynamics of material in an active compartmental system with one compartment and one pipe coming out of and returning into the compartment, where the delay τ is the length of time required in the process in which some material is produced, while the delay ω represents the transit time for the material flow to pass through the pipe (see [**2**] for details).

In this paper, we follow Smith and Thieme [**8**], [**10**] to consider the restriction of the solution semiflow to a dense subset X of C. The subset X chosen here is different from the one used by Smith and Thieme in [**8**], [**10**] for delay differential equations because of the peculiarity of neutral equations. We obtain fairly general sufficient conditions guaranteeing the strong order-preserving property of the solution semiflow in X. In particular, we show that the solution semiflow of (1.2) on X is strongly order-preserving if there exists $\mu \geq 0$ such that

$$\mu + L_1 - c\mu e^{\mu\tau} + \min\{L_2, 0\}e^{\mu\omega} > 0, \tag{1.4}$$

or if there exist $\mu \geq 0$, $\alpha_1, \alpha_2 \geq 0$ with $\alpha_1 + \alpha_2 = 1$ such that

$$\begin{cases} \mu + L_1 \geq 0, \\ \alpha_1(\mu + L_1)e^{-\mu\tau} - c\mu > 0, \\ \alpha_2(\mu + L_1)e^{-\mu\omega} + L_2 \geq 0, \end{cases} \tag{1.5}$$

where

$$L_1 = \inf_{(x,y)\in R^2} \frac{\partial g(x,y)}{\partial x}, \qquad L_2 = \inf_{(x,y)\in R^2} \frac{\partial g(x,y)}{\partial y}.$$

The established strong order-preserving property allows us to apply the powerful theory of convergence and stability theory developed in [**9**], [**11**] to neutral equations (1.1), (1.2) and (1.3). In particular, we will employ the principle of sequential limit set trichotomy to show that, under certain conditions, every solution of (1.3), starting from an initial function in X, is convergent to a single constant function. This extends corresponding results of [**3**], [**4**], [**13**] to the case where $\tau \neq \omega$. It should be mentioned that an example can be easily constructed in which periodic solutions of (1.3) exist [**7**]. Hence, certain conditions on c, τ, ω have to be imposed for solutions of equation (1.3) to converge to constant functions.

2. Monotonicity of the Solution Semiflow

We start with a description of a dense strongly ordered subspace of C. Define $C^1 = C^1([-r, 0]; R)$ and let

$$X = \{\phi \in C^1 : \ D\phi' = f(\phi)\}$$

be a metric space endowed with the topology of C^1, i.e., $d(\phi, \psi) = ||\phi - \psi||_{C^1} := \max\{||\phi - \psi||, ||\phi' - \psi'||\}$. Clearly, X is complete.

In order to show some properties of the space X, we need a particular function. For given real α, positive β and $0 < \gamma \le \min\{r_0, \frac{2\beta}{|\alpha|+1}\}$ define

$$\xi_{\alpha,\beta,\gamma}(s) = \begin{cases} 0, & \text{if } -r \le s \le -\gamma \\ \alpha\left(\frac{s^2}{2\gamma} + s + \frac{\gamma}{2}\right), & \text{if } -\gamma \le s \le 0. \end{cases} \tag{2.1}$$

It is straightforward to check that this $\xi_{\alpha,\beta,\gamma}$ has the following properties:

Lemma 2.1. *For any $\alpha \in R$, $\beta > 0$ and $0 < \gamma \le \min\{r_0, \frac{2\beta}{|\alpha|+1}\}$, the function $\xi = \xi_{\alpha,\beta,\gamma}$ satisfies $\xi \in C^1$, $||\xi|| < \beta$, $||\xi'|| = |\alpha|$, $\xi'(0) = \alpha$, $D\xi' = \alpha$. Moreover, for fixed $\beta > 0$, the mapping $\alpha \mapsto \xi_{\alpha,\beta,\gamma}$, where $0 < \gamma = \min\{r_0, \frac{2\beta}{|\alpha|+1}\}$, is continuous from R to C^1.*

By using Lemma 2.1 one can get the denseness of X in C.

Lemma 2.2. *X is dense in C.*

Proof. Let $\varepsilon > 0$ and $\phi \in C$ be given. Then there is $\psi \in C^1$ such that $||\phi - \psi|| < \varepsilon/2$ and ψ is constant on some interval $[-\delta, 0]$, $0 < \delta < r$. Let $\eta_\alpha = \xi_{\alpha,\varepsilon/2,\gamma}$ be defined as in (2.1), where $0 < \gamma = \min\{r_0, \frac{2\beta}{|\alpha|+1}, \frac{\delta}{2}\}$. Then, for any $\alpha \in R$, $\psi + \eta_\alpha$ is in C^1 and $||\phi - (\psi + \eta_\alpha)|| < \varepsilon$. It suffices to show that $\psi + \eta_\alpha \in X$, i.e., $D\psi' + \alpha = f(\psi + \eta_\alpha)$ for some α. $D\psi' + \alpha$ and $f(\psi + \eta_\alpha)$ are continuous in α by Lemma 2.1. The range of $D\psi' + \alpha$ is R as α varies, while the range of $f(\psi+\eta_\alpha)$ is bounded (at least for sufficiently small $\varepsilon > 0$ by condition (f)). Therefore, there is at least one α such that $\psi + \eta_\alpha \in X$. □

For any given constant $\mu \ge 0$, let us define an order $\le_\mu$ on X such that for ϕ, $\psi \in X$, $\phi \le_\mu \psi$ provided $\phi(s) \le \psi(s)$ and $\phi'(s) + \mu\phi(s) \le \psi'(s) + \mu\psi(s)$ on $[-r, 0]$, or, equivalently, $\phi(s) \le \psi(s)$ and $e^{\mu s}[\psi(s) - \phi(s)]$ is nondecreasing on $[-r, 0]$. If $\phi \le_\mu \psi$ and $\phi \ne \psi$ then we write that $\phi <_\mu \psi$. This order is compatible with the topology on X in the sense that for any sequences $\{\phi_n\}$, $\{\psi_n\}$ from $\phi_n \le_\mu \psi_n$, $\phi_n \to \phi$ and $\psi_n \to \psi$ it follows that $\phi \le_\mu \psi$. It is easy to see that X is normally ordered, i.e., there exists a constant $k > 0$ such that $d(\xi, \eta) \le kd(\phi, \psi)$ for all ξ, η, ϕ, ψ with $\phi <_\mu \psi$, $\phi \le_\mu \xi$, $\eta \le_\mu \psi$.

In the next lemma we show another relation between the order and the topology of X. For a $\phi \in X$, we say that ϕ can be approximated from above (below) in X if there exists a sequence $\{\phi_n\}$ in X satisfying $\phi <_\mu \phi_{n+1} <_\mu \phi_n$ ($\phi_n <_\mu \phi_{n+1} <_\mu \phi$) for $n \ge 1$, and $\phi_n \to \phi$ as $n \to \infty$.

Lemma 2.3. *If the assumption*

(L$^-$) *there is an $L \ge 0$ such that $\phi, \psi \in X$ and $\phi(s) \ge \psi(s)$, $s \in [-r, 0]$, imply*

$$f(\phi) - f(\psi) \ge -L||\phi - \psi||$$

holds and $\mu > L$, then each $\phi \in X$ can be approximated from above and below in X.

Proof. Let $\phi \in X$ be given. First we construct a $\phi_1 \in X$ such that $\phi <_\mu \phi_1$. Let $\delta_1 \in (0,1]$ and choose $\varepsilon_1 \in (0,\delta_1)$ such that $\mu(\delta_1-\varepsilon_1) > L(\delta_1+\varepsilon_1)$. We look for ϕ_1 in the form $\phi_1 = \phi+\delta_1+\eta_\alpha$, where $\eta_\alpha = \xi_{\alpha,\varepsilon_1,\gamma}$, $0 < \gamma = \min\{r_0, \frac{2\varepsilon_1}{|\alpha|+1}\}$. One has $\phi_1(s) > \phi(s)$, $-r \le s \le 0$, for any α because $||\eta_\alpha|| < \varepsilon_1$ and $\varepsilon_1 < \delta_1$. In order to have $\phi <_\mu \phi_1$, we need $(\phi_1-\phi)'+\mu(\phi_1-\phi) \ge 0$, that is $\eta'_\alpha+\mu(\delta_1+\eta_\alpha) \ge 0$. By using the definition of η_α, we see that the last inequality holds provided $\alpha+\mu(\delta_1-\varepsilon_1) \ge 0$. Thus it suffices to show that there is an $\alpha_1 \in R$ such that $\alpha_1 > -L(\delta_1+\varepsilon_1)$ and $\phi_1 = \phi+\delta_1+\eta_{\alpha_1} \in X$. ϕ_1 is in X if and only if $\alpha_1 = f(\phi+\delta_1+\eta_{\alpha_1}) - f(\phi)$, where we used $D\phi' = f(\phi)$ and $D\eta'_{\alpha_1} = \alpha_1$. The existence of an α_1 with $\alpha_1 = f(\phi+\delta_1+\eta_{\alpha_1}) - f(\phi)$ follows since $f(\phi+\delta_1+\eta_{\alpha_1}) - f(\phi)$ is bounded in α_1 by condition (f) (at least for sufficiently small δ_1). By assumption (L^-)

$$\alpha_1 = f(\phi+\delta_1+\eta_{\alpha_1}) - f(\phi) \ge -L||\delta_1+\eta_{\alpha_1}|| > -L(\delta_1+\varepsilon_1).$$

Therefore $\phi <_\mu \phi_1$.

Assume $\phi_n \in X$ is given such that $\phi <_\mu \phi_n$ and $\phi_n = \phi+\delta_n+\eta_{\alpha_n}$, where $\eta_{\alpha_n} = \xi_{\alpha_n,\varepsilon_n,\gamma}$, $0 < \gamma = \min\{r_0, \frac{2\epsilon_n}{|\alpha_n|+1}\}$, $0 < \varepsilon_n < \delta_n \le 1/n$, $\mu(\delta_n-\varepsilon_n) > L(\delta_n+\varepsilon_n)$, $\alpha_n > -L(\delta_n+\varepsilon_n)$.

We seek for ϕ_{n+1} in the form $\phi_{n+1} = \phi+\delta_{n+1}+\eta_\alpha$, where $\eta_\alpha = \xi_{\alpha,\varepsilon_{n+1},\gamma}$, $0 < \gamma = \min\{r_0, \frac{2\epsilon_{n+1}}{|\alpha|+1}\}$, $0 < \varepsilon_{n+1} < \delta_{n+1} \le 1/(n+1)$, $\mu(\delta_{n+1}-\varepsilon_{n+1}) > L(\delta_{n+1}+\varepsilon_{n+1})$, $\delta_{n+1}+\varepsilon_{n+1} < \delta_n-\varepsilon_n$. For this ϕ_{n+1}, the relation $\phi <_\mu \phi_{n+1}$ is satisfied if

$$(\phi_{n+1}-\phi)' + \mu(\phi_{n+1}-\phi) = \eta'_\alpha + \mu(\delta_{n+1}+\eta_\alpha) > 0.$$

This is true if $\alpha \ge -L(\delta_{n+1}+\varepsilon_{n+1})$ because of $\eta'_\alpha \ge \min\{0,\alpha\}$, $\eta_\alpha > -\varepsilon_1$ and $\mu(\delta_{n+1}-\varepsilon_{n+1}) > L(\delta_{n+1}+\varepsilon_{n+1})$. In the same way as above for α_1, one gets the existence of an α_{n+1} such that $\phi+\delta_{n+1}+\eta_{\alpha_{n+1}} \in X$. Since α_{n+1} satisfies $\alpha_{n+1} = f(\phi+\delta_{n+1}+\eta_{\alpha_{n+1}}) - f(\phi)$, by using the properties of $\eta_{\alpha_{n+1}}$ and condition (L^-), we obtain $\alpha_{n+1} > -L(\delta_{n+1}+\varepsilon_{n+1})$. So, letting $\alpha = \alpha_{n+1}$, one gets $\phi <_\mu \phi_{n+1}$.

We also have

$$\begin{aligned}(\phi_n-\phi_{n+1})' &+ \mu(\phi_n-\phi_{n+1}) \\ &\ge -L(\delta_n+\varepsilon_n) - |\alpha_{n+1}| + \mu(\delta_n-\varepsilon_n-\delta_{n+1}-\varepsilon_{n+1}) \\ &= -L(\delta_n+\varepsilon_n) + \mu(\delta_n-\varepsilon_n) - |\alpha_{n+1}| - \mu(\delta_{n+1}+\varepsilon_{n+1}).\end{aligned}$$

In order to show the relation $\phi_{n+1} <_\mu \phi_n$, the positivity of $-L(\delta_n+\varepsilon_n)+\mu(\delta_n-\varepsilon_n) - |\alpha_{n+1}| - \mu(\delta_{n+1}+\varepsilon_{n+1})$ is sufficient. If $\delta_{n+1} \to 0$ in the definition of ϕ_{n+1}, then the corresponding α_{n+1} also goes to 0 because of the continuity of f in C.

Thus, since $-L(\delta_n+\varepsilon_n)+\mu(\delta_n-\varepsilon_n)>0$, δ_{n+1} can be chosen so small that

$$-L(\delta_n+\varepsilon_n)+\mu(\delta_n-\varepsilon_n)-|\alpha_{n+1}|-\mu(\delta_{n+1}+\varepsilon_{n+1})>0,$$

that is $\phi_{n+1}<_\mu \phi_n$ will be satisfied.

Therefore, by induction we can construct $\{\phi_n\}$ in X such that $\phi<_\mu \phi_{n+1}<_\mu \phi_n$. The convergence $\phi_n\to\phi$ also holds in X, since $d(\phi_n,\phi)\le\max\{2\delta_n,|\alpha_n|\}$ and $\delta_n\to 0, \alpha_n\to 0$ as $n\to\infty$. This completes the proof of the approximation from above. The proof for the approximation from below is analogous. □

Lemma 2.4. *Assume that $T>0$ is a given constant. If $\phi\in C^1$, $y\in C([-r,T];R)$ and $h\in C^1([0,T];R)$ are given so that*

$$\begin{aligned} D(y_t)&=h(t) \qquad (0\le t\le T),\\ y_0&=\phi, \end{aligned}$$

and $D(\phi')=h'(0^+)$, then the mapping $[-r,T]\ni t\mapsto y(t)\in R$ belongs to $C^1([-r,T];R)$ and

$$|y|_{C^1[-r,T]}\le\max\left\{|\phi|_{C^1},\frac{1}{1-[\nu(-r_0)-\nu(-r)]}\,|h|_{C^1[0,T]}\right\}. \tag{2.2}$$

Proof. We have

$$y(t)=\int_{-r}^{-r_0} y(t+u)\,d\nu(u)+h(t) \qquad (t\in[0,T]), \tag{2.3}$$

from which and from our conditions it follows that $y\in C^1[-r,r_0]$. Repeating the same argument we get $y\in C^1[-r,T]$. From (2.3)

$$|y(t)|\le\int_{-r}^{-r_0}|y(t+u)|\,d\nu(u)+|h(t)|, \qquad 0\le t\le T.$$

Differentiating (2.3), one obtains

$$|y'(t)|\le\int_{-r}^{-r_0}|y'(t+u)|\,d\nu(u)+|h'(t)|, \qquad 0\le t\le T.$$

Thus

$$\sup_{t\in[0,T]}|y(t)|\le[\nu(-r_0)-\nu(-r)]\sup_{t\in[-r,T]}|y(t)|+\sup_{t\in[0,T]}|h(t)|$$

and

$$\sup_{t\in[0,T]}|y'(t)|\le[\nu(-r_0)-\nu(-r)]\sup_{t\in[-r,T]}|y'(t)|+\sup_{t\in[0,T]}|h'(t)|.$$

Using $0 \leq \nu(-r_0) - \nu(-r) < 1$, one obtains

$$\sup_{t\in[0,T]} |y(t)| \leq \max\left\{ ||\phi||, \frac{1}{1-[\nu(-r_0)-\nu(-r)]} \sup_{t\in[0,T]} |h(t)| \right\}$$

and

$$\sup_{t\in[0,T]} |y'(t)| \leq \max\left\{ ||\phi'||, \frac{1}{1-[\nu(-r_0)-\nu(-r)]} \sup_{t\in[0,T]} |h'(t)| \right\}.$$

Hence, taking into account $y_0 = \phi$, we get inequality (2.2). □

Lemma 2.5. *If $\phi \in X$, then $x_t(\phi) \in X$ for $t \in [0, \tau(\phi))$, where and in what follows, $\tau(\phi)$ denotes the escape time of the solution $x_t(\phi)$ of* (1.1) *with $x_0 = \phi$. If $\sup_{t\in[0,\tau(\phi))} ||x_t(\phi)|| < \infty$, then $\tau(\phi) = \infty$. Moreover, the mapping $[0,\infty) \times X \ni (t,\phi) \mapsto x_t(\phi) \in X$ defines a semiflow on X.*

Proof. Since $\phi \in X$, our equation on $[0, r_0]$ can be written as

$$x'(t) = f(x_t) + g(t),$$

where $g(t) = \int_{-r}^{-r_0} \phi'(t+s)\, d\nu(s)$. This is a delay differential equation and standard results imply the existence and uniqueness of the solution in phase space C. This argument can be repeated to get existence on $[r_0, 2r_0]$, $[2r_0, 3r_0]$ and so on. Clearly, the obtained solution satisfies $x_t(\phi) \in X$. It also comes from a standard result for delay differential equations that the boundedness of $x_t(\phi)$ in C on $[0, \tau(\phi))$ implies $\tau(\phi)) = \infty$.

Since the mapping $[0,\infty) \times C \ni (t,\phi) \mapsto x_t(\phi) \in C$ defines a semiflow on C, it suffices to check the continuity of $x_t(\phi)$ with respect to t and ϕ in the topology of X. As $x(\phi) \in C^1[-r, \tau(\phi))$ holds, the continuity in t is clear.

Let $\phi_1, \phi_2 \in X$, $\tau^* = \min\{\tau(\phi_1), \tau(\phi_2)\}$ and let T be arbitrarily given such that $0 < T < \tau^*$. Since f is Lipschitz continuous on the compact subset $\cup_{t\in[0,T]}\{x_t(\phi_1), x_t(\phi_2)\}$ of C, we can find a constant $M > 0$ such that

$$|f(x_t(\phi_2)) - f(x_t(\phi_1))| \leq M||x_t(\phi_2) - x_t(\phi_1)||, \quad t \in [0,T].$$

So, applying Lemma 2.4 with $y(t) = x(\phi_2)(t) - x(\phi_1)(t)$, $h(t) = D(\phi_2 - \phi_1) + \int_0^t [f(x_s(\phi_2)) - f(x_s(\phi_1))]\, ds$ and $\phi = \phi_2 - \phi_1$, we obtain

$$\begin{aligned} d(x_t(\phi_2), x_t(\phi_1)) &\leq |x(\phi_2) - x(\phi_1)|_{C^1[-r,T]} \\ &\leq \max\{|\phi_2 - \phi_1|_{C^1}, \frac{1}{1-[\nu(-r_0)-\nu(-r)]}(|D(\phi_2 - \phi_1)| \\ &\qquad + M(T+1) \sup_{t\in[0,T]} ||x_t(\phi_2) - x_t(\phi_1)||)\}. \end{aligned}$$

Therefore, our conclusion follows from the continuity of $x_t(\phi)$ with respect to ϕ in the topology of C. This completes the proof. □

Theorem 2.6. *If* $\int_{-r}^{-r_0} e^{-\mu s}\,d\nu(s) < 1$ *and the condition*

(M) $\mu[D(\phi_2) - D(\phi_1)] + f(\phi_2) - f(\phi_1) \geq 0$ *whenever* $\phi_1, \phi_2 \in X$ *and* $\phi_1 \leq_\mu \phi_2$

is satisfied, then the solution semiflow on X *is monotone, i.e., from* $\phi_1 \leq_\mu \phi_2$ *it follows that* $x_t(\phi_1) \leq_\mu x_t(\phi_2)$ *for* $t \in [0, \min\{\tau(\phi_1), \tau(\phi_2)\})$.

Proof. For any $\varepsilon > 0$ we define $f_\varepsilon(\phi) = f(\phi) + \varepsilon D(\phi)$ for $\phi \in C$, and denote by $x(t, \phi, \varepsilon)$ the unique solution of the initial value problem

$$\begin{aligned} \frac{d}{dt} D x_t &= f_\varepsilon(x_t), \\ x_0 &= \phi. \end{aligned}$$

By the well-known continuous dependence on initial data and right-hand functionals of solutions to neutral equations on C (see, e.g. [**5**]), for any $0 < T < \min\{\tau(\phi_1), \tau(\phi_2)\}$, there exists $\varepsilon_0 > 0$ such that if $0 < \varepsilon < \varepsilon_0$, then $x(t, \phi_i, \varepsilon)$ exists on $[0, T]$ for $i = 1, 2$.

Let $t_1 \in [0, T]$ be the maximum real number such that $x(t, \phi_1, \varepsilon) \leq x(t, \phi_2, \varepsilon)$ on $[0, t_1]$ and $e^{\mu t}[x(t, \phi_2, \varepsilon) - x(t, \phi_1, \varepsilon)]$ is nondecreasing on $[-r, t_1]$. We want to show that $t_1 = T$.

By way of contradiction, we assume $t_1 < T$. If $x(t_1, \phi_2, \varepsilon) = x(t_1, \phi_1, \varepsilon)$, then

$$0 = e^{\mu t_1}[x(t_1, \phi_2, \varepsilon) - x(t_1, \phi_1, \varepsilon)] \geq e^{\mu t}[x(t, \phi_2, \varepsilon) - x(t, \phi_1, \varepsilon)] \geq 0$$

for all $t \in [-r, t_1]$. Therefore, $\phi_1 = \phi_2$ on $[-r, 0]$. Hence by uniqueness, t_1 is not the maximum number satisfying the stated properties. So $x(t_1, \phi_2, \varepsilon) > x(t_1, \phi_1, \varepsilon)$. Let $y(t) = x(t, \phi_2, \varepsilon) - x(t, \phi_1, \varepsilon)$. By condition (M), we obtain

$$\begin{aligned}
&\frac{d}{dt}\left[e^{\mu t} D(x_t(\phi_2, \varepsilon) - x_t(\phi_1, \varepsilon))\right]|_{t=t_1} \\
&= e^{\mu t_1}(\mu + \varepsilon) D[x_{t_1}(\phi_2, \varepsilon) - x_{t_1}(\phi_1, \varepsilon)] + e^{\mu t_1}\left[f(x_{t_1}(\phi_2, \varepsilon)) - f(x_{t_1}(\phi_1, \varepsilon))\right] \\
&\geq \varepsilon e^{\mu t_1} D(x_{t_1}(\phi_2, \varepsilon) - x_{t_1}(\phi_1, \varepsilon)) \\
&= \varepsilon e^{\mu t_1}\left\{[x(t_1, \phi_2, \varepsilon) - x(t_1, \phi_1, \varepsilon)] - \int_{-r}^{-r_0} [x(t_1 + s, \phi_2, \varepsilon) - x(t_1 + s, \phi_1, \varepsilon)]\,d\nu(s)\right\} \\
&= \varepsilon e^{\mu t_1}\left\{y(t_1) - \int_{-r}^{-r_0} y(t_1 + s) e^{\mu s} e^{-\mu s}\,d\nu(s)\right\} \\
&\geq \varepsilon e^{\mu t_1} y(t_1)\left[1 - \int_{-r}^{-r_0} e^{-\mu s}\,d\nu(s)\right] \\
&> 0.
\end{aligned}$$

Therefore, since $\frac{d}{dt}[e^{\mu t} D(x_t(\phi_2, \varepsilon) - x_t(\phi_1, \varepsilon))]$ is continuous at $t = t_1$ (continuous from the right if $t_1 = 0$), there exists $h_0 > 0$ and $\alpha_0 > 0$ such that $h_0 < r_0$ and

$$\frac{d}{dt}\left[e^{\mu t} D(x_t(\phi_2, \varepsilon) - x_t(\phi_1, \varepsilon))\right] \geq \alpha_0, \qquad t \in [t_1, t_1 + h_0].$$

Therefore

$$e^{\mu t}D(x_t(\phi_2,\varepsilon)-x_t(\phi_1,\varepsilon))-e^{\mu s}D(x_s(\phi_2,\varepsilon)-x_s(\phi_1,\varepsilon))$$
$$\geq \alpha_0(t-s), \qquad t_1\leq s\leq t\leq t_1+h_0.$$

That is,

$$e^{\mu t}y(t)-e^{\mu s}y(s)\geq \alpha_0(t-s)+\int_{-r}^{-r_0}\left[e^{\mu t}y(t+\theta)-e^{\mu s}y(s+\theta)\right]\,d\nu(\theta)$$

for $t_1\leq s\leq t\leq t_1+h_0$. We note that $e^{\mu t}y(t)$ is nondecreasing on $[-r,t_1]$ and $t-r_0\leq t_1+h_0-r_0<t_1$. Therefore

$$\int_{-r}^{-r_0}\left[e^{\mu t}y(t+\theta)-e^{\mu s}y(s+\theta)\right]\,d\nu(\theta)\geq 0.$$

Consequently, $e^{\mu t}y(t)-e^{\mu s}y(s)\geq\alpha_0(t-s)$. This shows that $e^{\mu t}y(t)$ is increasing in $[t_1,t_1+h_0]$, contradicting the maximality of t_1.

So $x_t(\phi_1,\varepsilon)\leq_\mu x_t(\phi_2,\varepsilon)$ on $[0,T]$. Letting $\varepsilon\to 0$, we obtain the monotonicity.□

Theorem 2.7. *If* $\int_{-r}^{-r_0}e^{-\mu s}\,d\nu(s)<1$ *and the condition*

(SM) $\mu[D(\phi_2)-D(\phi_1)]+f(\phi_2)-f(\phi_1)>0$ *whenever* $\phi_1,\phi_2\in X$ *and* $\phi_1<_\mu\phi_2$

is satisfied, then the solution semiflow on X *is strongly order preserving, i.e., it is monotone, and whenever* ϕ, $\psi\in X$ *with* $\phi<_\mu\psi$, *there exist open sets* U *and* V *such that* $\phi\in U$, $\psi\in V$ *and* $x_r(U)\leq_\mu x_r(V)$ *(of course, we assume that both solutions* $x(\phi)$ *and* $x(\psi)$ *can be defined in an open interval containing* $[0,r]$*).*

Proof. Assume that both solutions $x(t,\phi)$ and $x(t,\psi)$ are defined on $[0,r]$. By Theorem 2.6, if $\phi<_\mu\psi$, then $x_t(\phi)\leq_\mu x_t(\psi)$ on $[0,r]$. It follows from the uniqueness of solutions and the nondecreasing property of $e^{\mu\theta}[x_t(\psi)(\theta)-x_t(\phi)(\theta)]$ on $[-r,0]$ that $x(t,\phi)<x(t,\psi)$ on $[0,r]$. In particular, $x_t(\phi)<_\mu x_t(\psi)$, $0\leq t\leq r$. Therefore, by condition (SM) and the compactness of $\cup_{t\in[0,r]}\{x_t(\phi),x_t(\psi)\}$ in C, we have

$$\beta_0:=\min_{t\in[0,r]}\{\mu[D(x_t(\psi)-x_t(\phi))]+f(x_t(\psi))-f(x_t(\phi))\}>0.$$

So

$$\frac{d}{dt}D(x_t(\psi)-x_t(\phi))+\mu D(x_t(\psi)-x_t(\phi))\geq\beta_0>0, \qquad t\in[0,r].$$

That is,

$$y'(t)+\mu y(t)-\int_{-r}^{-r_0}[y'(t+\theta)+\mu y(t+\theta)]\,d\nu(\theta)\geq\beta_0$$

on $[0, r]$, where $y(t) = x(t, \psi) - x(t, \phi)$. Hence, applying that $x_t(\phi) <_\mu x_t(\psi)$ and $x(t, \phi) < x(t, \psi)$ for $0 \le t \le r$, one obtains

$$y(t) > 0, \; y'(t) + \mu y(t) > 0, \qquad 0 \le t \le r.$$

Then it follows that there exists an $\varepsilon > 0$ such that for the balls $B_1 = \{\eta \in X;\; d(\eta, x_r(\phi)) < \varepsilon\}$ and $B_2 = \{\eta \in X;\; d(\eta, x_r(\psi)) < \varepsilon\}$ in X the relation $B_1 \le_\mu B_2$ holds. By Lemma 2.5, $x_r(\cdot)$ is continous from X to X provided solutions exist on $[0, r]$. Therefore we can find open sets U and V in X with $\phi \in U$ and $\psi \in V$ such that $x_r(U) \subset B_1$ and $x_r(V) \subset B_2$. This implies $x_r(U) \le_\mu x_r(V)$ and the proof is complete. □

Theorems 2.6 and 2.7 allow us to apply the powerful theory of strongly order-preserving semiflows developed in [**9**] and [**11**] to the solution semiflow on X of equation (1.1). We will ilustrate this by an application of the principle of sequential limit set trichotomy to the problem of asymptotic constancy of equation (1.3).

3. Asymptotic Constancy

We now consider the following scalar neutral functional differential equation

$$\frac{d}{dt}[x(t) - cx(t-\tau)] = g(x(t), x(t-\omega)) \tag{3.1}$$

where $0 \le c < 1$, $\tau > 0$, $\omega \ge 0$, $g : R^2 \to R$ is locally Lipschitz continuous and

$$L_1 = \inf_{(x,y)\in R^2} \frac{\partial g(x,y)}{\partial x} > -\infty, \quad L_2 = \inf_{(x,y)\in R^2} \frac{\partial g(x,y)}{\partial y} > -\infty.$$

Then g satisfies the following one-sided global Lipschitz condition:
(L) If $x_1 \le x_2$ and $y_1 \le y_2$, then $g(x_2, y_2) - g(x_1, y_1) \ge L_1(x_2 - x_1) + L_2(y_2 - y_1)$.

For the above equation, $D(\phi) = \phi(0) - c\phi(-\tau)$, $f(\phi) = g(\phi(0), \phi(-\omega))$ for $\phi \in C([-r, 0]; R)$, where $r = \max\{\tau, \omega\}$. We now fix $\mu \ge 0$. Then for any $\phi_2 \ge_\mu \phi_1$, we have

$$\begin{aligned} &\mu[D(\phi_2) - D(\phi_1)] + f(\phi_2) - f(\phi_1) \\ &\quad \ge \mu[\phi_2(0) - \phi_1(0)] - c\mu[\phi_2(-\tau) - \phi_1(-\tau)] \\ &\qquad + L_1[\phi_2(0) - \phi_1(0)] + L_2[\phi_2(-\omega) - \phi_1(-\omega)] \\ &\quad = (\mu + L_1)[\phi_2(0) - \phi_1(0)] - c\mu[\phi_2(-\tau) - \phi_1(-\tau)] + L_2[\phi_2(-\omega) - \phi_1(-\omega)]. \end{aligned}$$

Note that $\phi_2 \ge_\mu \phi_1$ implies that

$$\phi_2(0) - \phi_1(0) \ge [\phi_2(-\theta) - \phi_1(-\theta)]e^{-\mu\theta}, \quad \theta = \tau, \; \omega.$$

Then with $L_2^- = \min\{L_2, 0\}$, for $\phi_2 \geq_\mu \phi_1$ we have

$$\begin{aligned}
&\mu[D(\phi_2) - D(\phi_1)] + f(\phi_2) - f(\phi_1) \\
&\quad \geq (\mu + L_1)[\phi_2(0) - \phi_1(0)] - c\mu[\phi_2(-\tau) - \phi_1(-\tau)] + L_2^-[\phi_2(-\omega) - \phi_1(-\omega)] \\
&\quad \geq [\mu + L_1 - c\mu e^{\mu\tau} + L_2^- e^{\mu\omega}][\phi_2(0) - \phi_1(0)].
\end{aligned}$$

Therefore, condition (M) holds if

$$\mu + L_1 - c\mu e^{\mu\tau} + L_2^- e^{\mu\omega} \geq 0. \tag{3.2}$$

Notice also that if $\phi_2 >_\mu \phi_1$ then $\phi_2(0) > \phi_1(0)$ and hence (SM) is satisfied if

$$\mu + L_1 - c\mu e^{\mu\tau} + L_2^- e^{\mu\omega} > 0. \tag{3.3}$$

We next consider the case where $\mu + L_1 \geq 0$. Then for any nonnegative constants α_1, α_2 with $\alpha_1 + \alpha_2 = 1$, we have for $\phi_2 \geq_\mu \phi_1$ that

$$\begin{aligned}
\mu[D(\phi_2) - D(\phi_1)] &+ f(\phi_2) - f(\phi_1) \\
&\geq [\alpha_1(\mu + L_1) - c\mu e^{\mu\tau}][\phi_2(0) - \phi_1(0)] \\
&\quad + [\alpha_2(\mu + L_1)e^{-\mu\omega} + L_2][\phi_2(-\omega) - \phi_1(-\omega)].
\end{aligned}$$

So, (M) is satisfied if

$$\begin{cases} \mu + L_1 \geq 0, \\ \alpha_1(\mu + L_1)e^{-\mu\tau} - c\mu \geq 0, \\ \alpha_2(\mu + L_1)e^{-\mu\omega} + L_2 \geq 0, \end{cases} \tag{3.4}$$

and (SM) is satisfied if

$$\begin{cases} \mu + L_1 \geq 0, \\ \alpha_1(\mu + L_1)e^{-\mu\tau} - c\mu > 0, \\ \alpha_2(\mu + L_1)e^{-\mu\omega} + L_2 \geq 0. \end{cases} \tag{3.5}$$

Note also that if $\omega \leq \tau$ then $\phi_2 \geq_\mu \phi_1$ implies

$$\phi_2(-\tau) - \phi_1(-\tau) \leq e^{\mu(\tau-\omega)}[\phi_2(-\omega) - \phi_1(-\omega)].$$

So $\phi_2 \geq_\mu \phi_1$ implies that

$$\begin{aligned}
&\mu[D(\phi_2) - D(\phi_1)] + f(\phi_2) - f(\phi_1) \\
&\quad \geq [(\mu + L_1)e^{-\mu\omega} - c\mu e^{\mu(\tau-\omega)} + L_2][c\phi_2(-\omega) - \phi_1(-\omega)]
\end{aligned}$$

and hence (M) is also satisfied if

$$(3.6)\qquad \begin{cases} \mu + L_1 \geq 0, \\ (\mu + L_1)e^{-\mu\omega} - c\mu e^{\mu(\tau-\omega)} + L_2 \geq 0. \end{cases}$$

More sets of sufficient conditions for (M) and (SM) can be given similarly.

Similarly, we can show that if ϕ, $\psi \in C$ with $\phi(s) \geq \psi(s)$ for $s \in [-r, 0]$ then $f(\phi) - f(\psi) \geq (L_1^- + L_2^-)||\phi - \psi||$. Hence, (L^-) holds with $L = -(L_1^- + L_2^-)$.

In the case where $g(x, y) = -h(x) + h(y)$, $(x, y) \in R^2$, (L$^-$) is satisfied if $-\infty < \inf_{x\in R} h'(x) \leq \sup_{x\in R} h'(x) < +\infty$. Therefore, (SM) is satisfied if either (3.3) or (3.5) holds with $L_1 = -\sup_{x\in R} h'(x)$ and $L_2 = \inf_{x\in R} h'(x)$. In particular, (3.3) holds if $0 \leq \inf_{x\in R} h'(x) \leq \sup_{x\in R} h'(x) < +\infty$ and c is sufficiently small.

Note also that for equation (3.1) with $g(x, y) = -h(x) + h(y)$, $(x, y) \in R^2$, every constant function is a solution. The following result shows that one can apply the principle of sequential limit set trichotomy (Proposition 3.1 in [**9**]) to prove that, under certain conditions, every solution of (3.1) converges to a constant.

Theorem 3.1. *Consider equation* (3.1) *with* $g(x, y) = -h(x) + h(y)$, $(x, y) \in R^2$. *Assume that there exists a constant* $\mu > 0$ *so that one of the conditions*

$$(3.7)\qquad \inf_{x\in R} h'(x) \geq 0,\ \ \mu(1 - ce^{\mu\tau}) - \sup_{x\in R} h'(x) > 0;$$

$$(3.8)\qquad \inf_{x\in R} h'(x) < 0,\ \ \mu(1 - ce^{\mu\tau}) - \max\{\sup_{x\in R} h'(x), 0\} + \inf_{x\in R} h'(x)e^{\mu\omega} > 0$$

is satisfied. Then for any $\phi \in X := \{\phi \in C^1 :\ \phi'(0) - c\phi'(-\tau) = -h(\phi(0)) + h(\phi(-\omega))\}$ *there exists a constant* $m = m_\phi$ *so that* $\lim_{t\to\infty} x(t, \phi) = m_\phi$.

Proof. It is easy to show that either (3.7) or (3.8) implies (3.3) with $L_1 = -\sup_{x\in R} h'(x)$ and $L_2 = \inf_{x\in R} h'(x)$. Therefore, (3.1) satisfies (SM). Moreover, $\mu > -(L_1^- + L_2^-)$.

In order to apply Proposition 3.1 of [**9**], the following compactness hypothesis is required:

(C) For each $\phi \in X$, $\{x_t(\phi) :\ t \geq 0\}$ is precompact in X and, in addition, for each compact subset K of X, $\cup_{\phi\in K}\omega(\phi)$ is precompact in X.

This condition holds, in particular, if for any bounded subset B of X, the set $\{x_t(\phi) :\ \phi \in B,\ t \geq 0\}$ is precompact in X (see [**9**]). For a given $\phi \in X$ we have

$$-\infty < K_\phi \leq K^\phi < +\infty,$$

where

$$K_\phi = \frac{1}{\mu}\min\{0, \min_{\theta\in[-r,0]} \phi'(\theta)\} + \min_{\theta\in[-r,0]} \phi(\theta),$$
$$K^\phi = \frac{1}{\mu}\max\{0, \max_{\theta\in[-r,0]} \phi'(\theta)\} + \max_{\theta\in[-r,0]} \phi(\theta).$$

Clearly, $\phi(\theta) \le K^\phi$ for $\theta \in [-r,0]$. Moreover

$$\frac{d}{d\theta}[(K^\phi - \phi(\theta))e^{\mu\theta}] = [\mu(K^\phi - \phi(\theta)) - \phi'(\theta)]e^{\mu\theta} \ge 0, \quad \theta \in [-r,0].$$

Therefore, $\phi \le_\mu K^\phi$. Similarly, $\phi \ge_\mu K_\phi$. So by Theorem 2.6, we have

$$K_\phi \le_\mu x_t(\phi) \le_\mu K^\phi, \quad t \ge 0,$$

since every constant function is a solution of (3.1). Consequently,

$$K_\phi \le x(t,\phi) \le K^\phi, \quad t \ge 0,$$

and

$$\mu(K_\phi - K^\phi) \le \mu[K_\phi - x(t,\phi)] \le x'(t,\phi) \le \mu[K^\phi - x(t,\phi)] \le \mu(K^\phi - K_\phi)$$

on $[0,\infty)$. If B is a bounded subset of X, then $K^B = \sup_{\phi\in B} K^\phi < \infty$, $K_B = \inf_{\phi \in B} K_\phi > -\infty$ and

$$K_B \le x(t,\phi) \le K^B,\ \mu(K_B - K^B) \le x'(t,\phi) \le \mu(K^B - K_B) \quad (t \ge 0,\ \phi \in B).$$

Hence it follows that $\{x_t(\phi):\ t \ge 0,\ \phi \in B\}$ is precompact in C. From $x \in C^1([-r,\infty);R)$ we obtain $D({x'}_{t+h} - {x'}_t) = f(x_{t+h}) - f(x_t)$, $t \ge 0$, $h \ge 0$. Hence, applying the above inequalities and Theorem 4.1 of [**5, p. 287**] (D is stable in the sense of [**5**], it follows that there is a $b > 0$ such that

$$||x'_{t+h}(\phi) - x'_t(\phi)|| \le b||x'_h(\phi) - \phi'|| + bh\mu(K^B - K_B) \sup_{x\in R} |h'(x)|$$

for any $t \ge 0$ and $\phi \in B$. Therefore, $\{x'_t(\phi):\ t \ge 0,\ \phi \in B\}$ is uniformly bounded and equicontinuous in C. Consequently, $\{x_t(\phi):\ t \ge 0,\ \phi \in B\}$ is precompact in X.

By Lemma 2.3, ϕ can be approximated from below in X by a sequence. Therefore, by the principle of sequential limit set trichotomy (Proposition 3.1 in [**9**]), we can find a sequence $\{\phi_n\}$ such that $\phi_n <_\mu \phi_{n+1} <_\mu \phi$ for $n \ge 1$ with $\phi_n \to \phi$ as $n \to \infty$ and satisfying one of the following: (a) $\omega(\phi) = u_\phi$ for some $u_\phi \in X$; (b) $\omega(\phi_n)$ are identical for all n and consist of a single point (equilibrium) and $\omega(\phi_n) <_\mu \omega(\phi)$; (c) $\omega(\phi_n)$ are all identical to $\omega(\phi)$ for all n, where $\omega(\phi)$ is the omega limit set of the orbit $\{x_t(\phi)\}_{t\ge 0}$. Since in case (a) $u_\phi = \omega(\phi)$ is a single point in X, the invariance of $\omega(\phi)$ implies that $u_\phi \equiv m_\phi$ for some constant m_ϕ. So it suffices to exclude (b) and (c). This can be done by proving that if (3.7) or (3.8) is satisfied and ϕ, $\psi \in X$ with $\psi >_\mu \phi$, then $\omega(\psi) \cap \omega(\phi) = \emptyset$. Note that (3.1) with $g(x,y) = -h(x) + h(y)$ is equivalent to

$$\frac{d}{dt}[x(t) - cx(t-\tau) + \int_{-\omega}^0 h(x(t+s))\,ds] = 0,\ t \ge 0.$$

Therefore, any point $\phi^* \in \omega(\phi)$ satisfies

$$\phi^*(0) - c\phi^*(-\tau) + \int_{-\omega}^{0} h(\phi^*(s))\,ds = \phi(0) - c\phi(-\tau) + \int_{-\omega}^{0} h(\phi(s))\,ds.$$

Similar result holds for $\psi^* \in \omega(\psi)$. Consequently, if $\omega(\psi) \cap \omega(\phi) = \emptyset$ fails, then

$$\phi(0) - c\phi(-\tau) + \int_{-\omega}^{0} h(\phi(s))\,ds = \psi(0) - c\psi(-\tau) + \int_{-\omega}^{0} h(\psi(s))\,ds. \tag{3.9}$$

But $\inf_{x\in R} h'(x) < 0$ and $\psi >_\mu \phi$ imply that

$$\begin{aligned}
&\psi(0) - c\psi(-\tau) + \int_{-\omega}^{0} h(\psi(s))\,ds - [\phi(0) - c\phi(-\tau) + \int_{-\omega}^{0} h(\phi(s))\,ds] \\
&\quad \geq [1 - ce^{\mu\tau} + \inf_{x\in R} h'(x) \int_{-\omega}^{0} e^{-\mu s}\,ds][\psi(0) - \phi(0)] \\
&\quad > \mu^{-1}[\mu(1 - ce^{\mu\tau}) + \inf_{x\in R} h'(x) e^{\mu\omega}][\psi(0) - \phi(0)] \\
&\quad > 0,
\end{aligned}$$

by using (3.8). This clearly contradicts to (3.9). Similarly, if $\inf_{x\in R} h'(x) \geq 0$, by using condition (3.7), we will also get a contradiction to (3.9). This shows that $\omega(\phi) \cap \omega(\psi) = \emptyset$ and the proof is complete.

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