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FACTORABLE CONGRUENCES AND STRICT REFINEMENT

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ABSTRACT. We show that universal algebras with factorable congruences such as rings with 1 and semirings with 0 and 1 enjoy some of the properties of universal algebras whose congruence lattices are distributive, such as the strict refinement property and a variant of Jónsson's lemma.

A universal algebra A is said to have factorable congruences if whenever $A \cong B \times C$ and θ is a congruence on A, then $\theta = \varphi \times \psi$, where φ is a congruence on B and ψ is a congruence on C $(((b,c),(b',c')) \in \varphi \times \psi$ iff $(b,b') \in \varphi$ and $(c,c') \in \psi)$. In other words, if $A = B \times C$ and f is a homomorphism of A onto a universal algebra D, there are homomorphisms g of B onto a universal algebra E and h of C onto a universal algebra F such that $D \cong E \times F$ and $f = i \circ (g,h)$, where i is an isomorphism of $E \times F$ onto D; i.e., the operations of forming homomorphic images and finite Cartesian products permute. Varieties of universal algebras in which every algebra has factorable congruences are characterized via Mal'cev conditions by Fraser and Horn [**9**]. These varieties include varieties whose congruence lattices are distributive such as lattices and not necessarily associative Boolean rings [**13**], and varieties with uniform congruence schemes [**10**], [**5**]. Any algebra with binary terms $+$, $\cdot$ and constants 0, 1 satisfying the identities $x \cdot 1 = x + 0 = 0 + x = x$, $x \cdot 0 = 0$ has factorable congruences [**9**]. This includes rings with 1, near-rings with 1, and semirings with 0 and 1. We prove that universal algebras with factorable congruences enjoy the strict refinement property and every directly indecomposable homomorphic image of a Cartesian product of a family $\mathcal{K}$ of algebras with factorable congruences is a homomorphic image of an ultraproduct of $\mathcal{K}$. A variety of universal algebras $\mathcal{V}$ is said to have the Apple property [**2**], if the congruence lattice of every finite directly indecomposable member of $\mathcal{V}$ has a unique coatom. In [**2**], Berman and Blok prove a number of theorems concerning finite universal algebras with factorable congruences and satisfying the Apple property. We will show here that some of these theorems hold

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for universal algebras with factorable congruences and not necessarily satisfying the Apple property.

General references in universal algebra include Burris and Sankappanavar [**4**], Cohn [**7**], Grätzer [**12**], Mal'cev [**15**] and McKenzie, McNulty and Taylor [**17**]; also Bell and Slomson [**1**] for models and ultraproducts. Unless stated otherwise, the notations are those of McKenzie, McNulty and Taylor [**17**]. The word "algebra" will mean "universal algebra". We will use the same symbol for an algebra and its carrier set. For any algebra A, $\Delta(A)$ will denote the identity relation $\{(x,x) : x \in A\}$.

B. Jónsson's classic lemma [**14**] states that in a variety of algebras whose congruence lattices are distributive, if B is a subdirectly irreducible algebra in the variety generated by a family $\mathcal{K}$ of algebras, then B is a homomorphic image of a subalgebra of an ultraproduct of algebras in $\mathcal{K}$. Although algebras with factorable congruences do not generally satisfy Jónsson's lemma, we show that such algebras satisfy an interesting variant of the lemma.

Theorem 1. *Suppose A_i, $i \in I$ and B are algebras of a given similarity type. Let $A = \prod\{A_i : i \in I\}$ and let f be a homomorphism of A onto B. If A has factorable congruences and B is directly indecomposable, then B is a homomorphic image of an ultraproduct of $\{A_i : i \in I\}$.*

Proof. Let $\mathcal{U}$ be the set of all subsets S of I such that for any a, $b \in A$, $a \mid S = b \mid S$ implies $f(a) = f(b)$. Then $\mathcal{U}$ is an ultrafilter on I. Indeed, $I \in \mathcal{U}$, so $\mathcal{U}$ is not empty and $\emptyset \notin \mathcal{U}$, otherwise, $f(a) = f(b)$ for all a, $b \in A$ and the algebra B is trivial. If $S \in \mathcal{U}$ and $S \subseteq T \subseteq I$, then $T \in \mathcal{U}$. Let S, $T \in \mathcal{U}$, a, $b \in A$ and $a \mid S \bigcap T = b \mid S \bigcap T$. Define $c \in A$ by $c(i) = a(i)$ if $i \in S$, $c(i) = b(i)$, if $i \notin S$. Thus $c \mid S = a \mid S$ and so $f(a) = f(c)$. As $a \mid S \bigcap T = b \mid S \bigcap T$, $c \mid T = b \mid T$ and so $f(c) = f(b)$. Hence $\mathcal{U}$ is closed under intersection. This shows that $\mathcal{U}$ is a proper filter on I. We will be through once we show that for any $S \subseteq I$, either S or its complement S' belongs to $\mathcal{U}$ since $\{(a,b) : \{i : a(i) = b(i)\} \in \mathcal{U}\} \subseteq \ker(f)$. As $A = \prod\{A_i : i \in I\} \cong C \times D$, where $C = \prod\{A_i : i \in S\}$ and $D = \prod\{A_i : i \in S'\}$, and A has factorable congruences, the algebra B is a homomorphic image of $C \times D \cong A$, then $B \cong E \times F$ and there are homomorphisms g of C onto E and h of D onto F such that if we identify A with $C \times D$ and B with $E \times F$, the mapping f is identifiable with (g,h). Since B is directly indecomposable exactly one of the algebras E, F is trivial. If, for instance F is trivial, then $B \cong E$. If a, $b \in A$ and $a \mid S = b \mid S$, then there are $c \in C$, d, $e \in D$ such that $a = (c,d)$ and $b = (c,e)$ and $f(a) = (g(c), h(d)) = (g(c), h(e)) = f(b)$ and $S \in \mathcal{U}$. Similarly, if E is trivial, then $S' \in \mathcal{U}$. $\square$

Thus if all the algebras A_i in Theorem 1 are finite and there is only finitely many isomorphism types, then B is a homomorphic image of one of the algebras A_i.

We will consider now the refinement properties of algebras with factorable congruences. First we give a brief description of the definitions and some of the equivalent forms of the strict refinement property all of which is in [**8**], [**6**], [**17**]. Let A be an algebra and let θ, θ_i, $i \in I$ be congruences on A and for every $i \in I$, $\theta \subseteq \theta_i$. The congruence θ is said to be the product of the family $\{\theta_i : i \in I\}$ if the mapping $f : A/\theta \to \prod\{A/\theta_i : i \in I\}$ given by $f(a/\theta)(i) = a/\theta_i$ is an isomorphism of the quotient algebra A/θ onto $\prod\{A/\theta_i : i \in I\}$, where a/θ is the congruence class of a under θ. That θ is the product of the family $\{\theta_i : i \in I\}$ is expressed as $\theta = \prod\{\theta_i : i \in I\}$. The product of congruences α, β, if it exists, is denoted by $\alpha \times \beta$. A congruence θ is the product of $\{\theta_i : i \in I\}$ iff $\theta = \bigcap\{\theta_i : i \in I\}$ and for any family of elements $a_i \in A$, $i \in I$, there is $a \in A$ such that $(a, a_i) \in \theta_i$ for every $i \in I$. If $0 = \Delta(A) = \alpha \times \beta$, then α, β are called a factor congruence pair and α is called a factor congruence. An algebra A is said to have the strict refinement property, if for any families of congruences θ_i, $i \in I$ and ψ_j , $j \in J$ on A such that $0 = \Delta(A) = \prod\{\theta_i : i \in I\} = \prod\{\psi_j : j \in J\}$, $\theta_i = \prod\{\theta_i \vee \psi_j : j \in J\}$ for every $i \in I$. The set of factor congruences of A is denoted by $\mathrm{FR}(A)$. We will state here as a theorem a number of conditions equivalent to the strict refinement property.

Theorem 2 (Chang, Jónsson and Tarski [**6**]). *For any algebra A the following conditions are equivalent:*

(i) *A has the strict refinement property.*
(ii) *A has the strict refinement property for finite index sets I and J.*
(iii) *The set of factor congruences $\mathrm{FR}(A)$ forms a Boolean sublattice of the congruence lattice $\mathrm{Con}\,A$.*

Rings with 1 satisfy the strict refinement property [**8**] and [**17**], so does every algebra whose congruence lattice is distributive. We show that all algebras with factorable congruences satisfy the strict refinement property.

Theorem 3. *Let A be an algebra with factorable congruences. Then*

(i) *The factor congruences of A are neutral elements of the congruence lattice of A.*
(ii) *The factor congruences of A form a Boolean sublattice of the congruence lattice of A.*
(iii) *The factor congruences of A are the complemented congruences of A that are contained in the centralizer of the congruence lattice of A in the monoid of binary relations of A with respect to the relative product.*

Proof. Let A be an algebra with factorable congruences and let ρ, σ be a factor congruence pair on A. Then $A \cong A/\rho \times A/\sigma$. If $\theta \in \mathrm{Con}\,A$, then $\theta = (\rho \vee \theta) \wedge (\sigma \vee \theta)$ as θ is factorable, [**9**], [**17**]. Thus θ can be identified with $\varphi \times \psi$, where $\varphi \in \mathrm{Con}(A/\rho)$ and $\psi \in \mathrm{Con}(A/\sigma)$. This means that ρ is a neutral element of $\mathrm{Con}\,A$ [**3**], [**11**]. This proves (i). Since an element a of a lattice L is neutral iff for any elements b, c of L the set $\{a, b, c\}$ generates a distributive sublattice of L, the neutral elements in $\operatorname{Con} A$ form a Boolean sublattice. So, the factor congruences form a distributive sublattice of $\operatorname{Con} A$ as every factor congruence has a complement which is a factor congruence. This shows (ii). Let ρ , σ be a factor congruence pair on A and let $\theta \in \operatorname{Con} A$. Then A can be identified with $B \times C$ where $B = A/\rho$ and $C = A/\sigma$ and then $\theta = \varphi \times \psi$ where φ is a congruence on B and ψ is a congruence on C. The factor congruence ρ can be identified with $\triangle(B) \times C^2$. Now $\theta \circ \rho = (\varphi \times \psi) \circ (\triangle(B) \times C^2) = \varphi \times C^2 = (\triangle(B) \times C^2) \circ (\varphi \times \psi) = \rho \circ \theta$. This shows (iii) in one direction. Suppose ρ is a congruence on A with a complement σ such that $\rho \circ \theta = \theta \circ \rho$ for all $\theta \in \operatorname{Con} A$. Then the pair of congruences ρ, σ satisfies $\rho \wedge \sigma = 0$, $\rho \vee \sigma = 1$ and $\rho \circ \sigma = \sigma \circ \rho$. Thus ρ and σ are a factor pair. □

Corollary 4. *Every algebra with factorable congruences has the strict refinement property.*

This follows from Theorems 2 and 3 since for any algebra A with factorable congruences, $\mathrm{FR}\,(A)$ is a Boolean sublattice of $\operatorname{Con} A$.

Corollary 5. *Suppose A is a direct product of finitely many simple algebras and A has factorable congruences. Then A is arithmetical.*

Proof. Suppose $A = A_1 \times \cdots \times A_k$ where every A_i is simple. The congruence lattice of A is a direct product of the congruence lattices of simple algebras. Thus $\operatorname{Con} A = \mathrm{FR}\,(A)$ since every congruence θ on A is a product of congruences θ_i on the simple algebra A_i and so $\theta_i \in \left\{\Delta\left(A_i\right), A_i^2\right\}$; θ_i is trivially a factor congruence on A_i. Thus $\operatorname{Con} A$ is Boolean by Theorems 2 and 3 and so A has permutable congruences by (iii) of Theorem 3. □

In [**2**], it is shown that a finite algebra A with factorable congruences and satisfying the Apple property has a unique direct factorization as a product of directly indecomposables and if $\alpha_1, \ldots, \alpha_m$ are all the distinct coatoms of $\operatorname{Con} A$, and $\alpha = \alpha_1 \cap \cdots \cap \alpha_m$, then $A/\alpha \cong A/\alpha_1 \times \cdots \times A/\alpha_m$. This shows that in a variety $\mathcal{V}$ of algebras with factorable congruences and satisfying the Apple property, the class of all finite simple algebras is multiplicative as explained in the following:

Definition 6 ([**2**]). A class $\mathcal{K}$ of algebras is said to be multiplicative if whenever an algebra A is representable as an irredundant subdirect product of $B_1, \ldots, B_n \in \mathcal{K}$, then $A \cong B_1 \times \cdots \times B_n$. The algebra A is an irredundant subdirect product of $B_1, \ldots, B_n$ if A is subdirectly embedded in $\prod \{B_i : 1 \leq i \leq n\}$ and for all i no projection of A onto $\prod \{B_j : 1 \leq j \leq n, i \neq j\}$ is one-to-one.

In any variety of algebras with permutable congruences, the class of all simple algebras is multiplicative. The following theorem was stated ([**2**, Lemma 4.5]) for locally finite varieties of algebras with factorable congruences and satisfying the Apple property. However, it holds for any algebras with factorable congruences.

Theorem 7. *Suppose A is an algebra with factorable congruences and $\mathcal{K}$ is a set of directly indecomposable algebras. If A is isomorphic to a subdirect product of algebras in $\mathcal{K}$, then so is every direct factor of A.*

Proof. The proof given in Lemma 4.5 of Berman and Blok [**2**] uses only the facts that the algebra A has factorable congruences and members of $\mathcal{K}$ are directly indecomposable. For the sake of completeness, we will sketch the proof. Suppose α_i , $i \in I$ is a family of directly indecomposable congruences of the algebra A such that $0 = \bigcap\{\alpha_i : i \in I\}$ and let $0 = \beta \times \gamma$. As A has factorable congruences, $\alpha_i = (\alpha_i \vee \beta) \times (\alpha_i \vee \gamma)$. Since α_i is directly indecomposable, either $\alpha_i = \alpha_i \vee \beta$ and $\alpha_i \vee \gamma = 1$ or $\alpha_i = \alpha_i \vee \gamma$ and $\alpha_i \vee \beta = 1$. Thus every α_i contains exactly one of the congruences β, γ. Let $\delta = \bigcap\{\alpha_i : \alpha_i \supseteq \beta, i \in I\}$. Then $\delta \wedge \gamma = \delta \times \gamma$. But $\delta \wedge \gamma \subseteq \bigcap\{\alpha_i : i \in I\} = 0$. Hence β, γ, δ are factor congruences of A. From Theorem 3, $\mathrm{FR}\,(A)$ is Boolean. Since β, δ are complements of γ, then $\beta = \delta$ and so the direct summand A/β is a subdirect product of the A/α_i for which $\beta \subseteq \alpha_i$.□

We will give here two modifications of theorems proved in [**2**] for varieties of algebras with factorable congruences and satisfying the Apple property (Theorem 4.3 and Corollary 4.4). The proofs are also modifications of the corresponding theorems in [**2**]. For any variety $\mathcal{V}$ of algebras, the subvariety of $\mathcal{V}$ generated by all the finite simple algebras is denoted by $\mathcal{V}_0$.

Theorem 8. *Let $\mathcal{V}$ be a locally finite variety of algebras with factorable congruences. Suppose the class of all finite simple algebras in $\mathcal{V}$ is multiplicative and every subalgebra of a finite simple algebra of $\mathcal{V}$ is a direct product of simple algebras. Then the variety $\mathcal{V}_0$ is arithmetical and every finite member of $\mathcal{V}_0$ is a direct product of simple algebras and every finite algebra in $\mathcal{V}_0$ generates a directly representable variety.*

Proof. By the characterization of arithmetical varieties (Pixley [**18**], [**20**]), the variety $\mathcal{V}_0$ is arithmetical if its free algebra of rank 3 is. Let F be a free algebra of $\mathcal{V}_0$ of rank n where n is a positive integer. As $\mathcal{V}$ is locally finite F is finite and F is a subdirect product of subalgebras of finite simple algebras from $\mathcal{V}$. As every subalgebra of a finite simple algebra in $\mathcal{V}$ is a direct product of simple algebras, the algebra F is a subdirect product of finite simple algebras. Since the class of all finite simple algebras in $\mathcal{V}$ is multiplicative and F is finite F is a direct product of finite simple algebras. Since $\mathcal{V}$ has factorable congruences, F is arithmetical by Corollary 5. If $A \in \mathcal{V}_0$ is finite, then A is a homomorphic image of a free algebra of some appropriate finite rank. Since every free algebra of $\mathcal{V}_0$ of finite rank is a direct product of a finite number of simple algebras and $\mathcal{V}$ has factorable congruences, A is a direct product of homomorphic images of simple algebras, and so A is a direct product of simple algebras. Let $\mathcal{W}$ be the variety generated by A. Suppose B is a finite algebra in $\mathcal{W}$. Let $\mathcal{L}$ be the set of all simple quotients of subalgebras of A. Then $\mathcal{L}$ is finite and B is a direct product of algebras from $\mathcal{L}$.

The set $\mathcal{L}$ generates $\mathcal{W}$. Let G be a free algebra of finite rank in $\mathcal{W}$ such that B is a homomorphic image of G. Thus G is a subdirect product of subalgebras of members of $\mathcal{L}$. Since every subalgebra of an algebra in $\mathcal{L}$ is a direct product of simple algebras isomorphic to algebras in $\mathcal{L}$, so G is a subdirect product of algebras from $\mathcal{L}$. As the class of finite simple algebras in $\mathcal{V}$ is multiplicative and G is finite, G is a direct product of algebras from $\mathcal{L}$. Thus B is a direct product of homomorphic images of algebras from $\mathcal{L}$. Since every member of $\mathcal{L}$ is simple, B is a direct product of algebras from $\mathcal{L}$. This shows that $\mathcal{W}$ is directly representable (Mckenzie [**16**].) This can also be shown by applying Jónsson's lemma [**14**]. $\square$

Theorem 9. *Let $\mathcal{V}$ be a locally finite variety of algebras with factorable congruences. Suppose the class of all finite simple algebras in $\mathcal{V}$ is multiplicative and every subalgebra of a finite simple algebra in $\mathcal{V}$ is simple. Then every finite algebra in $\mathcal{V}_0$ generates a ternary discriminator variety.*

Proof. This follows from Theorem 8 and a result of Pixley [**19**], [**20**] stating that any arithmetical variety generated by finitely many hereditarily simple finite algebras is a discriminator variety. $\square$

Theorem 8 can be proved along similar lines as in the corresponding proof in [**2**]. The statements in Theorems 8 and 9 are more general than the corresponding statements in [**2**] as attested by the following:

Example 10. Let $\mathcal{V}$ be the variety of all rings with 1 satisfying the identities $2x = 0$, $x^2 + x^4 = 0$, $(xy + yx)^2 = 0$. Then $\mathcal{V}$ is a locally finite variety that does not satisfy the Apple property and $\mathcal{V}_0$ is the variety of all Boolean rings.

Proof. Let A be a finite simple ring in $\mathcal{V}$. Then A is isomorphic to the ring of all $n \times n$ matrices over a division ring F, by the Wedderburn Artin Theorem. As F is finite, F is a field. Since $F \in \mathcal{V}$, F is isomorphic to the ring $\mathbb{Z}_2$ of integers modulo 2. We need to show that $n > 1$ is impossible. We will show a contradiction for $n = 2$. Let $x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $y = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Then $(xy + yx)^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \neq 0$. This shows that $\mathcal{V}_0$ is the variety of all Boolean rings. Since rings with 1 are rings with factorable congruences and have permuting congruences and $\mathbb{Z}_2$ has precisely one subring, namely $\mathbb{Z}_2$, the variety $\mathcal{V}$ satisfies all the conditions of Theorem 9. We will be through once we show a finite directly indecomposable ring $B \in \mathcal{V}$ with two distinct maximal ideals. Let B be the ring whose additive group is the elementary abelian group of exponent 2 with basis $\{a, b, c\}$ and $a^2 = a$, $b^2 = b$, $c^2 = 0$, $ab = ba = 0$, $ca = c$, $cb = 0$, $ac = 0$, $bc = c$. In the ring B, the identity element is $1 = a + b$ and $\{0, c\}$ is the unique minimal ideal of B. Thus B is subdirectly irreducible. The ideals $\{0, a, c, a + c\}$ and $\{0, b, c, b + c\}$ are distinct maximal ideals of B. Thus B does not satisfy the Apple property. $\square$

Now, we look at connections between direct factorizations of algebras and their lattices of factor congruences.

Theorem 11. *Let A_i, $i \in I$ be a family of algebras of the same similarity type and $A = \prod\{A_i : i \in I\}$. If A has the strict refinement property, then* $\mathrm{FR}(A) \cong \prod\{\mathrm{FR}(A_i) : i \in I\}$.

Proof. Let $A = \prod\{A_i : i \in I\}$. Then there are congruences $\rho_i \in \mathrm{FR}(A_i)$, $i \in I$ such that $0 = \prod\{\rho_i : i \in I\}$ and $A_i \cong A/\rho_i$ for every $i \in I$. If $\rho \in \mathrm{FR}(A)$ and ρ' is the complement of ρ in $\mathrm{FR}(A)$, then by the strict refinement property, $\rho = \prod\{\rho_i \vee \rho : i \in I\}$, $\rho' = \prod\{\rho_i \vee \rho' : i \in I\}$ and $\rho_i = (\rho_i \vee \rho) \wedge (\rho_i \vee \rho')$, $i \in I$. The mapping $\rho \longrightarrow \rho \vee \rho_i$ is a homomorphism of the lattice $\mathrm{FR}(A)$ onto the lattice of factor congruences of A containing ρ_i. This lattice is isomorphic to the lattice $\mathrm{FR}(A_i)$. Composing these homomorphisms we get a family of homomorphisms $f_i : \mathrm{FR}(A) \to \mathrm{FR}(A_i)$, $i \in I$; $f_i(\rho)$ is the congruence on A_i corresponding to $\rho \vee \rho_i$. The family of mappings f_i, $i \in I$ separate the elements of $\mathrm{FR}(A)$. Indeed, if $f_i(\rho) = f_i(\sigma)$, then $\rho_i \vee \rho = \rho_i \vee \sigma$ and so, $\rho = \prod\{\rho_i \vee \rho : i \in I\} = \prod\{\rho_i \vee \sigma : i \in I\} = \sigma$. Let σ_i, $\sigma_i' \supseteq \rho_i$, σ_i, $\sigma_i' \in \mathrm{FR}(A)$ and $\rho_i = \sigma_i \times \sigma_i'$, $i \in I$; i.e., σ_i, σ_i' correspond to a factor congruence pair on A_i. Then $\sigma = \prod\{\sigma_i : i \in I\}$ and $\sigma' = \prod\{\sigma_i' : i \in I\}$ exist and $\sigma \times \sigma' = \prod\{\sigma_i : i \in I\} \times \prod\{\sigma_i' : i \in I\} = \prod\{\sigma_i \times \sigma_i' : i \in I\} = \prod\{\rho_i : i \in I\} = 0$. Thus σ, σ' is a factor congruence pair on A. Thus $\sigma \vee \rho_i \leq \sigma_i$ for all $i \in I$. Hence $\sigma = \prod\{\rho_i \vee \sigma : i \in I\} \leq \prod\{\sigma_i : i \in I\} = \sigma$. We need to show that $\rho_i \vee \sigma = \sigma_i$ for every $i \in I$. Fix $j \in I$. Then $\sigma = \prod\{\rho_i \vee \sigma : i \in I\} = (\rho_j \vee \sigma) \times \prod\{\rho_i \vee \sigma : i \in I, i \neq j\} \leq \sigma_j \times \prod\{\rho_i \vee \sigma : i \in I, i \neq j\} \leq \prod\{\sigma_i : i \in I\} = \sigma$. Then $\lambda \times (\rho_j \vee \sigma) = \lambda \times \sigma_j = \sigma$, where $\lambda = \prod\{\rho_i \vee \sigma : i \in I, i \neq j\} \in \mathrm{FR}(A)$. As $\mathrm{FR}(A)$ is a Boolean lattice, $\rho_j \vee \sigma = \sigma_j$, since σ_j and $\rho_j \vee \sigma$ are complements of λ in the interval $[\sigma, 1]$. □

Theorem 12. *Let A be an algebra with the strict refinement property. If* $\mathrm{FR}(A) \cong L_0 \times L_1$*, then $A \cong A_0 \times A_1$ where $L_i \cong \mathrm{FR}(A_i)$, $i = 0, 1$.*

Proof. As $FR(A) \cong L_0 \times L_1$, there are ρ_0, $\rho_1 \in \mathrm{FR}(A)$ such that ρ_1 is the complement of ρ_0 and L_i is isomorphic to the sublattice of all $\rho \in \mathrm{FR}(A)$ containing ρ_i, $i = 0, 1$. Thus ρ_0, ρ_1 is a factor congruence pair on the algebra A. Putting $A_i = A/\rho_i$, $i = 0, 1$, $A \cong A_0 \times A_1$ and the factor congruences on A_i correspond to the factor congruences of A containing ρ_i. Thus $L_i \cong \mathrm{FR}(A_i)$, $i = 0, 1$. □

Hence the finite direct factorizations of the algebra A are in one-to-one correspondence with the finite direct factorizations of the Boolean algebra $\mathrm{FR}(A)$. Thus an algebra A with the strict refinement property is a finite direct product of directly indecomposable algebras iff its Boolean algebra $\mathrm{FR}(A)$ is finite. Theorem 11 shows that any factorization of A gives a corresponding factorization of $\mathrm{FR}(A)$. If the algebra A is a Cartesian product of directly indecomposable algebras, then $\mathrm{FR}(A)$ is isomorphic to the Boolean lattice of all subsets of some set; in other words, $\mathrm{FR}(A)$ is a complete atomic Boolean lattice. From the characterization of the strict refinement property (Theorem 2), the strict refinement property is satisfied in an algebra for finite decompositions iff it holds for infinite decompositions. However, infinite direct factorizations of algebras cannot generally be deduced from the infinite direct factorizations of their factor congruence lattices. In the following we give an example of an algebra A with factorable congruences such that $\mathrm{FR}\,(A)$ has infinite Cartesian factorizations, while A is not a Cartesian product of any infinite family of non-trivial algebras. The two element Boolean lattice will be denoted by $\mathbf{2}$.

Example 13. Let A be the ring of all functions with finite range from the set of natural numbers $\mathbb{N}$ into the ring of integers $\mathbb{Z}$. Then A is a commutative ring with 1, $\mathrm{FR}\,(A) \cong^{\mathbb{N}} \mathbf{2}$ and A is not the Cartesian product of any infinite family of non-trivial rings.

The factor congruences of A are determined by the idempotents of A. The idempotents of A are given by: For every subset S of $\mathbb{N}$, $e_S(i) = 1$ if $i \in S$ and $e_S(i) = 0$ if $i \notin S$ (If $e^2 = e$, then $e(i)^2 = e(i)$ and so $e(i) \in \{0, 1\}$.) The factor ideals; i.e., direct summands of A, are of the form eA, where e is an idempotent. Thus the lattice $\mathrm{FR}\,(A)$ is isomorphic to the lattice of all idempotents of A; i.e., $\mathrm{FR}\,(A) \cong^{\mathbb{N}} \mathbf{2}$. We need to show that A is not isomorphic to the Cartesian product of any infinite family of non-trivial rings. It is sufficient to show that $A \cong \prod \{A_n : n \in \mathbb{N}\}$ is impossible if every A_n is non-trivial. Suppose $A \cong \prod \{A_n : n \in \mathbb{N}\}$ and A_n is non-trivial. for every $n \in \mathbb{N}$. Then there are idempotents $e_n \in A$ such that $e_n \neq 0$, $e_n \neq 1$, $A_n \cong A/e_nA$ for every $n \in \mathbb{N}$ and $0 = \prod \{e_nA : n \in \mathbb{N}\}$. For every $n \in \mathbb{N}$, let $t_n \in \mathbb{N}$ satisfy $e_n(t_n) = 0$. Such number t_n exists since $e_n \neq 1$. Let $a_n \in A$ satisfy $a_n(t_n) = n$; for instance, the constant function n will do. Then there is $a \in A$ such that $a - a_n \in e_nA$ for every $n \in \mathbb{N}$. Hence $a(t_n) = a_n(t_n) = n$ for every $n \in \mathbb{N}$. This contradicts the fact that every element of the ring A is of finite range.

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